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On the Study of Singular Nonlinear Traveling Wave Equations: Dynamical System Approach

Jibin Li   Huihui Dai

（奇非线性行波方程研究的动力系统方法）

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Preface

Nonlinear wave phenomena are of great importance in the physical world and have been for a long time a challenging topic of research for both pure and applied mathematicians. There are numerous nonlinear evolution equations for which we need to analyze the properties of the solutions for time evolution of the system. As the first step, we should understand the dynamics of their traveling wave solutions.

There exists enormous literature on the study of nonlinear wave equations, in which exact explicit solitary wave, kink wave, periodic wave solutions, bifurcations and dynamical stabilities of these waves are discussed. To find exact traveling wave solutions for a given nonlinear wave system, a lot of methods have been developed such as the inverse scattering method, Darboux transformation method, Hirota bilinear method, algebraic-geometric method, tanh method, etc. What is the dynamical behavior of these exact traveling wave solutions? How do the traveling wave solutions depend on the parameters of the system? What is the reason of the smoothness change of traveling wave solutions? How to understand the dynamics of the so-called compacton and peakon solutions? These are very interesting and important problems. In recent years, these topics have seen significant advances and research is also very active.

The aim of this book is to give a more systematic account for the bifurcation theory method of dynamical systems to find traveling wave solutions with an emphasis on singular waves and understand their dynamics for some classes of the well-posedness of nonlinear evolution equations. Usually, most natural systems are modeled by nonlinear partial differential equations. To consider traveling wave solutions of a partial differential equation, the essential work is to investigate the dynamical behavior of the corresponding ordinary differential equation (traveling wave equation). Therefore, the theory of dynamical systems plays a pivotal role in the qualitative study.

In this book, we pursue the line of studying somewhat less general, but hopefully more typical nonlinear wave equations. Usually, there exists at least a singular straight line (or curve) such that the right hand of the corresponding traveling wave systems of these partial differential equations is discontinuous. It was first found by Dai (1998b) (see also Dai and Huo (2000)) that the presence of such a singular line in a phase plane can result in a variety of singular waves (i.e., compactons, peakons and periodic peakons, breaking waves and so on) to appear. The transformation technique of dynamical systems (first introduced in Li and Liu (2000)) allows us to achieve greater precision studies by using the method of dynamical bifurcation theory of the differentiable dynamics and to work out methods which lead to proofs within the present knowledge of analysis. We now call the study method as "three-step method", which will be introduced in Chapter 2.

The materials of this book are mainly taken from the published papers written by authors, their collaborators and students. Some results are new which will be appeared in future coming journals. We hope that this book can serve as a guide to what can be cleared about the dynamical ideas in studying the traveling wave solutions of some nonlinear wave equations and for correcting some mistakes in understanding the dynamical behavior of some exact explicit traveling wave solutions.

Any reader trying to understand the subject of this book is only required to know the elementary theory of dynamical systems and elementary knowledge of nonlinear wave equations.

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Contents

Chapter 1

Traveling Wave Equations of Some Physical Models

The mathematical modeling of important phenomena arising in physics and biology often leads to nonlinear wave equations. It is quite remarkable that many of these universal equations exhibit a regular behavior, typical of integrable partial differential systems (there exist Hamiltonian structures). And their traveling wave systems are also integrable ordinary differential equations, in which there exist some singular properties. In this chapter, we shall address some very interesting mathematical models which describe specific natural phenomena.

1.1 The model of nonlinear oscillations of hyperelastic rods

Suppose that a circular cylindrical rod undergoes only axisymmetric radial and longitudinal motion. The rod composes of compressible and isotropic homogeneous materials. In an undistorted reference configuration, the rod is a circular cylinder of radius a. Its motion can be described by

$$z = \tilde{z}(Z,T), \quad r = \tilde{r}(Z,T)R, \quad \theta = \Theta, \tag{1.1.1}$$

where z, r, θ are the cylindrical coordinates of a material point which has Z, R, Θ coordinates in the reference configuration and T is the time. For such a motion, the three principle invariants can be easily calculated and they are given by [1]

$$I_1 = 2r^2 + z_Z^2 + r_Z^2 R^2, \tag{1.1.2}$$

$$I_2 = r^2[r^2 + 2z_Z^2 + r_Z^2 R^2], \tag{1.1.3}$$

$$I_3 = r^4 z_Z^2. \tag{1.1.4}$$

[1] Henceforth, for the convenience of notations, we shall drop the $\tilde{}$, i.e. r means $\tilde{r}$.

For a compressible Mooney-Rivlin material, the Helmholtz free-energy function has the following form

$$\Phi = \frac{1}{2}\mu\left(\frac{1}{2}+\beta\right)(I_1-3)+\frac{1}{2}\mu\left(\frac{1}{2}-\beta\right)(I_2-3)$$
$$+\frac{1}{2}\mu k(I_3-1)-\frac{1}{2}\mu\left(k+\frac{3}{2}-\beta\right)\ln I_3, \tag{1.1.5}$$

where $\mu > 0,\ k > 0,\ -\frac{1}{2} < \beta \leqslant \frac{1}{2}$ are material constants. If the material is incompressible, then, the third principle invariant $I_3 = 1$, and equation (1.1.5) yields the free-energy function for an incompressible Mooney-Rivlin material. And if further $\beta = 1/2$, equation (1.1.5) gives the free-energy function for an incompressible neo-Hookean material. Generally, if $\beta = 1/2$ in (1.1.5), Φ, then depending only on the first and the third principal invariants I_1 and I_3, may be regarded as the free-energy function for a compressible neo-Hookean material.

With the above free-energy function, the potential energy density can be calculated. The kinetic energy density can be calculated by using (1.1.1). The difference of these two quantities gives the Lagrangian. The rod equations are given by

$$\begin{cases} -\dfrac{\rho}{\mu}z_{TT}+\dfrac{1}{2}(1+2\beta)z_{ZZ}+\dfrac{1}{2}(2k+3-2\beta)\dfrac{z_{ZZ}}{z_Z^2}+(1-2\beta)r^2z_{ZZ} \\ \qquad +2(1-2\beta)rr_Zz_Z+kr^4z_{ZZ}+4kr^3r_Zz_Z=0, \\ (1+2\beta)r+(1-2\beta)r^3+2kr^3z_Z^2+(1-2\beta)rz_Z^2-\dfrac{(2k+3-2\beta)}{r} \\ \qquad -a^2\dfrac{1}{4}(1-2\beta)rr_Z^2-a^2\dfrac{1}{4}(1-2\beta)r^2r_{ZZ} \\ \qquad +\dfrac{\rho}{\mu}a^2\dfrac{1}{2}r_{TT}-a^2\dfrac{1}{4}(1+2\beta)r_{ZZ}=0, \end{cases} \tag{1.1.6}$$

where ρ is the material density. Equations (1.1.6) provide two coupled equations for z and r, which are the 1-dimensional rod system we shall work with. It can be seen that this system is very complicated and highly nonlinear.

We are interested in traveling wave solutions of (1.1.6). Let λ be the axial stretch z_Z. For traveling waves, we have

$$\lambda=\lambda(\xi),\quad r=r(\xi),\quad \xi=Z-cT, \tag{1.1.7}$$

where c is the propagating wave velocity. Noting that $z_{TT} = c^2\lambda_{\xi\xi}$, an integration of the first equation in (1.1.6) with respect to ξ yields

$$\left(\frac{1}{2}(1+2\beta) - \frac{c^2\rho}{\mu}\right)\lambda - \frac{1}{2}(2k+3-2\beta)\frac{1}{\lambda} + (1-2\beta)r^2\lambda + kr^4\lambda = g, \quad (1.1.8)$$

where g is an integration constant. The second equation in (1.1.6) becomes

$$\left(\frac{c^2a^2\rho}{2\mu} - \frac{a^2(1+2\beta)}{4} - \frac{a^2(1-2\beta)r^2}{4}\right)r_{\xi\xi} - \frac{a^2(1-2\beta)}{4}rr_\xi^2$$
$$-\frac{(2k+3-2\beta)}{r} + (1+2\beta)r + (1-2\beta)r^3 + ((1-2\beta)r + 2kr^3)\lambda^2 = 0. \quad (1.1.9)$$

Denote that

$$a_1 = \frac{c^2\rho}{\mu} - \frac{1+2\beta}{2}, \quad a_2 = \frac{2k+3-2\beta}{2} \quad (1.1.10)$$

and

$$\eta(r) = kr^4 + (1-2\beta)r^2 - a_1. \quad (1.1.11)$$

Thus, (1.1.8) becomes

$$\eta(r)\lambda^2 - g\lambda - a_2 = 0. \quad (1.1.12)$$

For $a_1 \leqslant 0, \eta(r) > 0$. Notice that λ should always be positive, so that

$$\lambda = \frac{g \pm \sqrt{g^2 + 4a_2\eta(r)}}{2\eta(r)}, \quad \text{for} \quad a_1 \leqslant 0, \ g \in R. \quad (1.1.13)$$

When $a_1 > 0, \eta(r)$ has a positive zero at $r = r_*$, where

$$r_* = \left(\frac{\sqrt{(1-2\beta)^2 + 4ka_1} - (1-2\beta)}{2k}\right)^{\frac{1}{2}}.$$

If for a wave that the value of r is always larger than r_*, which means that $\eta(r) > 0$, λ has the same form as (1.1.13). If for a wave there exists a point such that the value of r is equal to r_*, we have from (1.1.12) that

$$\lambda|_{r=r_*} = -\frac{a_1}{g}. \quad (1.1.14)$$

Hence in this case it is necessary to suppose that $g < 0$. If for a wave that the interval of the values of r contains r_*, then we have

$$\lambda = \frac{g + \sqrt{\Delta}}{2\eta(r)}, \quad \text{for} \quad g < 0, \quad (1.1.15)$$

where $\Delta = g^2 + 4a_2\eta(r) \geqslant 0$. If for a wave that the value of r is always smaller than r_*, which implies that $\eta(r) < 0$, and we allow $r \to r_*$, we also have (1.1.15). Therefore, for all $g \in R$ we have

$$\lambda^2 = \frac{g^2 + g\sqrt{\Delta} + 2a_2\eta(r)}{2\eta^2(r)} \tag{1.1.16}$$

and need to discuss the following three cases:

(i) $a_1 \leqslant 0$ and $g \in R$ for any value of $r > 0$;
(ii) $a_1 > 0$ and $g \geqslant 0$ for $r > r_*$;
(iii) $a_1 > 0$ and $g < 0$ for any value of $r > 0$.

Let $b_1 = \frac{a^2 a_1}{2}, b_2 = \frac{a^2(1-2\beta)}{4}$. Define that

$$\zeta(r) = (1-2\beta)r^4 + (1+2\beta)r^2 - 2a_2, \quad \theta(r) = 2kr^4 + (1-2\beta)r^2. \tag{1.1.17}$$

Substituting the expressions of λ^2 into (1.1.9), we obtain

$$(b_1 - b_2r^2)r_{\xi\xi} - b_2 r r_\xi^2 + \Phi(r) = 0, \tag{1.1.18}$$

where

$$\Phi(r) = \frac{\psi(r)}{2r\eta^2(r)}, \quad \text{for } r \neq r_*; \tag{1.1.19}$$

$$\psi(r) = 2\eta(r)[\eta(r)\zeta(r) + a_2\theta(r)] + g^2\theta(r) + g\,\theta(r)\sqrt{\Delta}. \tag{1.1.20}$$

When $r = r_*, \eta(r_*) = 0$, we have from $\lambda = \frac{-a_2}{g}$ that

$$\Phi(r_*) = \frac{g^2\zeta(r_*) + a_2^2\theta(r_*)}{r_* g^2}. \tag{1.1.21}$$

Notice that $\lim_{r\to r_*}\Phi(r) = \Phi(r_*)$. It implies that the vector field defined by (1.1.18) is continuous in the straight line $r = r_*$. The point $r = r_*$ is a removable discontinuity point of the function $\Phi(r)$.

In order to study the dynamical behavior of solutions of equation (1.1.18), introducing new variable $y = r_\xi$, we obtain the following planar system

$$\frac{dr}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{2b_2r^2\eta^2(r)y^2 - \psi(r)}{2r\eta^2(r)(b_1 - b_2r^2)}. \tag{1.1.22}$$

This is an integrable system with the first integral

$$H(r,y) = y^2(b_1 - b_2r^2) + (1+2\beta)r^2 + \frac{(1-2\beta)}{2}r^4 - 4a_2\ln r + a_2\ln|\eta(r)|$$

$$-\frac{g^2}{2\eta(r)}+\frac{\Delta^{\frac{3}{2}}}{2g\eta(r)}-\frac{2a_2\sqrt{\Delta}}{g}-\frac{a_2g}{|g|}\ln\frac{|\sqrt{\Delta}-|g||}{(\sqrt{\Delta}+|g|)}, \tag{1.1.23}$$

where $\Delta = g^2 + 4a_2\eta(r)$. Therefore, the system (1.1.22) can be written as follows

$$\frac{dr}{d\xi}=\mu(r)\frac{\partial H}{\partial y},\quad \frac{dy}{d\xi}=-\mu(r)\frac{\partial H}{\partial r}, \tag{1.1.24}$$

where $\mu(r)=\frac{1}{2(b_1-b_2r^2)}$ is the integral factor. Clearly, when $b_1-b_2r^2=0$, the right hand of the second equation of (1.1.22) is discontinuous. We need to treat this problem in next chapter.

1.2 Higher order wave equations of Korteweg-De Vries type

In 1995, A.S. Fokas proposed to study a class of physically important integrable equations including higher order wave equations of the Korteweg-De Vries Type. Consider the motion of a 2-dimensional, inviscid, incompressible fluid (water) lying above a horizontal flat bottom located at $y=-h_0$ (h_0 constant), and let there be air above the water. It turns out that, for such a system if the vorticity is zero initially, it remains zero. We analyze only such irrotational flows. This system is characterized by two parameters, $A=\frac{a}{h_0}$ and $B=\frac{h_0^2}{l^2}$, where a and l are typical values of the amplitude and of the wavelength of the waves. Let η and ϕ denote the position of the free surface and the velocity potential, respectively. Then $\eta(x,t)$ and $w(x,t)$ where $w=f_x$ and $\phi=\sum_0^\infty(-B)^m(1+A\eta)^{2m}f^{2m}/(2m)!$ (f^m denote the $m-$th derivative of f with respect to x) satisfy (see Whitham (1974))

$$\eta_t+w_x+A(\eta w)_x-\frac{1}{6}Bw_{xxx}-\frac{1}{2}AB(\eta w_{xx})_x-\frac{1}{2}A^2B(\eta^2w_{xx})_x+O(B^2)=0, \tag{1.2.1}$$

$$w_t+\eta_x+Aww_x+\frac{1}{2}B\eta_{xxx}+AB(\eta\eta_{xx}+w_x^2)_x+A^2B(2\eta^2w_x^2+\frac{1}{2}\eta^2\eta_{xx})_x+O(B^2)=0. \tag{1.2.2}$$

Suppose that $O(B)$ is less than $O(A)$ and the waves are unidirectional. Neglecting terms of $O(\alpha^4,\alpha^3\beta,\beta^2)$, equations (1.2.1) and (1.2.2) yield

$$\begin{aligned}&\eta_t+\eta_x+\alpha\eta\eta_x+\beta\eta_{xxx}+\rho_1\alpha^2\eta^2\eta_x+\alpha\beta(\rho_2\eta\eta_{xxx}+\rho_3\eta_x\eta_{xx})\\&+\rho_4\alpha^3\eta^3\eta_x+\alpha^2\beta[\rho_5\eta^2\eta_{xxx}+\rho_6\eta\eta_x\eta_{xx}+\rho_7\eta_x^3]=0,\end{aligned} \tag{1.2.3}$$

where $\alpha=\frac{3A}{2}$, $\beta=\frac{B}{6}$, $\rho_1=-\frac{1}{6}$, $\rho_2=\frac{5}{3}$, $\rho_3=\frac{23}{6}$, $\rho_4=\frac{1}{8}$, $\rho_5=\frac{7}{18}$, $\rho_6=$

$\frac{79}{36}$, $\rho_7 = \frac{45}{36}$. Neglecting terms of $O(\alpha^2, \alpha\beta)$, equation (1.2.3) reduces to the KdV equation

$$\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0. \qquad (1.2.4)$$

Neglecting terms of $O(\alpha^3, \alpha^2\beta)$, equation (1.2.3) reduces to the "more physically realistic form"

$$\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \rho_1\alpha^2\eta^2\eta_x + \alpha\beta(\rho_2\eta\eta_{xxx} + \rho_3\eta_x\eta_{xx}) = 0. \qquad (1.2.5)$$

We assume that ρ_i, $i = 1-7$ in (1.2.3) are considered as free parameters. Then, (1.2.5) and (1.2.3) are called second order and third order wave equations of KdV type, respectively.

A.S. Fokas also derived the following integrable generalizations of modified KdV equation

$$u_t + u_x + \nu u_{xxt} + \beta u_{xxx} + \alpha u u_x + \frac{1}{3}\alpha\nu(u u_{xxx} + 2u_x u_{xx}) = 0 \qquad (1.2.6)$$

and

$$u_t + u_x + \nu u_{xxt} + \beta u_{xxx} + \alpha u u_x + \frac{1}{3}\alpha\nu(u u_{xxx} + 2u_x u_{xx}) + 3\mu\alpha^2 u^2 u_x$$
$$+\nu\mu\alpha^2(u^2 u_{xxx} + u_x^3 + 4u u_x u_{xx}) + \nu^2\mu\alpha^2(u_x^2 u_{xxx} + 2u_x u_{xx}^2) = 0. \qquad (1.2.7)$$

First, we consider the traveling wave equation of the second order wave equations (1.2.5) of KdV type. Letting $\eta(x,t) = \phi(x-ct) = \phi(\xi)$, where c is the wave speed and $\xi = x - ct$, substituting $\phi(x-ct)$ into (1.2.5), we obtain

$$(1-c)\phi' + \frac{1}{2}\alpha(\phi^2)' + \beta\phi''' + \frac{1}{3}\alpha^2\rho_1(\phi^3)' + \alpha\beta(\rho_2(\phi\phi'')' + \frac{1}{2}(\rho_3 - \rho_2)((\phi')^2)') = 0, \qquad (1.2.8)$$

where "$'$" is the derivative with respect to ξ. Integrating once with respect to ξ, we have the following traveling wave equation of (1.2.5)

$$\beta(1+\alpha\rho_2\phi)\phi'' + \frac{1}{2}\alpha\beta(\rho_3 - \rho_2)(\phi')^2 + \frac{1}{3}\alpha^2\rho_1\phi^3 + \frac{1}{2}\alpha\phi^2 + (1-c)\phi + g = 0, \qquad (1.2.9)$$

where g is the integral constant. (1.2.9) is equivalent to the following 2-dimensional system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{3\alpha\beta(\rho_3 - \rho_2)y^2 + 2\alpha^2\rho_1\phi^3 + 3\alpha\phi^2 + 6(1-c)\phi + g}{6\beta(1 + \alpha\rho_2\phi)}. \qquad (1.2.10)$$

We next assume that $g = 0$. Let $\rho_3 = p\rho_2$, where p is a real number. Then, for $\rho_3 \neq -2\rho_2$, $\rho_3 \neq \pm\rho_2$ i.e., $p \neq -2$, $p \neq \pm 1$, system (1.2.10) has the following first integrals

$$y^2 = h(1 + \alpha\rho_2\phi)^{1-p} + \frac{A_0 + B_0\phi + C_0\phi^2 + D_0\phi^3}{3p(p+2)(p^2-1)\alpha^2\beta\rho_2^4}, \tag{1.2.11}$$

where $A_0 = 6[\rho_2^2(1-c)(p+1)(p+2) + 2\rho_1 - (p+2)\rho_2]$, $B_0 = -\alpha(p-1)\rho_2 A_0$, $C_0 = 3p(p-1)\alpha^2\rho_2^2[2\rho_1 - (p+2)\rho_2]$, $D_0 = -2p\rho_1\alpha^3\rho_2^3(p^2-1)$.
when $p = -2$, i.e., $\rho_3 = -2\rho_2$,

$$y^2 = h(1+\alpha\rho_2\phi)^3 - \frac{A_1 + B_1\phi + C_1\phi^2 + 6\rho_1(1+3\alpha\rho_2\phi + 3\alpha^2\rho_2^2\phi^2)\ln(1+2\alpha\rho_2\phi)}{9\beta\alpha^2\rho_2^4}, \tag{1.2.12}$$

where $A_1 = 11\rho_1 - 3\rho_2^2(1-c) - 3\rho_2$, $B_1 = -9\alpha(\rho_2^3(1-c) + \rho_2^2 - 3\rho_1\rho_2)$, $C_1 = 9\alpha^2\rho_2^3(2\rho_1 - \rho_2)$;
when $p = -1$, i.e., $\rho_3 = -\rho_2$,

$$y^2 = h(1+\alpha\rho_2\phi)^2 - \frac{A_2 + B_2\phi + C_2\phi^2 + D_2\phi^3 + (E_2 + F_2\phi + G_2\phi^2)\ln(1+2\alpha\rho_2\phi)}{6\beta\alpha^2\rho_2^4}, \tag{1.2.13}$$

where $A_2 = -10\rho_1 - 6\rho_2(1-c) + 9\rho_2$, $B_2 = \alpha(-8\rho_1\rho_2 - 12\rho_2^3(1-c) + 12\rho_2^2)$, $C_2 = 8\alpha^2\rho_1\rho_2^2$, $D_2 = 4\alpha^3\rho_1\rho_2^3$, $E_2 = 6\rho_2 - 12\rho_1$, $F_2 = 12\alpha(\rho_2^2 - 2\rho_1\rho_2)$, $G_2 = 6\alpha^2(\rho_2^3 - 2\rho_1\rho_2^2)$;
when $p = 1$, i.e., $\rho_3 = \rho_2$,

$$(18\beta\alpha^2\rho_2^4)y^2 + \alpha\rho_2[4\alpha^2\rho_2^2\phi^3 + 3\alpha\rho_2(3\rho_2^2 - 2\rho_1)\phi^2 + (9\rho_2^2(1-c) - 18\rho_2 + 12\rho_1)\phi]$$
$$-6(6\rho_2^2(1-c) - 3\rho_2 - 2\rho_1)\ln(1+\alpha\rho_2\phi) = h, \tag{1.2.14}$$

where h is an arbitrary constant.

We see from (1.2.11) that if $1-p = 2k$, (k is an integer) or p is an irrational number, then we must consider the case $1 + \alpha\rho_2\phi > 0$, i.e., $\phi > \phi_s = -\dfrac{1}{\alpha\rho_2}$.

System (1.2.10) is a planar dynamical system defined in the 7-parameter space $(\alpha, \beta, c, \rho_1, \rho_2, \rho_3, g)$.

Second, we investigate the traveling wave equation of third order wave equations (1.2.3) of KdV type. Substituting $\eta = \phi(x - ct)$ into (1.2.3) and letting $y = \phi'(\xi)$, $z = \phi''(\xi)$, where "$'$" is the derivative with respect to ξ, we have the following 3-dimensional traveling wave system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = z,$$

$$\frac{dz}{d\xi} = -\frac{[\alpha\beta(\alpha\rho_7 y^2 + (\rho_3 + \alpha\rho_6\phi)z) + \alpha^3\rho_4\phi^3 + \alpha^2\rho_1\phi^2 + \alpha\phi + (1-c)]y}{\beta(1+\alpha\rho_2\phi + \alpha^2\rho_5\phi^2)}. \tag{1.2.15}$$

There are two groups of parameter conditions (I) $\rho_6 = 2(\rho_5 + \rho_7)$ and (II) $\rho_7 = 0$ such that system (1.2.15) can be reduced to two 2-dimensional integrable systems.

We only consider the case (I). Then, we obtain from (1.2.15) that

$$(1-c)\phi' + \frac{1}{2}\alpha(\phi^2)' + \beta\phi''' + \frac{1}{3}\alpha^2\rho_1(\phi^3)' + \alpha\beta(\rho_2(\phi\phi'')' + \frac{1}{2}\alpha\beta(\rho_3 - \rho_2)((\phi')^2)'$$

$$+\frac{1}{4}\alpha^3\rho_4(\phi^4)' + \alpha^2\beta(\rho_5(\phi^2\phi'')' + \rho_7(\phi(\phi')^2)') = 0.$$

Integrating once with respect to ξ, we have the following traveling wave equation of (1.2.3)

$$\beta(1+\alpha\rho_2\phi + \alpha^2\rho_5\phi^2)\phi'' + \left(\frac{1}{2}\alpha\beta(\rho_3 - \rho_2) + \alpha^2\beta\rho_7\phi\right)(\phi')^2$$

$$+\frac{1}{4}\alpha^3\rho_4\phi^4 + \frac{1}{3}\alpha^2\rho_1\phi^3 + \frac{1}{2}\alpha\phi^2 + (1-c)\phi = 0, \tag{1.2.16}$$

where we take the integral constant $g = 0$. (1.2.16) is equivalent to the following 2-dimensional system

$$\frac{d\phi}{d\xi} = y,$$

$$\frac{dy}{d\xi} = -\frac{6\alpha\beta((\rho_3 - \rho_2) + 2\alpha\rho_7\phi)y^2 + 3\alpha^3\rho_4\phi^4 + 4\alpha^2\rho_1\phi^3 + 6\alpha\phi^2 + 12(1-c)\phi}{12\beta(1+\alpha\rho_2\phi + \alpha^2\rho_5\phi^2)}. \tag{1.2.17}$$

Write that

$$S(\phi) = 1 + \alpha\rho_2\phi + \alpha^2\rho_5\phi^2,$$

$$F(\phi) = f(\phi)\phi = (3\alpha^3\rho_4\phi^3 + 4\alpha^2\rho_1\phi^2 + 6\alpha\phi + 12(1-c))\phi.$$

Thus, (1.2.17) can be rewritten to the form

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{6\alpha\beta((\rho_3 - \rho_2) + 2\alpha\rho_7\phi)y^2 + F(\phi)}{12\beta S(\phi)}. \tag{1.2.18}$$

Clearly, system (1.2.18) is a planar dynamical system defined in the 10-parameter space $(\alpha, \beta, c, \rho_i), i = 1-7$. Corresponding to different parameter subspace, it has different rich and complicated dynamical behavior.

The system (1.2.17) has a first integral for $\rho_2^2 - 4\rho_5 > 0$,

$$y^2 S(\phi)^{\frac{\rho_7}{\rho_5}} \exp\left(A \text{ are } \tanh\left(\frac{\rho_2 + 2\alpha\rho_5\phi}{\sqrt{\rho_2^2 - 4\rho_5}}\right)\right)$$

$$+\frac{1}{6\beta}\int S(\phi)^{\frac{\rho_7}{\rho_5}-1} F(\phi) \exp\left(-A \text{ are } \tanh\left(\frac{\rho_2 + 2\alpha\rho_5\phi}{\sqrt{\rho_2^2 - 4\rho_5}}\right)\right) d\phi = h, \quad (1.2.19)$$

where $A = \dfrac{2[\rho_5(\rho_2 - \rho_3) + \rho_2\rho_7]}{\rho_5\sqrt{\rho_2^2 - 4\rho_5}}$; for $\rho_2^2 - 4\rho_5 < 0$,

$$y^2 S(\phi)^{\frac{\rho_7}{\rho_5}} \exp\left(-iA \text{ are} \tan\left(\frac{\rho_2 + 2\alpha\rho_5\phi}{\sqrt{4\rho_5 - \rho_2^2}}\right)\right)$$

$$+\frac{1}{6\beta}\int S(\phi)^{\frac{\rho_7}{\rho_5}-1} F(\phi) \exp\left(iA \text{ are} \tan\left(\frac{\rho_2 + 2\alpha\rho_5\phi}{\sqrt{4\rho_5 - \rho_2^2}}\right)\right) d\phi = h \qquad (1.2.20)$$

and for $\rho_2^2 - 4\rho_5 = 0$,

$$y^2(2 + \alpha\rho_2\phi)^{\frac{8\rho_7}{\rho_2^2}} \exp\left(\frac{4[\rho_2(\rho_2 - \rho_3) + 4\rho_7]}{\rho_2^2(2 + \alpha\rho_2\phi)}\right)$$

$$+\frac{2}{3\beta}\int F(\phi)(2 + \alpha\rho_2\phi)^{\frac{8\rho_7 - 2\rho_2^2}{\rho_2^2}} \exp\left(\frac{4[\rho_2(\rho_2 - \rho_3) + 4\rho_7]}{\rho_2^2(2 + \alpha\rho_2\phi)}\right) d\phi = h. \quad (1.2.21)$$

We see form (1.2.20) and (1.2.21) that to obtain an explicit integral formula for general parameters ρ_i, $i = 1 - 7$, it is very difficult.

Finally, we consider the traveling wave equation of (1.2.7). Substituting $\eta = \phi(x - ct)$ into (1.2.7) and letting $y = \phi'(\xi)$, similarly, we have the following two order equation

$$\left[(\beta - c\nu) + \frac{1}{3}\alpha\nu\phi + \nu\mu\alpha^2\phi^2 + \mu\nu^2\alpha^2(\phi')^2\right]\phi'' + (1 - c)\phi + \frac{1}{2}\alpha\phi^2 + \mu\alpha^2\phi^3$$

$$+\frac{1}{6}\nu\alpha(\phi')^2 + \mu\nu\alpha^2\phi(\phi')^2 + g = 0, \qquad (1.2.22)$$

where g is an integral constant. This equation is equivalent to the 2-dimensional system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{g + (1 - c)\phi + \frac{1}{2}\alpha\phi^2 + \mu\alpha^2\phi^3 + \frac{1}{6}\nu\alpha y^2 + \mu\nu\alpha^2\phi y^2}{(\beta - c\nu) + \frac{1}{3}\alpha\nu\phi + \nu\mu\alpha^2\phi^2 + \mu\nu^2\alpha^2 y^2}, \qquad (1.2.23)$$