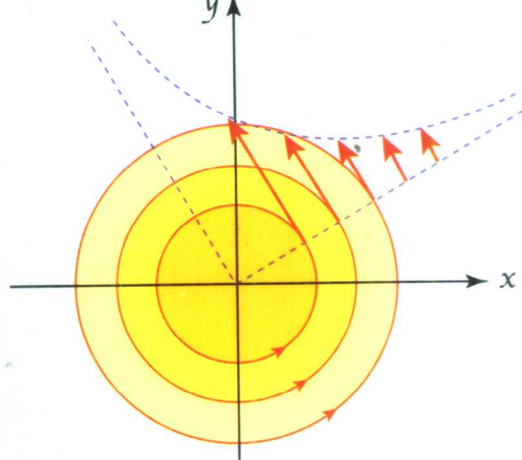


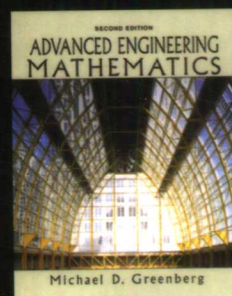
高等学校教材系列



# 高等工程数学

(第二版) (英文版)

Advanced Engineering  
Mathematics  
Second Edition



[美] Michael D. Greenberg 著



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## 内 容 简 介

本书系统地介绍了工程数学的基本原理与实践应用。全书共分5部分。第1部分介绍一阶、二阶及高阶线性方程,拉普拉斯变换,微分方程数值解,相平面和非线性微分方程;第2部分研究线性代数方程系统,高斯消去法,向量空间,矩阵与线性方程,本征值问题;第3部分阐述标量场与向量场理论,多变量方程,三维向量,曲线,面,体;第4部分分析傅里叶级数,偏微分方程,傅里叶积分,傅里叶变换,扩散方程,波动方程,拉普拉斯方程;第5部分描述复变函数方程,保角映射,复变函数积分,泰勒级数,洛朗级数,残数定理。

本书适合作为高等院校数学专业或工程学专业本科生或研究生的教材,也可供教师和工程师学习和参考。对于自学者,也是一本难得的参考书。

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# Preface

## Purpose and Prerequisites

This book is written primarily for a single- or multi-semester course in applied mathematics for students of engineering or science, but it is also designed for self-study and reference. By self-study we do not necessarily mean outside the context of a formal course. Even within a course setting, if the text can be read independently and understood, then more pedagogical options become available to the instructor.

The prerequisite is a full year sequence in calculus, but the book is written so as to be usable at both the undergraduate level and also for first-year graduate students of engineering and science. The flexibility that permits this broad range of use is described below in the section on Course Use.

## Changes from the First Edition

Principal changes from the first edition are as follows:

- 1. Part I on ordinary differential equations.** In the first edition we assumed that the reader had previously completed a first course in ordinary differential equations. However, differential equations is traditionally the first major topic in books on advanced engineering mathematics so we begin this edition with a seven chapter sequence on ordinary differential equations. Just as the book becomes increasingly sophisticated from beginning to end, these seven chapters are written that way as well, with the final chapter on nonlinear equations being the most challenging.
- 2. Incorporation of a computer-algebra-system.** Several powerful computer environments are available, such as *Maple*, *Mathematica*, and *MATLAB*. We selected *Maple*, as a representative and user-friendly software. In addition to an Instructor's Manual, a brief student supplement is also available, which presents parallel discussions of *Mathematica* and *MATLAB*.
- 3. Revision of existing material and format.** Pedagogical improvements that evolved through eight years of class use led to a complete rewriting rather than minor modifications of the text. The end-of-section exercises are refined and expanded.

## Format

The book is comprised of five parts:

- I Ordinary Differential Equations
- II Linear Algebra
- III Multivariable Calculus and Field Theory
- IV Fourier Methods and Partial Differential Equations
- V Complex Variable Theory

This breakdown is explicit only in the Contents, to suggest the major groupings of the chapters. Within the text there are no formal divisions between parts, only between chapters.

Each chapter begins with an introduction and (except for the first chapter) ends with a chapter review. Likewise, each section ends with a review called a closure, which is often followed by a section on computer software that discusses the *Maple* commands that are relevant to the material covered in that section; see, for example, pages 29–31. Subsections are used extensively to offer the instructor more options in terms of skipping or including material.

## Course Use at Different Levels

To illustrate how the text might serve at different levels, we begin by outlining how we have been using it for courses at the University of Delaware: a sophomore/junior level mathematics course for mechanical engineers, and a first-year graduate level two-semester sequence in applied mathematics for students of mechanical, civil, and chemical engineering, and materials science. We denote these courses as U, G1, and G2, respectively.

**Sophomore/junior level course (U).** This course follows the calculus/differential equations sequence taught in the mathematics department. We cover three main topics:

*Linear Algebra:* Chapter 8, Sections 9.1–9.5 (plus a one lecture overview of Secs. 9.7–9.9), 10.1–10.6, and 11.1–11.3. The focus is  $n$ -space and applications, such as the mass-spring system in Sec. 10.6.2, Markov population dynamics in Sec. 11.2, and orthogonal modes of vibration in Sec. 11.3.

*Field Theory:* Chapters 14 and 16. The heart of this material is Chapter 16. Having skipped Chapter 15, we distribute a one page “handout” on the area element formula (18) in Sec. 15.5 since that formula is needed for the surface integrals that occur in Chapter 16. Emphasis is placed on the physical applications in the sections on the divergence theorem and irrotational fields since those applications lead to two of the three chief partial differential equations that will be studied in the third part of the course—the diffusion equation and the Laplace equation.

*Fourier Series and PDE's:* Sections 17.1–17.4, 18.1, 18.3, 18.6.1, 19.1–19.2.2, 20.1–20.3.1, 20.5.1–20.5.2. Solutions are by separation of variables, using only the half- and quarter-range Fourier series, and by finite differences.

**First semester of graduate level course (G1).** Text coverage is as follows: Sections 4.4–4.6, 5.1–5.6, Chapter 9, Secs. 11.1–11.4, 11.6, 13.5–13.8, 14.6, 15.4–15.6, Chapter 16, Secs. 17.3, 17.6–17.11, 18.1–18.3.1, 18.3.3–18.4, 19.1–19.2, 20.1–20.4. As in “U” we do cover the important Chapter 16, although quickly. Otherwise, the approach complements that in “U.” For instance, in Chapter 9, “U” focuses on  $n$ -space, but “G1” focuses on generalized vector space (Sec. 9.6), to get ready for the Sturm–Liouville theory (Section 17.7); in Chapter 11 we emphasize the more advanced sections on diagonalization and quadratic forms, as well as Section 11.3.2 on the eigenvector expansion method in finite-dimensional space, so we can use that method to solve nonhomogeneous partial differential equations in later chapters. Likewise, in covering Chapter 17 we assume that the student has worked with Fourier series before so we move quickly, emphasizing the vector space approach (Sec. 17.6), the Sturm–Liouville theory, and the Fourier integral and transform. When we come

to partial differential equations we use Sturm–Liouville eigenfunction expansions (rather than the half- and quarter-range formulas that suffice in “U”), integral transforms, delta functions, and Bessel and Legendre functions. In solving the diffusion equation in “U” we work only with the homogeneous equation and constant end conditions, but in “G1” we discuss the nonhomogeneous equation and nonconstant end conditions, uniqueness, and so on; these topics are discussed in the exercises.

**Second semester of graduate level course (G2).** In the second semester we complete the partial differential equation coverage with the methods of images and Green’s functions, then turn to complex variable theory, the variational calculus, and an introduction to perturbation methods. For Green’s functions we use a “handout,” and for the variational calculus and perturbation methods we copy the relevant chapters from M.D. Greenberg, *Foundations of Applied Mathematics* (Englewood Cliffs, NJ: Prentice Hall, 1978). (If you are interested in using any of these materials please contact the College Mathematics Editor office at Prentice-Hall, Inc., One Lake Street, Upper Saddle River, NJ 07458.)

Text coverage is as follows: Chapters 21–24 on complex variable theory; then we return to PDE’s, first covering Secs. 18.5–18.6, 19.3–19.4, and 20.3.2–20.4 that were skipped in “G1”; “handouts” on Green’s functions, perturbation methods, and the variational calculus.

**Shorter courses and optional Sections.** A number of sections and subsections are listed as Optional in the Contents, as a guide to instructors in using this text for shorter or more introductory courses. In the chapters on field theory, for example, one could work only with Cartesian coordinates, and avoid the more difficult non-Cartesian case, by omitting those optional sections. We could have labeled the Sturm–Liouville theory section (17.7) as optional but chose not to, because it is such an important topic. Nonetheless, if one wishes to omit it, as we do in “U,” that is possible, since subsequent use of the Sturm–Liouville theory in the PDE chapters is confined to optional sections and exercises.

Let us mention Chapter 4, in particular, since its development of series solutions, the method of Frobenius, and Legendre and Bessel functions might seem more detailed than you have time for in your course. One minimal route is to cover only Sections 4.2.2 on power series solutions of ordinary differential equations (ODE’s) and 4.4.1 on Legendre polynomials, since the latter does not depend on the more detailed Frobenius material in Section 4.3. Then one can have Legendre functions available when the Laplace equation is studied in spherical coordinates. You might also want to cover Bessel functions but do not want to use class time to go through the Frobenius material. In my own course (“G1”) I deal with Bessel functions by using a “handout” that is simpler and shorter, which complements the more thorough treatment in the text.

## Exercises

Exercises are of different kinds and arranged, typically, as follows. First, and usually near the beginning of the exercise group, are exercises that follow up on the text or fill in gaps or relate to proofs of theorems stated in that section, thus engaging the student more fully in the reading (e.g., Exercises 1–3 in Section 7.2, Exercise 8 in Section 16.8). Second, there are usually numerous “drill type” exercises that ask the reader to mimic steps or calculations that are essentially like those demonstrated in the text (e.g., there are 19 matrices to invert by hand in Exercise 1 of Section 10.6, and again by computer software in Exercise 3).

Third, there are exercises that require the use of a computer, usually employing software that is explained at the end of the section or in an earlier section; these vary from drill type (e.g., Exercise 1, Section 10.6) to more substantial calculations (e.g., Exercise 15, Section 19.2). Fourth, there are exercises that involve physical applications (e.g., Exercises 8, 9, and 12 of Section 16.10, on the stream function, the entropy of an ideal gas, and integrating the equation of motion of fluid mechanics to obtain the Bernoulli equation). And, fifth, there are exercises intended to extend the text and increase its value as a reference book. In these, we usually guide the student through the steps so that the exercise becomes more usable for subsequent reference or self-study (e.g., see Exercises 17–22 of Section 18.3). Answers to selected exercises (which are denoted in the text by underlining the exercise number) are provided at the end of the book; a more complete set is available for instructors in the Instructor's Manual.

### Specific Pedagogical Decisions

In Chapter 2 we consider the linear first-order equation and then the case of separable first-order equations. It is tempting to reverse the order, as some authors do, but we prefer to elevate the linear/nonlinear distinction, which grows increasingly important in engineering mathematics; to do that, it seems best to begin with the linear equation.

It is stated, at the beginning of Chapter 3 on linear differential equations of second order and higher, that the reader is expected to be familiar with the theory of the existence and uniqueness of solutions of linear algebraic equations, especially the role of the determinant of the coefficient matrix, even though this topic occurs later in the text. The instructor is advised to handle this need either by assigning, as prerequisite reading, the brief summary of the needed information given in Appendix B or, if a tighter blending of the differential equation and linear algebra material is desired, by covering Sections 8.1–10.6 before continuing with Chapter 3. Similarly, it is stated at the beginning of Chapter 3 that an elementary knowledge of the complex plane and complex numbers is anticipated. If the class does not meet that prerequisite, then Section 21.2 should be covered before Chapter 3. Alternatively, we could have made that material the first section of Chapter 3, but it seemed better to keep the major topics together—in this case, to keep the complex variable material together.

Some authors prefer to cover second-order equations in one chapter and then higher-order equations in another. My opinion about that choice is that: (i) it is difficult to grasp clearly the second-order case (especially insofar as the case of repeated roots is concerned) without seeing the extension to higher order, and (ii) the higher-order case can be covered readily, so that it becomes more efficient to cover both cases simultaneously.

Finally, let us explain why Chapter 8, on systems of linear algebraic equations and Gauss elimination, is so brief. Just as one discusses the real number axis before discussing functions that map one real axis to another, it seems best to discuss vectors before discussing matrices, which map one vector space into another. But to discuss vectors, span, linear dependence, bases, and expansions, one needs to know the essentials regarding the existence and uniqueness of solutions of systems of linear algebraic equations. Thus, Chapter 8 is intended merely to suffice until, having introduced matrices in Chapter 10, we can provide a more complete discussion.

## Appendices

Appendix A reviews partial fraction expansions, needed in the application of Laplace and Fourier transforms. Appendix B summarizes the theory of the existence and uniqueness of solutions of linear algebraic equations, especially the role of the determinant of the coefficient matrix, and is a minimum prerequisite for Chapter 3. Appendices C through F are tables of transforms and conformal maps.

## Instructor's Manual

An Instructor's Manual will be available to instructors from the office of the Mathematics Editor, College Department, Prentice-Hall, Inc., 1 Lake Street, Upper Saddle River, NJ 07458. Besides solutions to exercises, this manual contains additional pedagogical ideas for lecture material and some additional coverage, such as the Fast Fourier Transform, that can be used as "handouts."

## Acknowledgements

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I'm grateful to my wife, Yisraela, for her deep support and love when this task looked like more than I could handle, and for assuming many of my responsibilities, to give me the needed time. I dedicate this book to her.

Most of all, I am grateful to the Lord for bringing this book back to life and watching over all aspects of its writing and production: "From whence cometh my help? My help cometh from the Lord, who made heaven and earth." (Psalm 121)

Michael D. Greenberg

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