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（影印版）1

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Topology I

General Survey

拓扑学 I

总论

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

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Topology

Sergei P. Novikov

Translated from the Russian
by Boris Botvinnik and Robert Burns

Contents

Introduction

In the present essay, we attempt to convey some idea of the skeleton of topology, and of various topological concepts. It must be said at once that, apart from the necessary minimum, the subject-matter of this survey does not include that subdiscipline known as "general topology" — the theory of general spaces and maps considered in the context of set theory and general category theory. (Doubtless this subject will be surveyed in detail by others.) With this qualification, it may be claimed that the "topology" dealt with in the present survey is that mathematical subject which in the late 19th century was called *Analysis Situs*, and at various later periods separated out into various subdisciplines: "Combinatorial topology", "Algebraic topology", "Differential (or smooth) topology", "Homotopy theory", "Geometric topology".

With the growth, over a long period of time, in applications of topology to other areas of mathematics, the following further subdisciplines crystallized out: the global calculus of variations, global geometry, the topology of Lie groups and homogeneous spaces, the topology of complex manifolds and algebraic varieties, the qualitative (topological) theory of dynamical systems and foliations, the topology of elliptic and hyperbolic partial differential equations. Finally, in the 1970s and 80s, a whole complex of applications of topological methods was made to problems of modern physics; in fact in several instances it would have been impossible to understand the essence of the real physical phenomena in question without the aid of concepts from topology.

Since it is not possible to include treatments of all of these topics in our survey, we shall have to content ourselves here with the following general remark: Topology has found impressive applications to a very wide range of problems concerning qualitative and stability properties of both mathematical and physical objects, and the algebraic apparatus that has evolved along with it has led to the reorientation of the whole of modern algebra.

The achievements of recent years have shown that the modern theory of Lie groups and their representations, along with algebraic geometry, which subjects have attained their present level of development on the basis of an ensemble of deep algebraic ideas originating in topology, play a quite different role in applications: they are applied for the most part to the exact formulaic investigation of systems possessing a deep internal algebraic symmetry. In fact this had already been apparent earlier in connexion with the exact solution of problems of classical mechanics and mathematical physics; however it became unequivocally clear only in modern investigations of systems that are, in a certain well-defined sense, integrable. It suffices to recall for instance the method of inverse scattering and the (algebro-geometric) finite-gap integration of non-linear field systems, the celebrated solutions of models of statistical physics and quantum field theory, self-dual gauge fields, and string theory. (One particular aspect of this situation is, however, worthy of note, namely the need for a serious "effectivization" of modern algebraic geometry, which would return the subject in spirit to the algebraic geometry of the 19th century, when it was regarded as a part of formulaic analysis.

This survey constitutes the introduction to a series of essays on topology, in which the development of its various subdisciplines will be expounded in greater detail.

Introduction to the English Translation

This survey was written over the period 1983-84, and published (in Russian) in 1986. The English translation was begun in 1993. In view of the appearance in topology over the past decade of several important new ideas, I have added an appendix summarizing some of these ideas, and several footnotes, in order to bring the survey more up-to-date.

I am grateful to several people for valuable contributions to the book: to M. Stanko, who performed a huge editorial task in connexion with the Russian edition; to B. Botvinnik for his painstaking work as scientific editor of the English edition, in particular as regards its modernization; to R. Burns for making a very good English translation at high speed; and to C. Shochet for advice and help with the translation and modernization of the text at the University of Maryland. I am grateful also to other colleagues for their help with modernizing the text.

Sergei P. Novikov,
November, 1995

Chapter 1
The Simplest Topological Properties

Topology is the study of topological properties or *topological invariants* of various kinds of mathematical objects, starting with rather general geometrical figures. From the topological point of view the name "geometrical figures" signifies: general polyhedra (polytopes) of various dimensions (*complexes*); or continuous or smooth "surfaces" of any dimension situated in some Euclidean space or regarded as existing independently (*manifolds*); or sometimes subsets of a more general nature of a Euclidean space or manifold, or even of an infinite-dimensional space of functions. Although it is not possible to give a precise general definition of "topological property" ("topological invariant") of a geometrical figure (or more general geometrical structure), we may describe such a property intuitively as one which is, generally speaking, "stable" in some well-defined sense, i.e. remains unaltered under small changes or deformations (*homotopies*) of the geometrical object, no matter how this is given to us. For instance for a general polytope (*complex*) the manner in which the polytope is given may be, and often is, changed by means of an operation of subdivision, whereby each face of whatever dimension is subdivided into smaller parts, and so converted into a more complex polyhedron, the subdivision being carried out in such a way as to be compatible on that portion of their boundaries shared by each pair of faces. In this way the whole polyhedron becomes transformed formally into a more complicated one with a larger number of faces of each dimension. The various topological properties, or numerical or algebraic invariants, should be the same for the subdivided complex as for the original.

The simplest examples. 1) Everyone is familiar with the elementary result called "Euler's Theorem", which, so we are told, was in fact known prior to Euler:

For any closed, convex polyhedron in 3*-dimensional Euclidean space* $\mathbb{R}^3$*, the number of vertices less the number of edges plus the number of (2-dimensional) faces, is* 2.

Thus the quantity $V - E + F$ is a topological invariant in that it is the same for any subdivision of a convex polyhedron in $\mathbb{R}^3$.

2) Another elementary observation of a topological nature, also dating back to Euler, is the so-called "problem of the three pipelines and three wells" . Here one is given three points a_1, a_2, a_3 in the plane $\mathbb{R}^2$ (three "houses") and three other points A_1, A_2, A_3 ("wells"), and it turns out that it is not possible to join each house a_i to each well A_j by means of a non-self-intersecting path ("pipeline") in such a way that no two of the 9 paths intersect in the plane. (Of course, this *is* possible in $\mathbb{R}^3$.) In topological language this conclusion may be rephrased as follows: Consider the one-dimensional complex (or *graph*) consisting of 6 vertices a_i, A_j, and 9 edges x_{ij}, $i, j = 1, 2, 3$, where the "boundary" of each edge, denoted by ∂x_{ij}, is given by $\partial x_{ij} = \{a_i, A_j\}$. The conclusion is that this one-dimensional complex cannot be situated in the plane $\mathbb{R}^2$ without incurring self-intersections. This represents a topological property of the given complex. □

These two observations of Euler may be considered as the archetypes of the basic ideas of combinatorial topology, i.e. of the topological theory of polyhedra and complexes established much later by Poincaré. It is important to bear in mind that the use of combinatorial methods to define and investigate topological properties of geometrical figures represents just one interpretation of such properties, providing a convenient and rigorous approach to the formulation of these concepts at the first stage of topology, though of course remaining useful for certain applications. However those same topological properties admit of alternative formulations in various different situations, for instance in the contexts of differential geometry and mathematical analysis. For an example, let us return to the general convex polyhedron of Example 1 above. By smoothing off its corners and edges a little, we obtain a general smooth, closed, convex surface in $\mathbb{R}^3$, the boundary of a convex solid. Denote this surface by M^2. At each point x of this surface the Gaussian curvature $K(x)$ is defined, as also the area-element $d\sigma(x)$, and we have the following formula of Gauss:

$$\frac{1}{2\pi}\iint_{M^2} K(x)d\sigma(x) = 2. \tag{0.1}$$

In the sequel it will emerge that this formula reflects the same topological property as does Euler's theorem concerning convex polyhedra. (Euler's theorem can be deduced quickly from the Gauss formula (0.1) by continuously deforming a suitable surface into the given convex polyhedron and taking into account the relationship between the integral of the Gaussian curvature and the solid angles at the vertices.) Note that the formula (0.1) holds also for nonconvex closed surfaces "without holes". A third interpretation, as it turns out, of the same general topological property (which we have still not formulated!) lies hidden in the following observation, attributed to Maxwell: Consider an island with shore sloping steeply away from the island's edge into the sea, and whose surface has no perfectly planar or linear features; then the number of peaks plus the number of pits less the number of passes is exactly 1. This may be easily transformed into an assertion about closed surfaces in $\mathbb{R}^3$ by formally extending the island's surface underneath so that it is convex everywhere under the water (i.e. by imagining the island to be "floating", with a convex underside satisfying the same assumption as the surface). The resulting floating island then has one further pit, namely the deepest point on it. We conclude that for a closed surface in $\mathbb{R}^3$ satisfying the above assumption, the number of peaks (points of locally maximum height) plus the number of pits (local minimum points) less the number of passes (saddle points) is equal to 2, the same number as appears in both Euler's theorem and the Gauss formula (0.1) for surfaces without holes.

What if the polyhedron or closed surface in $\mathbb{R}^3$ or floating island is more complicated? With an arbitrary closed surface M^2 in $\mathbb{R}^3$ we may associate an integer, its "genus" $g \geq 0$, naively interpreted as the "number of holes". Here we have the Gauss-Bonnet formula

$$\frac{1}{2\pi}\iint_{M^2} K(x)d\sigma(x) = 2 - 2g, \tag{0.2}$$

and the theorems of Euler and Maxwell become modified in exactly the same way: the number 2 is replaced by $2 - 2g$. Since Poincaré it has become clear that these results prefigure general relationships holding for a very wide class of geometrical figures of arbitrary dimension.

Gauss also discovered certain topological properties of non-self-intersecting (i.e. *simple*) *closed curves in* $\mathbb{R}^3$. *It is well known that a simple, closed,* continuous (or if you like smooth, or piecewise smooth, or even piecewise linear) curve separates the plane $\mathbb{R}^2$ into two parts with the property that it is impossible to get from one part to the other by means of a continuous path avoiding the given curve. The ideally rigorous formulation of this intuitively obvious fact in the context of an explicit system of axioms for geometry and analysis carries the title "The Jordan Curve Theorem" (although of course in fact it is, in somewhat simplified form, already included in the axiom system; if one is not concerned with economy in the axiom system, then it might just as well be included as one of the axioms). The same conclusion (as for a simple, closed, continuous curve) holds also for any "complete" curve in $\mathbb{R}^2$, i.e. a simple, continuous, unboundedly extended, non-closed curve both of those ends go off to infinity, without nontrivial limit points in the finite plane. This principle generalizes in the obvious way to n-dimensional space: a closed hypersurface in $\mathbb{R}^n$ separates it into two parts. In fact a local version of this principle is basic to the general topological definition of dimension (by induction on n).

There is however another less obvious generalization of this principle, having its most familiar manifestation in 3-dimensional space $\mathbb{R}^3$. Consider two continuous (or smooth) simple closed curves (loops) in $\mathbb{R}^3$ which do not intersect:

$$\gamma_1(t) = \left(x_1^1(t), x_1^2(t), x_1^3(t)\right), \qquad \gamma_1(t+2\pi) = \gamma_1(t),$$

$$\gamma_2(\tau) = \left(x_2^1(\tau), x_2^2(\tau), x_2^3(\tau)\right), \quad \gamma_2(\tau+2\pi) = \gamma_2(\tau).$$

Consider a "singular disc" D_i bounded by the curve γ_i, i.e. a continuous map of the unit disc into $\mathbb{R}^3$: $x_i^\alpha = x_i^\alpha(r,\phi)$, $i = 1,2$, $\alpha = 1,2,3$, where $0 \le r \le 1$, $0 \le \phi \le 2\pi$, sending the boundary of the unit disc onto γ_i:

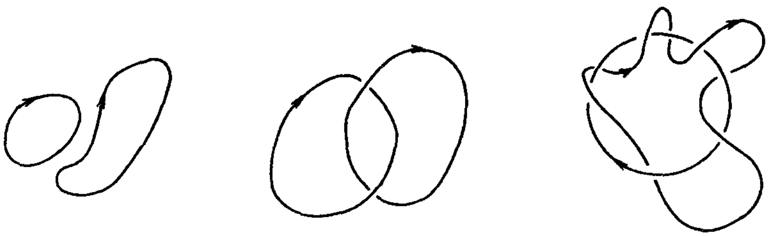
$$x_i^\alpha(r,\phi)|_{r=1} = x_i^\alpha(\phi), \quad \alpha = 1,2,3,$$

where $\phi = t$ for $i = 1$, and $\phi = \tau$ for $i = 2$.

Definition 0.1 Two curves γ_1 and γ_2 in $\mathbb{R}^3$ are said to be *nontrivially linked* if the curve γ_2 meets every singular disc D_1 with boundary γ_1 (or, equivalently, if the curve γ_1 meets every singular disc D_2 with boundary γ_2).

Simple examples are shown in Figure 1.1. In n-dimensional space $\mathbb{R}^n$ certain pairs of closed surfaces may be linked, namely submanifolds of dimensions p and q where $p + q = n - 1$. In particular a closed curve in $\mathbb{R}^2$ may be linked with a pair of points (a "zero-dimensional surface") – this is just the original principle that a simple closed curve separates the plane.

Gauss introduced an invariant of a link consisting of two simple closed curves γ_1, γ_2 in $\mathbb{R}^3$, *namely the signed number of turns of one of the curves* around the other, the *linking coefficient* $\{\gamma_1, \gamma_2\}$ of the link. His formula for this is



(a) Unlinked curves (b) Linking coefficient 1 (c) Linking coefficient 4

Fig. 1.1

$$N = \{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \oint\limits_{\gamma_1} \oint\limits_{\gamma_2} \frac{([d\gamma_1(t), d\gamma_2(t)], \gamma_1 - \gamma_2)}{|\gamma_1(t) - \gamma_2(t)|^3}, \tag{0.3}$$

where [,] denotes the vector (or cross) product of vectors in $\mathbb{R}^3$ and (,) the Euclidean scalar product. Thus this integral always has an integer value N. If we take one of the curves to be the z–axis in $\mathbb{R}^3$ and the other to lie in the (x, y)–plane, then the formula (0.3) gives the net number of turns of the plane curve around the z–axis.

It is interesting to note that the linking coefficient (0.3) may be zero even though the curves are nontrivially linked (see Figure 1.2). Thus its having non-zero value represents only a sufficient condition for nontrivial linkage of the loops.

Elementary topological properties of paths and homotopies between them played an important role in complex analysis right from the very beginning of that subject in the 19th century. They without doubt represent one of

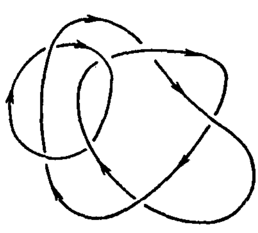


Fig. 1.2. The linking coefficient = 0, yet the curves are non-trivially linked

the most important features of the theory of functions of a complex variable, instrumental to the effectiveness and success of that theory in all of its applications. A complex analytic function $f(z)$ is often defined and single-valued only in a part of the complex plane, i.e. in some region $U \subset \mathbb{R}^2$ free of poles, branch points, etc. The Cauchy integral around each closed contour $\gamma \subset U$ yields a "topological" functional of the contour:

$$I_f(\gamma) = \oint_\gamma f(z)dz, \qquad (0.4)$$

in the sense that the integral remains unchanged under continuous homotopies (deformations) of the curve γ within the region U, i.e. by deformations of γ avoiding the singular points of the function. It is this very latitude – the possibility of deforming the closed contour without affecting the integral – which opens up enormous opportunities for varied application.

More complicated topological phenomena appeared in the 19th century – in essence beginning with Abel and Riemann – in connexion with the investigation of functions $f(z)$ of a complex variable, given only implicitly by an equation

$$F(z, w) = 0, \quad w = f(z), \qquad (0.5)$$

or else by means of analytic continuation throughout the plane, of a function originally given as analytic and single–valued only in some portion of the plane. The former situation arises in especially sharp form, as became clear after Riemann and Poincaré, in the context of Abel's resolution of the well-known problem of the insolubility of general algebraic equations by radicals, where the function $F(z, w)$ is a polynomial in two variables:

$$F(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0. \qquad (0.6)$$

Such a polynomial equation has, in general, finitely many isolated branch points $z_1, \cdots, z_m$ in the plane, away from which it has exactly n distinct roots $w_j(z)$, $z \neq z_k$ $(k = 1, \ldots, m)$. Here the region U is just the plane $\mathbb{R}^2$ with the m branch points removed:

$$U = \mathbb{R}^2 \setminus \{z_1, \ldots, z_m\}.$$

It turns out that in general the branch points cannot be merely ignored, for the following reason. In some neighborhood of each point z_0 that is not a branch point, the equation (0.6) determines exactly n distinct functions $w_j(z)$ such that $F(z, w_j(z)) = 0$. If, however, we attempt to continue any one w_j of these functions analytically outside that neighborhood, we encounter a difficulty of the following sort: if we continue w_j along a path which goes round some of the branch points and back to the point z_0, it may happen that we obtain

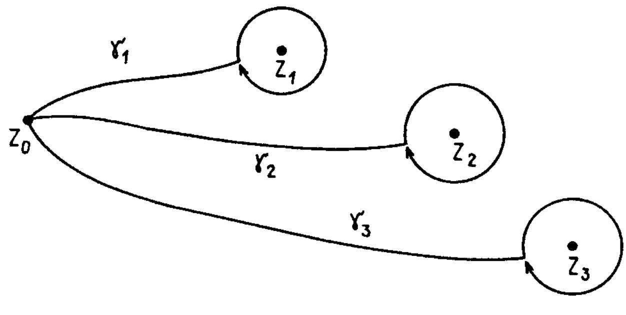


Fig. 1.3

nontrivial "monodromy", i.e. that we arrive at one of the other solutions at z_0:

$$w_s(z_0) \neq w_j(z_0), \quad s \neq j.$$

Proceeding more systematically, consider all possible loops $\gamma(t)$, $a \leq t \leq b$, in the region $U = \mathbb{R}_2 \setminus \{z_1, \cdots, z_m\}$, with $\gamma(a) = \gamma(b) = z_0$. Each such loop determines a permutation of the branches of the function $w(z)$: if we start at the branch given by $w_j(z)$ and continue around the loop from a to b, then we arrive when $t = b$ at the branch defined by w_s, so that the loop $\gamma(t)$ determines a permutation $j \rightarrow s$ of the branches (or sheets) above z_0:

$$\gamma \rightarrow \sigma_\gamma, \quad \sigma_\gamma(j) = s.$$

The inverse path γ^{-1} (i.e. the path traced backwards from b to a) yields the inverse permutation ${\sigma_\gamma}^{-1} : s \rightarrow j$, and the superposition $\gamma_1 \cdot \gamma_2$ of two paths γ_1 (traced out from time a to time b) and γ_2 (from b to c), i.e. the path obtained by following γ_1 by γ_2, corresponds to the product of the corresponding permutations:

$$\sigma_{\gamma_1 \cdot \gamma_2} = \sigma_{\gamma_2} \circ \sigma_{\gamma_1}, \qquad \sigma_{\gamma^{-1}} = (\sigma_\gamma)^{-1}. \tag{0.7}$$

In the general, non-degenerate, situation the permutations of the form σ_γ generate the full symmetric group of permutations of n symbols. (This is the underlying reason for the general insolubility by radicals of the algebraic equation (0.6) for $n \geq 5$.) To see this, note that the "basic" path γ_j, $j = 1, \cdots, m$, which starts from z_0, encircles the single branch point z_j, and then proceeds back to z_0 along the same initial segment (see Figure 1.3) corresponds, in the typical situation of maximally non-degenerate branch points, to the interchange of two sheets (i.e. σ_{γ_j} is just a transposition of two indices). The claim then follows from the fact that the transpositions generate all permutations.

It is noteworthy that the permutation σ_γ is unaffected if the loop γ is subjected to a continuous homotopy within U, throughout which its beginning and end remain fixed at z_0. This is analogous to the preservation of the Cauchy integral under homotopies (see (0.4) above), but is algebraically more complicated: the dependence of the permutation σ_γ on the path γ is non-commutative, in contrast with the Cauchy integral:

$$\sigma_{\gamma_1\cdot\gamma_2} = \sigma_{\gamma_2} \circ \sigma_{\gamma_1} \neq \sigma_{\gamma_2\cdot\gamma_1}, \quad I_f(\gamma_1 \cdot \gamma_2) = I_f(\gamma_1) + I_f(\gamma_2). \tag{0.8}$$

This sort of consideration leads naturally to a group with elements the *homotopy classes* of continuous loops $\gamma(t)$ beginning and ending at a particular point $z_0 \in U$, for any region, or indeed any manifold, complex or *topological space* U. This group is called the *fundamental group* of U (with base point z_0) and is denoted by $\pi_1(U, z_0)$. The *Riemann surface* defined by $F(z, w) = 0$ thus gives rise to a homomorphism – monodromy – from the fundamental group to the group of permutations of its "sheets", i.e. the branches of the function $w(z)$ in a neighborhood of $z = z_0$:

$$\sigma : \pi_1(U, z_0) \to S_n, \tag{0.9}$$

where S_n denote the symmetric group on n symbols, and U is as before – a region of $\mathbb{R}^2$.

For transcendental functions F, on the other hand, the equation $F(z, w) = 0$ may determine a many-valued function $w(z)$ with infinitely many sheets ($n = \infty$). Here the simplest example is

$$F(z, w) = \exp w - z = 0, \quad U = \mathbb{R}^2 \setminus 0, \quad w = \ln z.$$

In this example the sheets are numbered in a natural way by means of the integers: taking $z_0 = 1$, we have $w_k = \ln z_0 = 2\pi i k$, where k ranges over the integers. The path $\gamma(t)$ with $|\gamma| = 1$, $\gamma(0) = \gamma(2\pi) = 1$, going round the point $z = 0$ in the clockwise direction exactly once, yields the monodromy $\gamma \to \sigma_\gamma$, $\sigma_\gamma(k) = k - 1$.

An interesting topological theory where the non-abelianness of the fundamental group $\pi(U, z_0)$ plays an important role is that of *knots*, i.e. smooth (or, if preferred, piecewise smooth, or piecewise linear) simple, closed curves $\gamma(t) \subset \mathbb{R}^3$, $\gamma(t + 2\pi) = \gamma(t)$, or, more generally, the theory of *links*, as introduced above, a link being a finite collection of simple, closed, non-intersecting curves $\gamma_1, \ldots, \gamma_k \subset \mathbb{R}^3$. For $k > 1$, one has the matrix with entries the linking coefficients $\{\gamma_i, \gamma_j\}$, $i \neq j$, given by the formula (0.3), which however does not determine all of the topological invariants of the link. In the case $k = 1$, that of a knot, there is no such coefficient available. Let γ be a knot and U the complementary region of $\mathbb{R}^3$:

$$U = \mathbb{R}^3 \setminus \gamma. \tag{0.10}$$

It turns out that the fundamental group $\pi_1(U, z_0)$, where z_0 is any point of U, is abelian precisely when the given knot γ can be deformed by means of a smooth homotopy-of-knots (i.e. by an "isotopy", as it is called) into the trivial knot, i.e. into the unknotted circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$, where the circle S^1 lies in