

# Graduate Texts in Mathematics

**Jean-Pierre Serre**

## **Linear Representations of Finite Groups**

**有限群的线性表示**

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Jean-Pierre Serre

# Linear Representations of Finite Groups

Translated from the French by  
Leonard L. Scott



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## Preface

This book consists of three parts, rather different in level and purpose:

The first part was originally written for quantum chemists. It describes the correspondence, due to Frobenius, between linear representations and characters. This is a fundamental result, of constant use in mathematics as well as in quantum chemistry or physics. I have tried to give proofs as elementary as possible, using only the definition of a group and the rudiments of linear algebra. The examples (Chapter 5) have been chosen from those useful to chemists.

The second part is a course given in 1966 to second-year students of l'École Normale. It completes the first on the following points:

- (a) degrees of representations and integrality properties of characters (Chapter 6);
- (b) induced representations, theorems of Artin and Brauer, and applications (Chapters 7–11);
- (c) rationality questions (Chapters 12 and 13).

The methods used are those of linear algebra (in a wider sense than in the first part): group algebras, modules, noncommutative tensor products, semisimple algebras.

The third part is an introduction to Brauer theory: passage from characteristic 0 to characteristic  $p$  (and conversely). I have freely used the language of abelian categories (projective modules, Grothendieck groups), which is well suited to this sort of question. The principal results are:

- (a) The fact that the decomposition homomorphism is surjective: all irreducible representations in characteristic  $p$  can be lifted “virtually” (i.e., in a suitable Grothendieck group) to characteristic 0.
- (b) The Fong–Swan theorem, which allows suppression of the word “virtually” in the preceding statement, provided that the group under consideration is  $p$ -solvable.

I have also given several applications to the Artin representations.

## Preface

I take pleasure in thanking:

Gaston Berthier and Josiane Serre, who have authorized me to reproduce Part I, written for them and their students in *Quantum Chemistry*;

Yves Balasko, who drafted a first version of Part II from some lecture notes;

Alexandre Grothendieck, who has authorized me to reproduce Part III, which first appeared in his Séminaire de Géométrie Algébrique, I.H.E.S., 1965/66.

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**REPRESENTATIONS  
AND CHARACTERS**



# CHAPTER 1

## Generalities on linear representations

### 1.1 Definitions

Let  $V$  be a vector space over the field  $\mathbb{C}$  of complex numbers and let  $\text{GL}(V)$  be the group of *isomorphisms* of  $V$  onto itself. An element  $a$  of  $\text{GL}(V)$  is, by definition, a linear mapping of  $V$  into  $V$  which has an inverse  $a^{-1}$ ; this inverse is linear. When  $V$  has a finite basis  $(e_i)$  of  $n$  elements, each linear map  $a: V \rightarrow V$  is defined by a square matrix  $(a_{ij})$  of order  $n$ . The coefficients  $a_{ij}$  are complex numbers; they are obtained by expressing the images  $a(e_j)$  in terms of the basis  $(e_i)$ :

$$a(e_j) = \sum_i a_{ij} e_i.$$

Saying that  $a$  is an isomorphism is equivalent to saying that the determinant  $\det(a) = \det(a_{ij})$  of  $a$  is not zero. The group  $\text{GL}(V)$  is thus identifiable with the group of *invertible square matrices of order  $n$* .

Suppose now  $G$  is a *finite* group, with identity element  $1$  and with composition  $(s, t) \mapsto st$ . A *linear representation* of  $G$  in  $V$  is a homomorphism  $\rho$  from the group  $G$  into the group  $\text{GL}(V)$ . In other words, we associate with each element  $s \in G$  an element  $\rho(s)$  of  $\text{GL}(V)$  in such a way that we have the equality

$$\rho(st) = \rho(s) \cdot \rho(t) \quad \text{for } s, t \in G.$$

[We will also frequently write  $\rho_s$  instead of  $\rho(s)$ .] Observe that the preceding formula implies the following:

$$\rho(1) = 1, \quad \rho(s^{-1}) = \rho(s)^{-1}.$$

When  $\rho$  is given, we say that  $V$  is a *representation space* of  $G$  (or even simply, by abuse of language, a *representation* of  $G$ ). In what follows, we

## Chapter 1: Representations and characters

restrict ourselves to the case where  $V$  has *finite dimension*. This is not a very severe restriction. Indeed, for most applications, one is interested in dealing with a *finite number of elements*  $x_i$  of  $V$ , and can always find a *subrepresentation* of  $V$  (in a sense defined later, cf. 1.3) of finite dimension, which contains the  $x_i$ : just take the vector subspace generated by the images  $\rho_s(x_i)$  of the  $x_i$ .

Suppose now that  $V$  has finite dimension, and let  $n$  be its dimension; we say also that  $n$  is the degree of the representation under consideration. Let  $(e_i)$  be a basis of  $V$ , and let  $R_s$  be the matrix of  $\rho_s$  with respect to this basis. We have

$$\det(R_s) \neq 0, \quad R_{st} = R_s \cdot R_t \quad \text{if } s, t \in G.$$

If we denote by  $r_{ij}(s)$  the coefficients of the matrix  $R_s$ , the second formula becomes

$$r_{ik}(st) = \sum_j r_{ij}(s) \cdot r_{jk}(t).$$

Conversely, given invertible matrices  $R_s = (r_{ij}(s))$  satisfying the preceding identities, there is a corresponding linear representation  $\rho$  of  $G$  in  $V$ ; this is what it means to give a representation "in matrix form."

Let  $\rho$  and  $\rho'$  be two representations of the same group  $G$  in vector spaces  $V$  and  $V'$ . These representations are said to be *similar* (or *isomorphic*) if there exists a linear isomorphism  $\tau: V \rightarrow V'$  which "transforms"  $\rho$  into  $\rho'$ , that is, which satisfies the identity

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \quad \text{for all } s \in G.$$

When  $\rho$  and  $\rho'$  are given in matrix form by  $R_s$  and  $R'_s$  respectively, this means that there exists an invertible matrix  $T$  such that

$$T \cdot R_s = R'_s \cdot T, \quad \text{for all } s \in G,$$

which is also written  $R'_s = T \cdot R_s \cdot T^{-1}$ . We can *identify* two such representations (by having each  $x \in V$  correspond to the element  $\tau(x) \in V'$ ); in particular,  $\rho$  and  $\rho'$  have the same degree.

### 1.2 Basic examples

(a) A representation of *degree 1* of a group  $G$  is a homomorphism  $\rho: G \rightarrow C^*$ , where  $C^*$  denotes the multiplicative group of nonzero complex numbers. Since each element of  $G$  has finite order, the values  $\rho(s)$  of  $\rho$  are roots of unity; in particular, we have  $|\rho(s)| = 1$ .

If we take  $\rho(s) = 1$  for all  $s \in G$ , we obtain a representation of  $G$  which is called the *unit* (or *trivial*) representation.

(b) Let  $g$  be the order of  $G$ , and let  $V$  be a vector space of dimension  $g$ , with a basis  $(e_t)_{t \in G}$  indexed by the elements  $t$  of  $G$ . For  $s \in G$ , let  $\rho_s$  be

the linear map of  $V$  into  $V$  which sends  $e_i$  to  $e_{si}$ ; this defines a linear representation, which is called the *regular representation* of  $G$ . Its degree is equal to the order of  $G$ . Note that  $e_s = \rho_s(e_1)$ ; hence note that the images of  $e_1$  form a basis of  $V$ . Conversely, let  $W$  be a representation of  $G$  containing a vector  $w$  such that the  $\rho_s(w)$ ,  $s \in G$ , form a basis of  $W$ ; then  $W$  is isomorphic to the regular representation (an isomorphism  $\tau: V \rightarrow W$  is defined by putting  $\tau(e_s) = \rho_s(w)$ ).

(c) More generally, suppose that  $G$  acts on a finite set  $X$ . This means that, for each  $s \in G$ , there is given a permutation  $x \mapsto sx$  of  $X$ , satisfying the identities

$$1x = x, s(stx) = (st)x \quad \text{if } s, t \in G, x \in X.$$

Let  $V$  be a vector space having a basis  $(e_x)_{x \in X}$  indexed by the elements of  $X$ . For  $s \in G$  let  $\rho_s$  be the linear map of  $V$  into  $V$  which sends  $e_x$  to  $e_{sx}$ ; the linear representation of  $G$  thus obtained is called the *permutation representation* associated with  $X$ .

### 1.3 Subrepresentations

Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation and let  $W$  be a vector subspace of  $V$ . Suppose that  $W$  is *stable* under the action of  $G$  (we say also “invariant”), or in other words, suppose that  $x \in W$  implies  $\rho_s x \in W$  for all  $s \in G$ . The restriction  $\rho_s^W$  of  $\rho_s$  to  $W$  is then an isomorphism of  $W$  onto itself, and we have  $\rho_{st}^W = \rho_s^W \cdot \rho_t^W$ . Thus  $\rho^W: G \rightarrow \text{GL}(W)$  is a linear representation of  $G$  in  $W$ ;  $W$  is said to be a *subrepresentation* of  $V$ .

**EXAMPLE.** Take for  $V$  the regular representation of  $G$  [cf. 1.2 (b)], and let  $W$  be the subspace of dimension 1 of  $V$  generated by the element  $x = \sum_{s \in G} e_s$ . We have  $\rho_s x = x$  for all  $s \in G$ ; consequently  $W$  is a subrepresentation of  $V$ , isomorphic to the unit representation. (We will determine in 2.4 all the subrepresentations of the regular representation.)

Before going further, we recall some concepts from linear algebra. Let  $V$  be a vector space, and let  $W$  and  $W'$  be two subspaces of  $V$ . The space  $V$  is said to be the *direct sum* of  $W$  and  $W'$  if each  $x \in V$  can be written uniquely in the form  $x = w + w'$ , with  $w \in W$  and  $w' \in W'$ ; this amounts to saying that the intersection  $W \cap W'$  of  $W$  and  $W'$  is 0 and that  $\dim(V) = \dim(W) + \dim(W')$ . We then write  $V = W \oplus W'$  and say that  $W'$  is a complement of  $W$  in  $V$ . The mapping  $p$  which sends each  $x \in V$  to its component  $w \in W$  is called the *projection* of  $V$  onto  $W$  associated with the decomposition  $V = W \oplus W'$ ; the image of  $p$  is  $W$ , and  $p(x) = x$  for  $x \in W$ ; conversely if  $p$  is a linear map of  $V$  into itself satisfying these two properties, one checks that  $V$  is the direct sum of  $W$  and the *kernel*  $W'$  of  $p$ .

(the set of  $x$  such that  $px = 0$ ). A bijective correspondence is thus established between the *projections* of  $V$  onto  $W$  and the *complements* of  $W$  in  $V$ .

We return now to subrepresentations:

**Theorem 1.** *Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$  in  $V$  and let  $W$  be a vector subspace of  $V$  stable under  $G$ . Then there exists a complement  $W^0$  of  $W$  in  $V$  which is stable under  $G$ .*

Let  $W'$  be an arbitrary complement of  $W$  in  $V$ , and let  $p$  be the corresponding projection of  $V$  onto  $W$ . Form the average  $p^0$  of the conjugates of  $p$  by the elements of  $G$ :

$$p^0 = \frac{1}{g} \sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1} \quad (g \text{ being the order of } G).$$

Since  $p$  maps  $V$  into  $W$  and  $\rho_t$  preserves  $W$  we see that  $p^0$  maps  $V$  into  $W$ ; we have  $\rho_t^{-1}x \in W$  for  $x \in W$ , whence

$$p \cdot \rho_t^{-1}x = \rho_t^{-1}x, \quad \rho_t \cdot p \cdot \rho_t^{-1}x = x, \quad \text{and} \quad p^0x = x.$$

Thus  $p^0$  is a projection of  $V$  onto  $W$ , corresponding to some complement  $W^0$  of  $W$ . We have moreover

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s \quad \text{for all } s \in G.$$

Indeed, computing  $\rho_s \cdot p^0 \cdot \rho_s^{-1}$ , we find:

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0.$$

If now  $x \in W^0$  and  $s \in G$  we have  $p^0x = 0$ , hence  $p^0 \cdot \rho_s x = \rho_s \cdot p^0 x = 0$ , that is,  $\rho_s x \in W^0$ , which shows that  $W^0$  is stable under  $G$ , and completes the proof.  $\square$

*Remark.* Suppose that  $V$  is endowed with a *scalar product*  $(x|y)$  satisfying the usual conditions: linearity in  $x$ , semilinearity in  $y$ , and  $(x|x) > 0$  if  $x \neq 0$ . Suppose that this scalar product is *invariant* under  $G$ , i.e., that  $(\rho_s x | \rho_s y) = (x | y)$ ; we can always reduce to this case by replacing  $(x|y)$  by  $\sum_{t \in G} (\rho_t x | \rho_t y)$ . Under these hypotheses the *orthogonal complement*  $W^0$  of  $W$  in  $V$  is a complement of  $W$  stable under  $G$ ; another proof of theorem 1 is thus obtained. Note that the invariance of the scalar product  $(x|y)$  means that, if  $(e_i)$  is an orthonormal basis of  $V$ , the matrix of  $\rho_s$  with respect to this basis is a *unitary matrix*.

Keeping the hypothesis and notation of theorem 1, let  $x \in V$  and let  $w$  and  $w^0$  be its projections on  $W$  and  $W^0$ . We have  $x = w + w^0$ , whence  $\rho_s x = \rho_s w + \rho_s w^0$ , and since  $W$  and  $W^0$  are stable under  $G$ , we have  $\rho_s w \in W$  and  $\rho_s w^0 \in W^0$ ; thus  $\rho_s w$  and  $\rho_s w^0$  are the projections of  $\rho_s x$ . It follows the representations  $W$  and  $W^0$  determine the representation  $V$ .



We say that  $V$  is the direct sum of  $W$  and  $W^0$ , and write  $V = W \oplus W^0$ . An element of  $V$  is identified with a pair  $(w, w^0)$  with  $w \in W$  and  $w^0 \in W^0$ . If  $W$  and  $W^0$  are given in matrix form by  $R_s$  and  $R_s^0$ ,  $W \oplus W^0$  is given in matrix form by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

The direct sum of an arbitrary finite number of representations is defined similarly.

## 1.4 Irreducible representations

Let  $\rho: G \rightarrow GL(V)$  be a linear representation of  $G$ . We say that it is *irreducible* or *simple* if  $V$  is not 0 and if no vector subspace of  $V$  is stable under  $G$ , except of course 0 and  $V$ . By theorem 1, this second condition is equivalent to saying  $V$  is *not the direct sum of two representations* (except for the trivial decomposition  $V = 0 \oplus V$ ). A representation of degree 1 is evidently irreducible. We will see later (3.1) that each nonabelian group possesses at least one irreducible representation of degree  $\geq 2$ .

The irreducible representations are used to construct the others by means of the direct sum:

**Theorem 2.** *Every representation is a direct sum of irreducible representations.*

Let  $V$  be a linear representation of  $G$ . We proceed by induction on  $\dim(V)$ . If  $\dim(V) = 0$ , the theorem is obvious (0 is the direct sum of the empty family of irreducible representations). Suppose then  $\dim(V) \geq 1$ . If  $V$  is irreducible, there is nothing to prove. Otherwise, because of th. 1,  $V$  can be decomposed into a direct sum  $V' \oplus V''$  with  $\dim(V') < \dim(V)$  and  $\dim(V'') < \dim(V)$ . By the induction hypothesis  $V'$  and  $V''$  are direct sums of irreducible representations, and so the same is true of  $V$ .  $\square$

*Remark.* Let  $V$  be a representation, and let  $V = W_1 \oplus \cdots \oplus W_k$  be a decomposition of  $V$  into a direct sum of irreducible representations. We can ask if this decomposition is *unique*. The case where all the  $\rho_s$  are equal to 1 shows that this is not true in general (in this case the  $W_i$  are lines, and we have a plethora of decompositions of a vector space into a direct sum of lines). Nevertheless, we will see in 2.3 that the *number* of  $W_i$  isomorphic to a given irreducible representation does not depend on the chosen decomposition.

## 1.5 Tensor product of two representations

Along with the direct sum operation (which has the formal properties of an addition), there is a "multiplication": the *tensor product*, sometimes called the *Kronecker product*. It is defined as follows: