Graduate Texts in Mathematics

Jean-Pierre Serre

Linear Representations of Finite Groups

有限群的线性表示

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Preface

This book consists of three parts, rather different in level and purpose:

The first part was originally written for quantum chemists. It describes the correspondence, due to Frobenius, between linear representations and characters. This is a fundamental result, of constant use in mathematics as well as in quantum chemistry or physics. I have tried to give proofs as elementary as possible, using only the definition of a group and the rudiments of linear algebra. The examples (Chapter 5) have been chosen from those useful to chemists.

The second part is a course given in 1966 to second-year students of l'École Normale. It completes the first on the following points:

(a) degrees of representations and integrality properties of characters (Chapter 6);
(b) induced representations, theorems of Artin and Brauer, and applications (Chapters 7–11);
(c) rationality questions (Chapters 12 and 13).

The methods used are those of linear algebra (in a wider sense than in the first part): group algebras, modules, noncommutative tensor products, semisimple algebras.

The third part is an introduction to Brauer theory: passage from characteristic 0 to characteristic p (and conversely). I have freely used the language of abelian categories (projective modules, Grothendieck groups), which is well suited to this sort of question. The principal results are:

(a) The fact that the decomposition homomorphism is surjective: all irreducible representations in characteristic p can be lifted "virtually" (i.e., in a suitable Grothendieck group) to characteristic 0.
(b) The Fong–Swan theorem, which allows suppression of the word "virtually" in the preceding statement, provided that the group under consideration is p-solvable.

I have also given several applications to the Artin representations.

I take pleasure in thanking:

Gaston Berthier and Josiane Serre, who have authorized me to reproduce Part I, written for them and their students in *Quantum Chemistry*;

Yves Balasko, who drafted a first version of Part II from some lecture notes;

Alexandre Grothendieck, who has authorized me to reproduce Part III, which first appeared in his Séminaire de Géométrie Algébrique, I.H.E.S., 1965/66.

Contents

I

REPRESENTATIONS AND CHARACTERS

CHAPTER 1

Generalities on linear representations

1.1 Definitions

Let V be a vector space over the field **C** of complex numbers and let GL(V) be the group of *isomorphisms* of V onto itself. An element a of GL(V) is, by definition, a linear mapping of V into V which has an inverse a^{-1}; this inverse is linear. When V has a finite basis (e_i) of n elements, each linear map $a\colon \mathrm{V} \to \mathrm{V}$ is defined by a square matrix (a_{ij}) of order n. The coefficients a_{ij} are complex numbers; they are obtained by expressing the images $a(e_j)$ in terms of the basis (e_i):

$$a(e_j) = \sum_i a_{ij} e_i.$$

Saying that a is an isomorphism is equivalent to saying that the determinant $\det(a) = \det(a_{ij})$ of a is not zero. The group GL(V) is thus identifiable with the group of *invertible square matrices of order n.*

Suppose now G is a *finite* group, with identity element 1 and with composition $(s, t) \mapsto st$. A *linear representation* of G in V is a homomorphism ρ from the group G into the group GL(V). In other words, we associate with each element $s \in \mathrm{G}$ an element $\rho(s)$ of GL(V) in such a way that we have the equality

$$\rho(st) = \rho(s) \cdot \rho(t) \quad \text{for } s, t \in \mathrm{G}.$$

[We will also frequently write ρ_s instead of $\rho(s)$.] Observe that the preceding formula implies the following:

$$\rho(1) = 1, \quad \rho(s^{-1}) = \rho(s)^{-1}.$$

When ρ is given, we say that V is a *representation space* of G (or even simply, by abuse of language, a *representation* of G). In what follows, we restrict ourselves to the case where V has *finite dimension*. This is not a very severe restriction. Indeed, for most applications, one is interested in dealing with a *finite number of elements* x_i of V, and can always find a *subrepresentation* of V (in a sense defined later, cf. 1.3) of finite dimension, which contains the x_i: just take the vector subspace generated by the images $\rho_s(x_i)$ of the x_i.

Suppose now that V has finite dimension, and let n be its dimension; we say also that n is the degree of the representation under consideration. Let (e_i) be a basis of V, and let R_s be the matrix of ρ_s with respect to this basis. We have

$$\det(\mathrm{R}_s) \neq 0, \qquad \mathrm{R}_{st} = \mathrm{R}_s \cdot \mathrm{R}_t \quad \text{if } s, t \in \mathrm{G}.$$

If we denote by $r_{ij}(s)$ the coefficients of the matrix R_s, the second formula becomes

$$r_{ik}(st) = \sum_j r_{ij}(s) \cdot r_{jk}(t).$$

Conversely, given invertible matrices $\mathrm{R}_s = (r_{ij}(s))$ satisfying the preceding identities, there is a corresponding linear representation ρ of G in V; this is what it means to give a representation "in matrix form."

Let ρ and ρ' be two representations of the same group G in vector spaces V and V′. These representations are said to be *similar* (or *isomorphic*) if there exists a linear isomorphism $\tau\colon \mathrm{V} \to \mathrm{V}'$ which "transforms" ρ into ρ', that is, which satisfies the identity

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \quad \text{for all } s \in \mathrm{G}.$$

When ρ and ρ' are given in matrix form by R_s and R'_s respectively, this means that there exists an invertible matrix T such that

$$\mathrm{T} \cdot \mathrm{R}_s = \mathrm{R}'_s \cdot \mathrm{T}, \quad \text{for all } s \in \mathrm{G},$$

which is also written $\mathrm{R}'_s = \mathrm{T} \cdot \mathrm{R}_s \cdot \mathrm{T}^{-1}$. We can *identify* two such representations (by having each $x \in \mathrm{V}$ correspond to the element $\tau(x) \in \mathrm{V}'$); in particular, ρ and ρ' have the same degree.

1.2 Basic examples

(a) A representation *of degree* 1 of a group G is a homomorphism $\rho\colon \mathrm{G} \to \mathbf{C}^*$, where $\mathbf{C}^*$ denotes the multiplicative group of nonzero complex numbers. Since each element of G has finite order, the values $\rho(s)$ of ρ are roots of unity; in particular, we have $|\rho(s)| = 1$.

If we take $\rho(s) = 1$ for all $s \in \mathrm{G}$, we obtain a representation of G which is called the *unit* (or *trivial*) representation.

(b) Let g be the order of G, and let V be a vector space of dimension g, with a basis $(e_t)_{t \in \mathrm{G}}$ indexed by the elements t of G. For $s \in \mathrm{G}$, let ρ_s be the linear map of V into V which sends e_t to e_{st}; this defines a linear representation, which is called the *regular representation* of G. Its degree is equal to the order of G. Note that $e_s = \rho_s(e_1)$; hence note that the images of e_1 form a basis of V. Conversely, let W be a representation of G containing a vector w such that the $\rho_s(w)$, $s \in G$, form a basis of W; then W is isomorphic to the regular representation (an isomorphism τ: V $\rightarrow$ W is defined by putting $\tau(e_s) = \rho_s(w)$).

(c) More generally, suppose that G acts on a finite set X. This means that, for each $s \in G$, there is given a permutation $x \mapsto sx$ of X, satisfying the identities

$$1x = x,\ s(tx) = (st)x \quad \text{if } s, t \in G,\ x \in X.$$

Let V be a vector space having a basis $(e_x)_{x \in X}$ indexed by the elements of X. For $s \in G$ let ρ_s be the linear map of V into V which sends e_x to e_{sx}; the linear representation of G thus obtained is called the *permutation representation* associated with X.

1.3 Subrepresentations

Let ρ: G $\rightarrow$ GL(V) be a linear representation and let W be a vector subspace of V. Suppose that W is *stable* under the action of G (we say also "invariant"), or in other words, suppose that $x \in W$ implies $\rho_s x \in W$ for all $s \in G$. The restriction ρ_s^W of ρ_s to W is then an isomorphism of W onto itself, and we have $\rho_{st}^W = \rho_s^W \cdot \rho_t^W$. Thus ρ^W: G $\rightarrow$ GL(W) is a linear representation of G in W; W is said to be a *subrepresentation* of V.

EXAMPLE. Take for V the regular representation of G [cf. 1.2 (b)], and let W be the subspace of dimension 1 of V generated by the element $x = \sum_{s \in G} e_s$. We have $\rho_s x = x$ for all $s \in G$; consequently W is a subrepresentation of V, isomorphic to the unit representation. (We will determine in 2.4 all the subrepresentations of the regular representation.)

Before going further, we recall some concepts from linear algebra. Let V be a vector space, and let W and W′ be two subspaces of V. The space V is said to be the *direct sum* of W and W′ if each $x \in V$ can be written uniquely in the form $x = w + w'$, with $w \in W$ and $w' \in W'$; this amounts to saying that the intersection $W \cap W'$ of W and W′ is 0 and that $\dim(V) = \dim(W) + \dim(W')$. We then write $V = W \oplus W'$ and say that W′ is a complement of W in V. The mapping p which sends each $x \in V$ to its component $w \in W$ is called the *projection* of V onto W associated with the decomposition $V = W \oplus W'$; the image of p is W, and $p(x) = x$ for $x \in W$; conversely if p is a linear map of V into itself satisfying these two properties, one checks that V is the direct sum of W and the *kernel* W′ of p

(the set of x such that $px = 0$). A bijective correspondence is thus established between the *projections* of V onto W and the *complements* of W in V.

We return now to subrepresentations:

Theorem 1. *Let* ρ: G $\to$ GL(V) *be a linear representation of* G *in* V *and let* W *be a vector subspace of* V *stable under* G. *Then there exists a complement* W^0 *of* W *in* V *which is stable under* G.

Let W′ be an arbitrary complement of W in V, and let p be the corresponding projection of V onto W. Form the average p^0 of the conjugates of p by the elements of G:

$$p^0 = \frac{1}{g}\sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1} \qquad (g \text{ being the order of G}).$$

Since p maps V into W and ρ_t preserves W we see that p^0 maps V into W; we have $\rho_t^{-1}x \in$ W for $x \in$ W, whence

$$p \cdot \rho_t^{-1}x = \rho_t^{-1}x, \qquad \rho_t \cdot p \cdot \rho_t^{-1}x = x, \qquad \text{and} \qquad p^0 x = x.$$

Thus p^0 is a projection of V onto W, corresponding to some complement W^0 of W. We have moreover

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s \quad \text{for all } s \in G.$$

Indeed, computing $\rho_s \cdot p^0 \cdot \rho_s^{-1}$, we find:

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{g}\sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{g}\sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0.$$

If now $x \in W^0$ and $s \in$ G we have $p^0 x = 0$, hence $p^0 \cdot \rho_s x = \rho_s \cdot p^0 x = 0$, that is, $\rho_s x \in W^0$, which shows that W^0 is stable under G, and completes the proof. □

Remark. Suppose that V is endowed with a *scalar product* $(x|y)$ satisfying the usual conditions: linearity in x, semilinearity in y, and $(x|x) > 0$ if $x \neq 0$. Suppose that this scalar product is *invariant* under G, i.e., that $(\rho_s x|\rho_s y) = (x|y)$; we can always reduce to this case by replacing $(x|y)$ by $\sum_{t \in G}(\rho_t x|\rho_t y)$. Under these hypotheses the *orthogonal complement* W^0 *of* W in V is a complement of W stable under G; another proof of theorem 1 is thus obtained. Note that the invariance of the scalar product $(x|y)$ means that, if (e_i) is an orthonormal basis of V, the matrix of ρ_s with respect to this basis is a *unitary matrix*.

Keeping the hypothesis and notation of theorem 1, let $x \in$ V and let w and w^0 be its projections on W and W^0. We have $x = w + w^0$, whence $\rho_s x = \rho_s w + \rho_s w^0$, and since W and W^0 are stable under G, we have $\rho_s w \in$ W and $\rho_s w^0 \in W^0$; thus $\rho_s w$ and $\rho_s w^0$ are the projections of $\rho_s x$. It follows the representations W and W^0 determine the representation V.

We say that V is the direct sum of W and W^0, and write $V = W \oplus W^0$. An element of V is identified with a pair (w, w^0) with $w \in W$ and $w^0 \in W^0$. If W and W^0 are given in matrix form by R_s and R_s^0, $W \oplus W^0$ is given in matrix form by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

The direct sum of an arbitrary finite number of representations is defined similarly.

1.4 Irreducible representations

Let $\rho: G \to GL(V)$ be a linear representation of G. We say that it is *irreducible* or *simple* if V is not 0 and if no vector subspace of V is stable under G, except of course 0 and V. By theorem 1, this second condition is equivalent to saying V *is not the direct sum of two representations* (except for the trivial decomposition $V = 0 \oplus V$). A representation of degree 1 is evidently irreducible. We will see later (3.1) that each nonabelian group possesses at least one irreducible representation of degree $\geqslant 2$.

The irreducible representations are used to construct the others by means of the direct sum:

Theorem 2. *Every representation is a direct sum of irreducible representations.*

Let V be a linear representation of G. We proceed by induction on $\dim(V)$. If $\dim(V) = 0$, the theorem is obvious (0 is the direct sum of the *empty family* of irreducible representations). Suppose then $\dim(V) \geqslant 1$. If V is irreducible, there is nothing to prove. Otherwise, because of th. 1, V can be decomposed into a direct sum $V' \oplus V''$ with $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)$. By the induction hypothesis V' and V'' are direct sums of irreducible representations, and so the same is true of V. □

Remark. Let V be a representation, and let $V = W_1 \oplus \cdots \oplus W_k$ be a decomposition of V into a direct sum of irreducible representations. We can ask if this decomposition is *unique*. The case where all the ρ_s are equal to 1 shows that this is not true in general (in this case the W_i are lines, and we have a plethora of decompositions of a vector space into a direct sum of lines). Nevertheless, we will see in 2.3 that the *number* of W_i isomorphic to a given irreducible representation does not depend on the chosen decomposition.

1.5 Tensor product of two representations

Along with the direct sum operation (which has the formal properties of an addition), there is a "multiplication": the *tensor product*, sometimes called the *Kronecker* product. It is defined as follows: