Functional and Impulsive Differential Equations of Fractional Order

Qualitative Analysis and Applications

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This book is dedicated
to Trayan and Alex

Preface

This book is an exposition of the most recent results related to the qualitative analysis of a variety of fractional-order equations.

The mathematical investigations of derivatives of non-integer order mark their beginning with the correspondence between Leibniz and L'Hospital in 1695 (see [Leibniz 1695], [Podlubny 1999]). Although fractional calculus has a more than three centuries long history, the subject of fractional differential equations has gained considerable popularity and importance during the past few decades, due mainly to its demonstrated applications to many real-world phenomena studies in physics, mechanics, chemistry, engineering, finance, etc.

There are several excellent books devoted to equations of fractional order. The book written by I. Podlubny (see [Podlubny 1999]) played an outstanding role in the development of the theory of fractional ordinary differential equations. It is the most cited and used book which is entirely devoted to a systematic presentation of the basic ideas and methods of fractional calculus in the theory and applications of such equations. The monograph of Kilbas, Srivastava and Trujillo (2006) provides uptodate developments on fractional differential and fractional integro-differential equations involving many different potentially useful operators of fractional calculus. In addition to these, the books of Abbas, Benchohra and N'Guérékata (2012), Baleanu, Diethelm, Scalas and Trujillo (2012), Caponetto, Dongola, Fortuna and Petráš (2010), Das (2011), Diethelm (2010), Hilfer (2000), Kiryakova (1994), Lakshmikantham, Leela and Vasundhara Devi (2009), Magin (2006), Miller and Ross (1993) are good sources for the theory of fractional operators and equations, as well as their numerous applications.

On the other hand, relatively recently functional differential equations of fractional order have started to receive an increasing interest [Baleanu, Sadati, Ghaderi, Ranjbar, Abdeljawad and Jarad 2010], [Banaś and Zając 2011], [Benchohra, Henderson, Ntouyas and Ouahab 2008], [El-Sayed, Gaaraf and

Hamadalla 2010], [Henderson and Ouahab 2009], [Lakshmikantham 2008]. Indeed, fractional operators are a very natural tool to model memory-dependent phenomena. Also, fractional calculus has been incorporated into impulsive differential equations [Ahmad and Nieto 2011], [Ahmad and Sivasundaram 2010], [Anguraj and Maheswari 2012], [Cao and Chen 2012], [Chauhan and Dabas 2011], [Fečkan, Zhou and Wang 2012]. Many interesting results on the fundamental theory of such equations have been reported. However, due to the lack of a book on these topics, many researchers remain unaware of this field.

The questions related to the qualitative theory of different classes of differential equations are the age-old problems of a great importance. The methods used in the qualitative investigation of their solutions, and their wide applications have all advanced to the extent that aspects in each of these areas have demanded individual attention.

The survey published in 2011 by Li and Zhang (see [Li and Zhang 2011]), is a very good overview on the recent stability results of fractional differential equations without impulses and delays, and the analytical methods used. It is seen that, at the time, a few stability results rely on a restrictive modeling of fractional differential systems: the basic hypothesis deals with commensurability, i.e. the fractional derivative orders have to be an integer multiple of minimal fractional order. In the last decades, many researchers have more interests in the stability of linear systems and some methods have emerged in succession. For example, there are the frequency domain methods, Linear Matrix Inequalities methods, and conversion methods [Li and Zhang 2011]. By contrast, the development of stability of nonlinear fractional differential systems even without impulsive perturbations was a bit slow.

During the last few years the authors' research in the area of the qualitative theory of different classes of functional and impulsive fractional-order equations have undergone rapid development. A string of extensive results on the stability, boundedness, asymptotic behavior and almost periodicity for these classes of equations have been obtained. The primary aim of this book is to gather under one cover many of these results which will be of prime importance for researchers on the topic. It fills a void by making available a source book which describes existing literature on the topic, methods and their development. The second motivation comes from the applicable point of view, since the qualitative properties have significant practical applications in the emerging areas such as optimal control, biology, mechanics, medicine, bio-technologies, electronics, economics, etc. For the applied scientists it is important to have an introduction to the qualitative theory of fractional equations, which could help in their initial steps to adopt the results and methods in their research.

The book consists of four chapters. It presents results for different classes of fractional equations, including fractional functional differential equations, fractional impulsive differential equations, fractional impulsive functional differential equations, which have not been covered by other books. It shows the manifestations of different constructive methods by demonstrating how these effective techniques can be applied to investigate qualitative properties of the solutions of fractional systems. Since many applications are also included, the demonstrated techniques and models can be used in training of students in mathematical modeling and as an instigation in the study and development of fractional-order models.

The book is addressed to a wide audience of professionals such as mathematicians, applied researchers and practitioners.

The authors are extremely grateful and very much indebted to Dr. Sandy Norman, Chair of the Department of Mathematics at the University of Texas at San Antonio for ensuring the opportunity for successful work on this book. In addition, the authors have the pleasure to express their sincere gratitude to all their co-authors, the work with whom expanded their experience. They are also thankful to all friends, colleagues and reviewers for their valuable comments and suggestions during the preparation of the manuscript.

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Contents

Chapter 1

Introduction

The present chapter will deal with an introduction of functional and impulsive equations of fractional order and the basic theory necessary for their qualitative investigations.

Section 1.1 will offer the main classes of fractional equations, investigated in the book. Some results from fractional calculus will be also discussed.

Section 1.2 is devoted to definitions of qualitative properties of fractional equations that are used throughout the book.

Section 1.3 will deal with different classes of Lyapunov functions. The definitions of their Caputo and Riemann–Liouville fractional derivatives will be also given.

Finally, in *Section* 1.4, fractional scalar and vector comparison results will be considered in terms of Lyapunov-like functions. Some auxiliary lemmas for Caputo and Riemann–Liouville fractional derivatives are also presented.

1.1 Preliminary Notes

In this section we shall make a brief description of the main classes of fractional equations that will be used in the book.

Let

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

be the *Gamma function* which converges in the right-half of the complex plane $Re(z) > 0$. There are many excellent books (see, for example [Diethelm 2010] and [Podlubny 1999]) in which the definition and main properties of the Gamma function are given, and here we will avoid a repetition.

We define the *fractional integral* of order α on the interval $[a,t]$ by

$$_a\mathscr{D}_t^{-\alpha} l(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} l(s)ds,$$

where $0 < \alpha < 1$ and l is an arbitrary integrable function (see [Gelfand and Shilov 1959], [Podlubny 1999]).

For an arbitrary real number p we denote the *Riemann–Liouville* and *Caputo* fractional derivatives of order p, respectively, as

$$_aD_t^p l(t) = \frac{d^{\lceil p \rceil}}{dt^{\lceil p \rceil}} \left[{}_a\mathscr{D}_t^{-(\lceil p \rceil - p)} l(t) \right]$$

and

$${}_a^cD_t^p l(t) = {}_a\mathscr{D}_t^{-(\lceil p \rceil - p)} \left[\frac{d^{\lceil p \rceil}}{dt^{\lceil p \rceil}} l(t) \right],$$

where $\lceil p \rceil$ stands for the smallest integer not less than p; and D and ${}^cD$ denote the Riemann–Liouville and Caputo fractional operators, respectively.

For the case $0 < p < 1$, which is investigated in our book, the relation between the two fractional derivatives is given by

$${}_a^cD_t^p l(t) = {}_aD_t^p \left[l(t) - l(a) \right].$$

A main difference between both fractional derivatives is that the Caputo derivative of a constant C is zero, i.e.

$${}_a^cD_t^p C = 0,$$

while the Riemann–Liouville fractional derivative of a constant C is given by

$${}_aD_t^p C = \frac{C}{\Gamma(1-p)} (t-a)^{-p}.$$

Therefore,

$${}_a^cD_t^p l(t) = {}_aD_t^p l(t) - \frac{l(a)}{\Gamma(1-p)} (t-a)^{-p}.$$

In particular, if $l(a) = 0$, then

$${}_a^cD_t^p l(t) = {}_aD_t^p l(t).$$

1.1.1 Fractional Functional Differential Equations

It is well-known that the delay differential equations of integer order are of great theoretical interest and form an important class as regards to their applications (see [Burton 1985], [Hale 1977], [Kolmanovskii and Myshkis 1999], [Kolmanovskii and Nosov 1986] and the references therein). The centrality of functional differential equations for theory and applications is witnessed by the current persistency of new contributions in this topic of interest [Bernfeld, Corduneanu and Ignatyev 2003], [Corduneanu and Ignatyev 2005], [Stamova and Stamov G 2014a]. Since 1960 many generalizations and extensions of such equations became a part of the literature ([Stamov 2012], [Stamova 2009]).

It is also known that the functional differential equations of fractional order have several applications. It has been proved that such type of equations are valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics (see [Babakhani, Baleanu and Khanbabaie 2012], [Bhalekar, Daftardar-Gejji, Baleanu and Magin 2011], [El-Sayed, Gaaraf and Hamadalla 2010], [Henderson and Ouahab 2009], [Stamova 2016], [Stamova and Stamov G 2013], [Wang, Huang and Shi 2011], [Wang, Yu and Wen 2014], [Wu, Hei and Chen 2013]). The efficient applications of fractional functional differential equations requires the finding of criteria for qualitative properties of their solutions.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $||.||$ and Ω be an open set in $\mathbb{R}^n$ containing the origin, $\mathbb{R}_+ = [0,\infty)$.

For a given $r > 0$ we suppose that $x \in C([t_0 - r, t_0 + A], \mathbb{R}^n)$, $t_0 \in \mathbb{R}$, $A > 0$. For any $t \in [t_0, t_0 + A)$, we denote by x_t an element of $C[[-r,0],\Omega]$ defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$.

Consider the system of fractional functional differential equations

$$ {}_{t_0}\mathscr{D}_t^{\alpha} x(t) = f(t, x_t), \tag{1.1} $$

where $f : \mathbb{R} \times C[[-r,0],\Omega] \to \mathbb{R}^n$ and ${}_{t_0}\mathscr{D}_t^{\alpha}$ denotes either Caputo or Riemann–Liouville fractional derivative of order α, $0 < \alpha < 1$. The class of equations (1.1) provides mathematical models for real-world problems in which the fractional rate of change depends on the influence of their hereditary effects.

Instead of an initial point value for an ordinary differential equation, for an initial condition related to (1.1) an initial function is required which is defined over the range of time delimited by the delay.

Let $\varphi_0 \in C[[-r,0],\Omega]$. We shall denote by $x(t) = x(t; t_0, \varphi_0)$ the solution of (1.1) with initial data $(t_0, \varphi_0) \in \mathbb{R} \times C[[-r,0],\Omega]$.

If the fractional derivative ${}_{t_0}\mathscr{D}_t^{\alpha} = {}_{t_0}^{\;c}D_t^{\alpha}$, then the initial condition is of the same type as for an integer-order equation:

$$ x_{t_0} = \varphi_0. \tag{1.2} $$

We shall assume that $f(t,\varphi)$ is smooth enough on $\mathbb{R} \times C[[-r,0],\Omega]$ to guarantee the existence and uniqueness of a solution $x(t; t_0, \varphi_0)$ of the initial value problem (IVP) (1.1), (1.2) on $[t_0 - r, \infty)$ for each initial function $\varphi_0 \in C[[-r,0],\Omega]$.

Then, the IVP (1.1), (1.2) is equivalent to the following Volterra fractional integral with memory (see [Lakshmikantham 2008]):

$$x_{t_0} = \varphi_0,$$
$$x(t) = \varphi_0(0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s) ds, \ t_0 \le t < \infty.$$

For some basic concept and theorems on the fundamental theory of such equations, we refer readers to [Benchohra, Henderson, Ntouyas and Ouahab 2008], [Henderson and Ouahab 2009], [Lakshmikantham 2008], and [Su and Feng 2012].

It was believed that the case ${}_{t_0}\mathscr{D}_t^{\alpha} = {}_{t_0}D_t^{\alpha}$ leads to initial conditions without physical meaning. This was contradicted by Heymans and Podlubny (2005) who studied several cases and gave physical meaning to the initial conditions of fractional differential equations with Riemann–Liouville derivatives. See also [Lazarević and Spasić 2009], [Li and Zhang 2011] and [Ortigueira and Coito 2010]. It is known that for a zero initial function the Riemann–Liouville and Caputo fractional derivatives coincide.

The system (1.1) is an universal type of fractional functional differential system. In the particular case, it contains a system of fractional differential equations when $r = 0$. Systems of the type (1.1) also include the following systems of fractional functional differential equations:

- systems of fractional differential-difference equations of the type

  $${}_{t_0}\mathscr{D}_t^{\alpha} x(t) = F(t, x(t), x(t - r_1(t)), x(t - r_2(t)), \ldots, x(t - r_p(t))),$$

  where $0 \le r_j(t) \le r$, $j = 1, 2, \ldots, p$;

- systems of fractional integro-differential equations of the type

  $${}_{t_0}\mathscr{D}_t^{\alpha} x(t) = \int_{-r}^{0} g(t, s, x(t+\theta)) d\theta;$$

- systems of fractional integro-differential equations with infinite delays

  $${}_{t_0}\mathscr{D}_t^{\alpha} x(t) = \int_{-\infty}^{t} k(t, \theta) f(t, x(\theta)) d\theta,$$

  where $k : \mathbb{R}^2 \to \mathbb{R}_+$ is continuous.

The system (1.1) is *linear*, when $f(t, x_t) = L(t, x_t) + h(t)$, where $L(t, x_t)$ is linear with respect to x_t.

Some systems of more general types are also included in the type (1.1) system.

1.1.2 Fractional Impulsive Differential Equations

The states of many evolutionary processes are often subjects to instantaneous perturbations and experience abrupt changes at certain moments of time. The duration of the changes is very short and negligible in comparison with the duration of the process considered, and can be thought of as "momentarily" changes or as impulses. Systems with short-term perturbations are often naturally described by impulsive differential equations (see for example [Bainov and Simeonov 1989], [Benchohra, Henderson and Ntouyas 2006], [Lakshmikantham, Bainov and Simeonov 1989], [Mil'man and Myshkis 1960], [Pandit and Deo 1982], [Samoilenko and Perestyuk 1995]). Such differential equations have became an active research subject in nonlinear science and have attracted more attention in many fields. For example, important impulsive differential equations have been introduced as mathematical models in population ecology (see [Ahmad and Stamov 2009a], [Ahmad and Stamova 2007a], [Ahmad and Stamova 2012], [Ahmad and Stamova 2013], [Ballinger and Liu 1997], [Dong, Chen and Sun 2006], [Georgescu and Zhang 2010], [Hu 2013], [Jiang and Lu 2007], [Liu and Rohlf 1998], [Stamov 2008], [Stamov 2009a], [Stamov 2012], [Xue, Wang and Jin 2007], [Yan 2003], [Yan 2010]), hematopoiesis (see [Kou, Adimy and Ducrot 2009]), pest control (see [Wang, Chen and Nieto 2010]), drug treatment (see [Liu, Yu and Zhu 2008], [Smith and Wahl 2004]), the chemostat (see [Sun and Chen 2007]) tumor-normal cell interaction (see [Dou, Chen and Li 2004]), plankton allelopathy (see [He, Chen and Li 2010]), vaccination (see [D'Onofrio 2002], [Qiao, Liu and Forys 2013]), radio engineering (see [Joelianto and Sutarto 2009]), communication security (see [Khadra, Liu and Shen 2003]), neural networks (see [Arbib 1987], [Stamov 2004], [Stamov 2009d], [Stamov 2012], [Zhao, Xia and Ding 2008]), etc. Also, in optimal control of economic systems, frequency-modulated signal processing systems, and some flying object motions, many systems are characterized by abrupt changes in their states at certain instants (see [Korn 1999], [Stamova and Stamov 2011], [Stamova and Stamov 2012], [Stamova and Stamov A 2013], [Stamova, Stamov and Simeonova 2013], [Yang 2001]). This type of impulsive phenomenon can also be found in the fields of information science, electronics, automatic control systems, computer networks, artificial intelligence, robotics and telecommunications. Many sudden and sharp changes occur instantaneously in these systems, in the form of impulses which cannot be well described by a pure continuous–time or discrete–time model [Li, Liao, Yang and Huang 2005].

A great progress in studying impulsive functional differential equations and their applications has also been made (see [Ahmad and Stamov 2009b], [Ahmad and Stamova 2007b], [Ahmad and Stamova 2008], [Hui and Chen 2005], [Li and Fan 2007], [Liu and Ballinger 2002], [Liu, Huang and Chen 2012], [Liu and Takeuchi 2007], [Liu, Teo and Hu 2005], [Liu and Wang 2007], [Liu and

Zhao 2012], [Long and Xu 2008], [Luo and Shen 2001], [Stamov 2009b], [Stamov 2009c], [Stamov 2010a], [Stamov 2010b], [Stamov 2012], [Stamov, Alzabut, Atanasov and Stamov 2011], [Stamov and Stamov 2013], [Stamov and Stamova 2001], [Stamov and Stamova 2007], [Stamova 2007], [Stamova 2008], [Stamova 2009], [Stamova 2010], [Stamova 2011a], [Stamova 2011b], [Stamova, Emmenegger and Stamov 2010], [Stamova, Ilarionov and Vaneva 2010], [Stamova and Stamov 2011], [Stamova and Stamov 2012], [Stamova and Stamov A 2013], [Stamova and Stamov T 2014a], [Stamova and Stamov T 2014b], [Stamova, Stamov and Li 2014], [Stamova, Stamov and Simeonova 2013], [Stamova, Stamov and Simeonova 2014], [Teng, Nie and Fang 2011], [Wang, Yu and Niu 2012], [Widjaja and Bottema 2005], [Xia 2007], [Zhou and Wan 2009]). Indeed, impulsive mathematical models with delays are found in almost every domain of applied sciences and they played a very important role in modern mathematical modelling of processes and phenomena studied in physics, population dynamics, chemical technology and economics.

Since, the tools of impulsive fractional differential equations are applicable to various fields of study, the investigation of the theory of such equations has been started quite recently (see [Ahmad and Nieto 2011], [Ahmad and Sivasundaram 2010], [Bai 2011], [Benchohra and Slimani 2009], [Cao and Chen 2012], [Chauhan and Dabas 2011], [Fečkan, Zhou and Wang 2012], [Kosmatov 2013], [Li, Chen and Li 2013], [Liu 2013], [Mahto, Abbas and Favini 2013], [Mophou 2010], [Rehman and Eloe 2013], [Stamov and Stamova 2014a], [Stamov and Stamova 2015b], [Stamova 2014a], [Stamova 2015], [Tariboon, Ntouyas and Agarwal 2015], [Wang, Ahmad and Zhang 2012], [Wang, Fečkan and Zhou 2011], [Wang and Lin 2014], [Wang, Zhou and Fečkan 2012], [Zhang, Zhang and Zhang 2014]). Also, in relation to the mathematical simulation in chaos, fluid dynamics and many physical systems, the investigation of impulsive fractional functional differential equations began (see [Anguraj and Maheswari 2012], [Chang and Nieto 2009], [Chen, Chen and Wang 2009], [Debbouche and Baleanu 2011], [Gao, Yang and Liu 2013], [Guo and Jiang 2012], [Mahto and Abbas, 2013], [Mahto, Abbas and Favini 2013], [Stamov 2015], [Stamov and Stamova 2014b], [Stamov and Stamova 2015a], [Stamova 2014b], [Stamova and Stamov G 2014b], [Wang 2012], [Xie 2014]).

We shall consider the following impulsive systems of fractional order:

I. *Impulsive systems of fractional differential equations.* Let $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\tau_k : \Omega \to \mathbb{R}$, $I_k : \Omega \to \mathbb{R}^n$, $k = \pm 1, \pm 2, \ldots$. Consider the impulsive fractional-order differential system

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha} x(t) = f(t, x(t)), t \neq \tau_k(x(t)), \\ \Delta x(t) = I_k(x(t)), t = \tau_k(x(t)), k = \pm 1, \pm 2, ..., \end{cases} \tag{1.3}$$

where $\Delta x(t) = x(t^+) - x(t)$.

The second part of the impulsive system (1.3) is called a *jump condition*. The functions I_k, that define the magnitudes of the impulsive perturbations, are the *impulsive functions*.

Let $t_0 \in \mathbb{R}$, $x_0 \in \Omega$. Denote by $x(t) = x(t;t_0,x_0)$ the solution of system (1.3) satisfying the initial condition

$$x(t_0^+;t_0,x_0) = x_0. \tag{1.4}$$

Note that, instead of the initial condition $x(t_0) = x_0$, we have imposed the limiting condition $x(t_0^+) = x_0$ which, in general, is natural for equation (1.3) since (t_0,x_0) may be such that $t_0 = \tau_k(x_0)$ for some k. Whenever $t_0 \neq \tau_k(x_0)$, for all k, we shall understand the initial condition $x(t_0^+) = x_0$ in the usual sense, i.e., $x(t_0) = x_0$.

We suppose that the functions f and I_k, τ_k, $k = \pm 1, \pm 2, \ldots$, are smooth enough on $\mathbb{R} \times \Omega$ and Ω, respectively, to guarantee existence, uniqueness and continuability of the solution $x(t) = x(t;t_0,x_0)$ of the equation (1.3) on the interval $[t_0,\infty)$ for all suitable initial data $x_0 \in \Omega$ and $t_0 \in \mathbb{R}$. We also assume that the functions $(E + I_k) : \Omega \to \Omega$, $k = \pm 1, \pm 2, \ldots$, where E is the identity in Ω.

In some qualitative investigations, we shall also consider such solutions of system (1.3) for which the continuability to the left of t_0 is guaranteed. The solutions $x(t;t_0,x_0)$ of system (1.3) are, in general, piecewise continuous functions with points of discontinuity of the first kind at which they are left continuous, i.e., at the moment t_{l_k} when the integral curve of the solution meets the hypersurfaces

$$\sigma_k = \Big\{(t,x) : \ t = \tau_k(x),\ x \in \Omega\Big\}$$

the following relations are satisfied:

$$x(t_{l_k}^-) = x(t_{l_k}) \ \text{ and } \ x(t_{l_k}^+) = x(t_{l_k}) + I_k(x(t_{l_k})).$$

The points t_{l_k}, at which the impulses occur, are the *moments of impulsive effect*. The impulsive moments for a solution of (1.3) depend on the solution, i.e. different solutions have different points of discontinuity. In general, $k \neq l_k$. In other words, it is possible that the integral curve of the IVP (1.3), (1.4) does not meet the hypersurface σ_k at the moment t_k. This leads to a number of difficulties in the investigation of such systems. We shall assume that for each $x \in \Omega$ and $k = \pm 1, \pm 2, \ldots$,

$$\tau_k(x) < \tau_{k+1}(x)$$

and

$$\tau_k(x) \to \infty \text{ as } k \to \infty\, (\tau_k(x) \to -\infty \text{ as } k \to -\infty), \text{ uniformly on } x \in \Omega,$$

and the integral curve of each solution of the system (1.3) meets each of the hypersurfaces $\{\sigma_k\}$ at most once.