CONTEMPORARY MATHEMATICS

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Lie Algebras, Vertex Operator Algebras, and Related Topics

Conference in Honor of J. Lepowsky and R. Wilson on
Lie Algebras, Vertex Operator Algebras, and Related Topics
August 14–18, 2015
University of Notre Dame, Notre Dame, IN

Katrina Barron
Elizabeth Jurisich
Antun Milas
Kailash Misra
Editors

American Mathematical Society

This volume contains the proceedings of the conference on Lie Algebras, Vertex Operator Algebras, and Related Topics, celebrating the 70th birthday of James Lepowsky and Robert Wilson, held from August 14–18, 2015, at the University of Notre Dame, Notre Dame, Indiana.

Since their seminal work in the 1970s, Lepowsky and Wilson, their collaborators, their students, and those inspired by their work, have developed an amazing body of work intertwining the fields of Lie algebras, vertex algebras, number theory, theoretical physics, quantum groups, the representation theory of finite simple groups, and more. The papers presented here include recent results and descriptions of ongoing research initiatives representing the broad influence and deep connections brought about by the work of Lepowsky and Wilson, and include a contribution by Yi-Zhi Huang summarizing some major open problems in these areas, in particular as they pertain to two-dimensional conformal field theory.

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Preface

This volume contains the proceedings of the international conference "Lie Algebras, Vertex Operator Algebras, and Related Topics", a tribute to James Lepowsky and Robert Wilson, held at the University of Notre Dame, Notre Dame, Indiana, August 14–18, 2015.

Since their seminal work in the 1970s, Lepowsky and Wilson, their collaborators, and their students have developed a transformative body of work intertwining the fields of Lie algebras, vertex algebras, number theory, theoretical physics, quantum groups, the representation theory of finite simple groups, and more. Their research and mentorship in the field have been instrumental in the development of the subject, and their influence continues through their work and the work of their students and collaborators. Jointly and individually they have supervised more than 30 Ph.D. students, several who are now distinguished researchers in their fields.

This volume features one expository article with open problems, and fourteen original research articles, dedicated to Jim and Robert reflecting just a bit of the broad and deep influences they have had, and continue to have in Lie algebras, vertex operator algebras and related areas of research.

We thank everyone who participated in the conference, those who helped plan and run the conference, and those who labored on this volume. The conference was made possible through the generous support of the National Science Foundation (NSF-DMS Conference Grant 1507305), a Participating Institutions Conference Award from the Institute for Mathematics and its Applications at the University of Minnesota, and through the generous support of the Center for Mathematics at the University of Notre Dame.

Without the hard work of the contributors and the referees, as well as the editorial staff of the American Mathematical Society, this volume would not have been feasible. Our thanks go to Christine M. Thivierge for her constant help and patience. We also thank the technical staff for allowing us to include a group picture of the participants.

We greatly appreciate the staff of the Mathematics Department and the Center for Mathematics at the University of Notre Dame for their help during the conference, in particular Lisa Driver, for her efficient and dedicated work on logistics.

We are grateful to Haisheng Li for helping us at the initial stage of the conference. Finally, we express our great appreciation to Yi-Zhi Huang—for providing invaluable help with the preparation of the NSF proposal, contributing to the proceedings, advising us on many aspects of the conference, and initiating the conference itself.

Katrina Barron, Elizabeth Jurisich
Antun Milas, Kailash Misra

Conference Participants

Photography by Zane and Aurora Exner.

Contents

Preface v

Conference Participants vi

Generalizations of Q-systems and orthogonal polynomials from representation theory
D. Addabbo and M. Bergvelt 1

Some applications and constructions of intertwining operators in logarithmic conformal field theory
D. Adamović and A.Milas 15

Kac–Moody groups and automorphic forms in low dimensional supergravity theories
L. Bao and L. Carbone 29

The Lusztig-Macdonald-Wall polynomial conjectures and q-difference equations
K. Bringmann, K. Mahlburg, and A. Milas 41

Uniqueness of representation–theoretic hyperbolic Kac–Moody groups over $\mathbb{Z}$
L. Carbone and F. Wagner 51

Coends in conformal field theory
J. Fuchs and C. Schweigert 65

Remarks on ϕ-coordinated modules for quantum vertex algebras
H. Li 83

The classification of chiral WZW models by $H^4_+(BG, \mathbb{Z})$
A. Henriques 99

Some open problems in mathematical two-dimensional conformal field theory
Y.-Z. Huang 123

On realization of some twisted toroidal Lie algebras
N. Jing, C. Mangum, and K. Misra 139

Twisted generating functions incorporating singular vectors in Verma modules and their localizations, I
J. Lepowsky and J. Yang 149

Characterization of the simple Virasoro vertex operator algebras with 2 and 3-dimensional space of characters
Y. Arike, K. Nagatomo, and Y. Sakai 175

Quasiconformal Teichmüller theory as an analytical foundation for two-dimensional conformal field theory
D. RADNELL, E. SCHIPPERS, and W. STAUBACH 205

Centralizing the centralizers
A. M. SEMIKHATOV 239

On Neeman's Gradient Flows
N. WALLACH 261

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Generalizations of Q-systems and orthogonal polynomials from representation theory

Darlayne Addabbo and Maarten Bergvelt

ABSTRACT. We briefly describe what tau-functions in integrable systems are. We then define a collection of tau-functions given as matrix elements for the action of $\widehat{GL_2}$ on two-component Fermionic Fock space. These tau-functions are solutions to a discrete integrable system called a Q-system.

We can prove that our tau-functions satisfy Q-system relations by applying the famous "Desnanot-Jacobi identity" or by using "connection matrices", the latter of which gives rise to orthogonal polynomials. In this paper, we will provide the background information required for computing these tau-functions and obtaining the connection matrices and will then use the connection matrices to derive our difference relations and to find orthogonal polynomials.

We generalize the above by considering tau-functions that are matrix elements for the action of $\widehat{GL_3}$ on three-component Fermionic Fock space, and discuss the new system of discrete equations that they satisfy. We will show how to use the connection matrices in this case to obtain "multiple orthogonal polynomials of type II".

1. Introduction

Integrable differential equations, such as the KdV equation,

$$u_t + u_{xxx} + 6uu_x = 0, \tag{1.1}$$

can be solved exactly by employing a change of variables to rewrite the equations more simply in bilinear form. In the case of the KdV equation, this change of variables is given by [**6**]

$$u = 2(\ln \tau)_{xx}. \tag{1.2}$$

This method of changing variables is called Hirota's method and the solutions of these differential equations under the change of variables are referred to as "tau-functions" (For more details on using Hirota's method to find solutions to the KdV equation, as well as many other examples, see [**6**].)

Interestingly, tau-functions are often equal to matrix elements for representations of infinite dimensional Lie groups (see, for example, [**12**] and [**8**]).

In this paper, we will discuss tau-functions that satisfy discrete integrable equations. We will first define tau-functions that are given as matrix elements for the

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action of $\widehat{GL_2}$ on two-component Fermionic Fock space and will discuss how to show that these tau-functions satisfy a Q-system.

More specifically, we will see that our $\widehat{GL_2}$ tau-functions satisfy

$$\tau_k^{(\alpha)}\tau_{k-2}^{(\alpha+2)} = \tau_{k-1}^{(\alpha+2)}\tau_{k-1}^{(\alpha)} - (\tau_{k-1}^{(\alpha+1)})^2, \tag{1.3}$$

for $k \geq 0$ and $\alpha \in \mathbb{Z}$. By applying a suitable change of variables, this can be shown to be equivalent to the defining relations for the $A_{\infty/2}$ Q-system which is discussed, for example, in [**4**]. These difference relations are found using "connection matrices" (defined below) and these connection matrices can also be used to obtain orthogonal polynomials.

Q-systems are discrete integrable systems that appear in various places in mathematics, for example, as the relations satisfied by characters of Kirillov-Reshetikhin modules (see [**10**], [**11**]) or as mutations in a cluster algebra (see [**9**], [**3**]).

Since Q-systems and orthogonal polynomials are already interesting, it is natural to ask what sort of discrete relations are satisfied by analogous tau-functions, given as matrix elements for the action of $\widehat{GL_3}$ on three-component Fermionic Fock space and what sort of orthogonal polynomials come from the corresponding connection matrices. In the following, we will describe how to define these new $\widehat{GL_3}$ tau-functions and how to use connection matrices to show that they satisfy the following system of equations, for all $k, \ell \geq 0$ and $\alpha, \beta \in \mathbb{Z}$,

$$(\tau_{k,\ell}^{(\alpha+1,\beta)})^2 = \tau_{k,\ell}^{(\alpha,\beta)}\tau_{k,\ell}^{(\alpha+2,\beta)} + \tau_{k+1,\ell+1}^{(\alpha,\beta)}\tau_{k-1,\ell-1}^{(\alpha+2,\beta)} - \tau_{k+1,\ell}^{(\alpha,\beta)}\tau_{k-1,\ell}^{(\alpha+2,\beta)} \tag{1}$$

$$\tau_{k,\ell}^{(\alpha+1,\beta)}\tau_{k,\ell-1}^{(\alpha+2,\beta)} = -\tau_{k-1,\ell-1}^{(\alpha+2,\beta)}\tau_{k+1,\ell}^{(\alpha+1,\beta)} + \tau_{k,\ell-1}^{(\alpha+1,\beta)}\tau_{k,\ell}^{(\alpha+2,\beta)} \tag{2}$$

$$(\tau_{k,\ell}^{(\alpha,\beta+1)})^2 = \tau_{k,\ell}^{(\alpha,\beta)}\tau_{k,\ell}^{(\alpha,\beta+2)} - \tau_{k,\ell-1}^{(\alpha,\beta+2)}\tau_{k,\ell+1}^{(\alpha,\beta)} - \tau_{k+1,\ell}^{(\alpha,\beta+2)}\tau_{k-1,\ell}^{(\alpha,\beta)} \tag{3}$$

$$\tau_{k-1,\ell}^{(\alpha,\beta+1)}\tau_{k,\ell+1}^{(\alpha,\beta)} = \tau_{k-1,\ell}^{(\alpha,\beta)}\tau_{k,\ell+1}^{(\alpha,\beta+1)} - \tau_{k-1,\ell+1}^{(\alpha,\beta)}\tau_{k,\ell}^{(\alpha,\beta+1)}. \tag{4}$$

We hope, similarly to Q-systems, that our new system of equations will also have connections to other areas of mathematics. We will briefly discuss progress we have made in analyzing this new system of equations.

Applying restrictions to the connection matrices in the $\widehat{GL_3}$ case, we find an analogous collection of orthogonal polynomials, which we will discuss. In our future work, we hope to investigate more general situations, obtained by dropping these restrictions.

(See [**1**] for more details on the computations of our tau-functions and the difference relations that they satisfy. Orthogonal polynomials, however are not discussed there.)

2. Calculating $\widehat{GL_2}$ Tau-Functions on Two-Component Fermionic Fock Space

Before we define our $\widehat{GL_2}$ tau-functions, we first define two-component Fermionic Fock space, $F^{(2)}$ and describe the action of $\widehat{GL_2}$ on this space. Here, we will omit most technical details. For more information, we refer the reader to [**15**] and [**1**]. In particular, all omitted details of the following background on Fermionic Fock space and the associated action of $\widehat{gl_2}$ can be found in [**1**].

Consider the vector space $H^{(2)} := \mathbb{C}^2 \otimes \mathbb{C}[z, z^{-1}]$. A basis of this space is given by elements, $e_a z^k$, $a = 0, 1$, $k \in \mathbb{Z}$, where $e_0 = \begin{pmatrix}1\\0\end{pmatrix}$ and $e_1 = \begin{pmatrix}0\\1\end{pmatrix}$. $F^{(2)}$ is then

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spanned by vectors,

$$w = w_0 \wedge w_1 \wedge w_2 \wedge \cdots,$$

where $w_i \in H^{(2)}$ and the w satisfy some restrictions that we will now discuss.

Let the vacuum vector be

$$v_0 := \begin{pmatrix}1\\0\end{pmatrix} \wedge \begin{pmatrix}0\\1\end{pmatrix} \wedge \begin{pmatrix}z\\0\end{pmatrix} \wedge \begin{pmatrix}0\\z\end{pmatrix} \wedge \cdots \in F^{(2)},$$

and define operators, $e(e_a z^k)$ and $i(e_a z^k)$ (called exterior and interior product operators, respectively), given by $e(e_a z^k)w = e_a z^k \wedge w$ and $i(e_a z^k)w = \beta$ if $w = e_a z^k \wedge \beta$. $F^{(2)}$ is the span of the vectors obtained by acting on v_0 by finitely many exterior and interior product operators. We can specify an order in which to act by these exterior and interior product operators and define "elementary wedges" as those wedges obtained by acting on v_0 by monomials of exterior and interior product operators, subject to this order. For more information, see [**15**] and [**1**]. The elementary wedges are defined in such a way that there exists a unique bilinear form on $F^{(2)}$, denoted $\langle v, w\rangle$ for $v, w \in F^{(2)}$, for which the elementary wedges are an orthonormal basis.

It is useful to introduce generating series, called fermion fields, for the exterior and interior product operators. Define

$$\psi_a^{\pm}(w) = \sum_{k\in\mathbb{Z}} {}_a\psi^{\pm}_{(k)} w^{-k-1}, \quad a = 0, 1,$$

where

$${}_a\psi^{+}_{(k)} = e(e_a z^k) \text{ and } {}_a\psi^{-}_{(k)} = i(e_a z^{-k-1}).$$

We can use fermion fields to express the action of $\widehat{gl_2} = gl_2 \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c$ on $F^{(2)}$. Let $E_{ab} \in gl_2$ $(a, b = 0, 1)$ be the matrices such that $E_{ab}e_c = \delta_{bc}e_a$ and let the current, $E_{ab}(w) = \sum_{k\in\mathbb{Z}} E_{ab} z^k w^{-k-1}$, be the generating series of elements in $\widehat{gl_2}$. When $a \neq b$, the series acts on $F^{(2)}$ by

$$E_{ab}(w) = \psi_a^{+}(w)\psi_b^{-}(w). \tag{2.1}$$

The action of $E_{ab}(w)$ in general requires using the normal ordered product, but we do not discuss this here since it is not needed in this paper (More details can be found in [**1**].).

In addition to the action of the Lie algebra, $\widehat{gl_2}$, $F^{(2)}$ also carries an action of the group $\widehat{GL_2}$, a central extension of the loop group, $\widetilde{GL_2}$. In particular, on $F^{(2)}$ we have the action of "fermionic translation operators" Q_0, Q_1 such that

$$\pi(Q_0) = \begin{bmatrix} z^{-1} & 0 \\ 0 & -1 \end{bmatrix}, \quad \pi(Q_1) = \begin{bmatrix} -1 & 0 \\ 0 & z^{-1} \end{bmatrix}$$

where π denotes the projection from $\widehat{GL_2}$ to $\widetilde{GL_2}$. We also define $T = Q_1 Q_0^{-1}$ such that

$$\pi(T) = -\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

Let $g_C \in \widehat{GL_2}$ be such that

$$\pi(g_C) = \begin{bmatrix} 1 & 0 \\ C(z) & 1 \end{bmatrix},$$

where $C(z) = \sum_{i\in\mathbb{Z}} \frac{c_i}{z^{i+1}}$, $c_i \in \mathbb{C}$. (Actually, for our purposes, it will sometimes be useful to take the c_is to be formal variables.)

We define our ("unshifted") tau-functions to be

$$\tau_k = \langle T^k v_0, g_C \cdot v_0 \rangle.$$

We also need to define "shifted" tau-functions, corresponding to the action of $g_C^{(\alpha)}$ on the vacuum vector, where

$$g_C^{(\alpha)} = Q_0^\alpha g_C Q_0^{-\alpha}, \quad \alpha \in \mathbb{Z}.$$

Denote by $C^{(\alpha)}$, the series $C^{(\alpha)}(z) = (-1)^\alpha \sum_{i\in\mathbb{Z}} \frac{c_{i+\alpha}}{z^{i+1}}$, so that $\pi(g_C^{(\alpha)}) = \begin{bmatrix} 1 & 0 \\ C^{(\alpha)}(z) & 1 \end{bmatrix}$.

We can calculate these tau-functions by noting that

$$g_C^{(\alpha)} = \exp \begin{bmatrix} 0 & 0 \\ C^{(\alpha)}(z) & 0 \end{bmatrix} = \exp(\mathrm{Res}_w(C^{(\alpha)}(w)E_{10}(w))),$$

where the action of the current, $E_{10}(w)$ on $F^{(2)}$, is given by (2.1). We then have the following formulas for our tau-functions, which are stated and proven in [**1**]:

THEOREM 2.1. (1) $\tau_k^{(\alpha)} = 0$ *for* $k < 0$.
(2) $\tau_0^{(\alpha)} = 1$.
(3) *When* $k > 0$,

$$\tau_k^{(\alpha)} = \frac{1}{k!} \mathrm{Res}_{\mathbf{w}}\Big(\prod_{1\le i<j\le k} (w_i - w_j)^2 \prod_{i=1}^{k} C^{(\alpha)}(w_i)\Big),$$

where $\mathrm{Res}_{\mathbf{w}}$ *denotes* $\mathrm{Res}_{w_1}(\mathrm{Res}_{w_2} \cdots (\mathrm{Res}_{w_k} \cdots) \cdots)$.
Alternatively, when $k > 0$ *we can write*

$$\tau_k^{(\alpha)} = (-1)^{k\alpha} \det \begin{bmatrix} c_\alpha & c_{\alpha+1} & \cdots & c_{\alpha+k-1} \\ c_{\alpha+1} & c_{\alpha+2} & \cdots & c_{\alpha+k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{\alpha+k-1} & c_{\alpha+k} & \cdots & c_{\alpha+2k-2} \end{bmatrix}.$$

We note in particular that these tau-functions are determinants of Hankel matrices and are thus especially well suited to applying the famous Desnanot-Jacobi identity [**2**]. We obtain

$$\tau_k^{(\alpha)} \tau_{k-2}^{(\alpha+2)} = \tau_{k-1}^{(\alpha+2)} \tau_{k-1}^{(\alpha)} - (\tau_{k-1}^{(\alpha+1)})^2, \tag{2.2}$$

for $k \ge 0$ and $\alpha \in \mathbb{Z}$.

3. Generalizing to $\widehat{GL_3}$

We now generalize the above to the $\widehat{GL_3}$ case. Here, in the same way that we did in the $\widehat{GL_2}$ case, we first define the action of $\widehat{gl_3}$ on three-component Fermionic Fock space, $F^{(3)}$. We define $H^{(3)} := \mathbb{C}^3 \otimes \mathbb{C}[z, z^{-1}]$, which has a basis given by $e_a z^k$, where $a = 0, 1, 2$, $k \in \mathbb{Z}$, and $e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The vacuum vector is

$$v_0 = \begin{pmatrix}1\\0\\0\end{pmatrix} \wedge \begin{pmatrix}0\\1\\0\end{pmatrix} \wedge \begin{pmatrix}0\\0\\1\end{pmatrix} \wedge \begin{pmatrix}z\\0\\0\end{pmatrix} \wedge \begin{pmatrix}0\\z\\0\end{pmatrix} \wedge \begin{pmatrix}0\\0\\z\end{pmatrix} \wedge \cdots \in F^{(3)}.$$

Similarly to the $\widehat{GL_2}$ case, in addition to the action of $\widehat{gl_3}$ on $F^{(3)}$, we have an associated action of $\widetilde{GL_3}$, a central extension of the loop group, $\widetilde{GL_3}$. We have as before, a projection map, π from $\widetilde{GL_3}$ to $\widetilde{Gl_3}$. We define fermionic translation operators, Q_0, Q_1, and Q_2 such that

$$\pi(Q_0) = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \pi(Q_1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \pi(Q_2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}.$$

We also need operators, $T_1 = Q_1 Q_0^{-1}$ and $T_2 = Q_2 Q_1^{-1}$ such that

$$\pi(T_1) = \begin{bmatrix} -z & 0 & 0 \\ 0 & -z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \quad \pi(T_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & -z^{-1} \end{bmatrix}.$$

The fermion fields and action of $\widehat{gl_3}$ are defined in a way completely analogous to how they were defined in our discussion of the $\widehat{GL_2}$ case. As before, we omit any discussion of the normal ordered product, since the details are not neccesary for this paper and can be found in [**1**].

We take a loop group element, $g_{C,D,E} \in \widehat{Gl_3}$, such that

$$\pi(g_{C,D,E}) = \begin{bmatrix} 1 & 0 & 0 \\ C(z) & 1 & 0 \\ D(z) & E(z) & 1 \end{bmatrix},$$

where $C(z) = \sum_{i\in\mathbb{Z}} \frac{c_i}{z^{i+1}}$, $D(z) = \sum_{i\in\mathbb{Z}} \frac{d_i}{z^{i+1}}$, and $E(z) = \sum_{i\in\mathbb{Z}} \frac{c_i}{z^{i+1}}$ and the c_is, d_is, and e_is are complex numbers or formal variables.

We define

$$\tau_{k,\ell} = \langle g_{C,D,E} \cdot v_0, T_1^k T_2^\ell v_0 \rangle,$$

and calculate the action of $g_{C,D,E}$ in the same way that we calculated the action of our previous group element on the two-component Fermionic Fock space, by expressing $g_{C,D,E}$ in terms of fermion fields.

In order to obtain our difference relations we must, as in the $\widehat{GL_2}$ case, introduce shifted tau-functions. Here, we have two independent shifts. We define the $\tau_{k,\ell}^{(\alpha,\beta)}$s to be the tau-functions corresponding to the action of the group element,

$$g_{C,D,E}^{(\alpha,\beta)} = Q_0^{-\alpha} Q_1^{-\beta} g_{C,D,E} Q_1^{\beta} Q_0^{\alpha}.$$

We comment that we do not need Q_2 to obtain all possible shifts since

$$Q_2 g_{C,D,E} Q_2^{-1} = Q_0^{-1} Q_1^{-1} g_{C,D,E} Q_1 Q_0,$$

so there really are only two independent shifts, as claimed.

The formula for our $\tau_{k,\ell}^{(\alpha,\beta)}$ functions is then: (The following is stated and proven in [**1**]).

THEOREM 3.1.

$$\tau_{k,\ell}^{(\alpha,\beta)} = \sum_{n_c+n_d=k, n_e+n_d=\ell} c_{n_c,n_d,n_e}$$

where

$$c_{n_c,n_d,n_e} = \frac{1}{n_c!n_d!n_e!} \operatorname{Res}_{\mathbf{x}} \operatorname{Res}_{\mathbf{y}} \operatorname{Res}_{\mathbf{z}} \left(\prod_{i=1}^{n_c} C^{(\alpha-\beta)}(x_i) \prod_{i=1}^{n_d} D^{(\alpha)}(y_i) \prod_{i=1}^{n_e} E^{(\beta)}(z_i) p_{n_c,n_d,n_e} \right)$$

where we use the same notation for residues as we did in the $\widehat{GL_2}$ case and

$$p_{n_c,n_d,n_e} = (-1)^{\frac{n_d(n_d+1)}{2}} \times$$

$$\frac{\prod_{1\le i<j\le n_c}(x_i-x_j)^2 \prod_{1\le i<j\le n_d}(y_i-y_j)^2 \prod_{1\le i<j\le n_e}(z_i-z_j)^2 \prod_{i=1}^{n_c}\prod_{j=1}^{n_d}(x_i-y_j) \prod_{i=1}^{n_d}\prod_{j=1}^{n_e}(y_i-z_j)}{\prod_{i=1}^{n_c}\prod_{j=1}^{n_e}(x_i-\underline{z_j})}.$$

Here, the underline means that we expand $\frac{1}{x_i-\underline{z_j}}$ in positive powers of z_j. We comment that the formula given here is analogous to the formula for our $\widehat{GL_2}$ tau-functions, but is much more complicated. In particular, the denomenator appearing in our formula here means that our tau-functions are, in general, infinite series of monomials in the c_is, d_is, and e_is.

4. Connection Matrices and Zero Curvature Equations

Although the Desnanot-Jacobi identity easily yields difference relations for our $\widehat{GL_2}$ tau-functions, it is unclear how one might use it to find difference relations for our $\widehat{GL_3}$ tau-functions. To find difference relations for our $\widehat{GL_3}$ tau-functions, we use "connection matrices" and show that certain zero curvature equations are satisfied. For this, we need the Birkhoff factorization of $T_2^{-\ell}T_1^{-k}g_{C,D,E}^{(\alpha,\beta)}$, which exists when $\tau_{k,\ell}^{(\alpha,\beta)} \neq 0$ ([**13**]). Here, we consider our c_is, d_is, and e_is to be formal variables and assume that the Birkhoff factorization exists for all $T_2^{-\ell}T_1^{-k}g_{C,D,E}^{(\alpha,\beta)}$ where k and ℓ are both nonnegative and $\alpha, \beta \in \mathbb{Z}$.

4.1. $\widehat{GL_2}$ Case. In this section, we will use zero curvature equations ([**5**]) to rederive our difference relations for the $\widehat{GL_2}$ case. The $\widehat{GL_3}$ case is similar, the details of which are included in [**1**].

For the $\widehat{GL_2}$ case, we have a factorization

$$T^{-k}g_C^{(\alpha)} = g_-^{[k](\alpha)}g_+^{[k](\alpha)},$$

where $g_-^{[k](\alpha)}$ is such that $\pi(g_-^{[k](\alpha)}) = I+$ terms involving only negative powers of z and $g_+^{[k](\alpha)}$ is such that $\pi(g_+^{[k](\alpha)}) = A_k^{(\alpha)}+$ terms involving only positive powers of z. Here, I denotes the identity matrix and $A_k^{(\alpha)}$ denotes a matrix that is independent of both z and z^{-1}. This factorization of $T^{-k}g_C^{(\alpha)}$ is called the Birkhoff factorization ([**13**]). From now on, we will work in the non-centrally extended loop group, i.e. in $\widetilde{GL_2}$, but omit π in our notation.

We define matrix Baker functions, $\Psi^{[k](\alpha)}$ by

$$\Psi^{[k](\alpha)} = T^k g_-^{[k](\alpha)}.$$

We then have connection matrices, $U_k^{(\alpha)}$, between these matrix Baker functions, given by

$$U_k^{(\alpha)} = (\Psi^{[k](\alpha)})^{-1}\Psi^{[k+1](\alpha)}.$$

The entries of these connection matrices have expressions in terms of the tau-functions. (Another fact about the connections matrices, which will be particularly important in our discussion of orthogonal polynomials in the next section, is that they are nonnegative in z, i.e. none of their entries have negative powers of z. [**1**])

We obtain the difference relations by factoring our connection matrices,

$$U_k^{(\alpha)} = V_k^{(\alpha)}(W_k^{(\alpha)})^{-1} = (W_k^{(\alpha-1)})^{-1}V_{k+1}^{(\alpha-1)},$$

where

$$V_k^{(\alpha)} = (g_-^{[k](\alpha)})^{-1}Q_0^{-1}g_-^{[k](\alpha+1)}$$

and

$$W_k^{(\alpha)} = (g_-^{[k+1](\alpha))})^{-1}Q_1^{-1}g_-^{[k](\alpha+1)}.$$

The equality of these two different expressions for the connection matrices are called "zero curvature equations" and they give us equalities satisfied by the tau-functions. These equalities imply the difference relations previously found using the Desnanot-Jacobi identity.

More explicitly, we have the following lemma, the proof of which is included in [**1**]:

LEMMA 4.1. [**1**]

$$V_k^{(\alpha)} = \begin{bmatrix} z + \frac{\tau_{k-1}^{(\alpha+1)}\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha+1)}\tau_k^{(\alpha)}} & \frac{\tau_{k-1}^{(\alpha+1)}}{\tau_k^{(\alpha+1)}} \\ -\frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} & -1 \end{bmatrix}, \quad W_k^{(\alpha)} = \begin{bmatrix} -1 & -\frac{\tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha)}} \\ \frac{\tau_{k+1}^{(\alpha+1)}}{\tau_k^{(\alpha+1)}} & z + \frac{\tau_k^{(\alpha)}\tau_{k+1}^{(\alpha+1)}}{\tau_{k+1}^{(\alpha)}\tau_k^{(\alpha+1)}} \end{bmatrix}.$$

Since $V_k^{(\alpha)}$ and $W_k^{(\alpha)}$ satisfy,

$$U_k^{(\alpha)} = V_k^{(\alpha)}(W_k^{(\alpha)})^{-1} = (W_k^{(\alpha-1)})^{-1}V_{k+1}^{(\alpha-1)},$$

we find that

$$U_k^{(\alpha)} = \begin{bmatrix} -z - \frac{\tau_k^{(\alpha)}\tau_{k+1}^{(\alpha+1)}}{\tau_{k+1}^{(\alpha)}\tau_k^{(\alpha+1)}} - \frac{\tau_{k-1}^{(\alpha+1)}\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha+1)}-\tau_k^{(\alpha)}} & -\frac{\tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha)}} \\ \frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} & 0 \end{bmatrix} = \begin{bmatrix} -z - \frac{\tau_k^{(\alpha)}\tau_{k+2}^{(\alpha-1)}}{\tau_{k+1}^{(\alpha)}\tau_{k+1}^{(\alpha-1)}} - \frac{\tau_k^{(\alpha-1)}\tau_{k+1}^{(\alpha)}}{\tau_{k+1}^{(\alpha-1)}\tau_k^{(\alpha)}} & -\frac{\tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha)}} \\ \frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} & 0 \end{bmatrix}.$$

This implies

$$(\tau_k^{(\alpha)})^2(\tau_{k+2}^{(\alpha-1)}\tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)}\tau_{k+1}^{(\alpha+1)}) = (\tau_{k+1}^{(\alpha)})^2(\tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)} - \tau_k^{(\alpha-1)}\tau_k^{(\alpha+1)}).$$

We notice that, if

$$\tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)} = \tau_k^{(\alpha+1)}\tau_k^{(\alpha-1)} - (\tau_k^{(\alpha)})^2$$

holds for some k, the above identity implies that it holds for $k+1$. Since $\tau_{-1}^{(\alpha)} = 0$ and $\tau_0^{(\alpha-1)} = 1$, $\tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)} = \tau_k^{(\alpha+1)}\tau_k^{(\alpha-1)} - (\tau_k^{(\alpha)})^2$ holds trivially for $k = 0$, and hence

$$\tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)} = \tau_k^{(\alpha+1)}\tau_k^{(\alpha-1)} - (\tau_k^{(\alpha)})^2$$

holds for all $k \geq 0$, which is exactly what we previously obtained using the Desnanot-Jacobi identity (1.3).

4.2. $\widehat{GL_3}$ **Case.** The matrix Baker functions in the $\widehat{GL_3}$ case are given in a completely analogous way to the $\widehat{GL_2}$ case. Here,

$$\Psi^{[k,\ell](\alpha,\beta)} = T_1^k T_2^\ell g_-^{[k,\ell](\alpha,\beta)},$$

where $g_-^{[k,\ell](\alpha,\beta)}$ is the part of the Birkhoff factorization of $T_2^{-\ell} T_1^{-k} g_{C,D,E}^{(\alpha,\beta)}$ that is a lift of I plus a 3×3 matrix whose entries have only negative powers of z. (As before, I denotes the identity matrix.)

We then have two sets of connection matrices, $U_{[k_+,\ell]}^{(\alpha,\beta)}$ and $U_{[k,\ell_+]}^{(\alpha,\beta)}$, that we factor in two different ways to obtain our zero curvature equations.

$$U_{[k_+,\ell]}^{(\alpha,\beta)} = (\Psi^{[k,\ell](\alpha,\beta)})^{-1} \Psi^{[k+1,\ell](\alpha,\beta)},$$

$$U_{[k,\ell_+]}^{(\alpha,\beta)} = (\Psi^{[k,\ell](\alpha,\beta)})^{-1} \Psi^{[k,\ell+1](\alpha,\beta)}.$$

Factoring these connection matrices and performing calculations similar to those above, we find that our $\widehat{GL_3}$ tau-functions satisfy the system of four difference equations listed in the introduction. Under suitable changes of variables, equations (2) and (4) can be shown to be T-system relations. T-systems are a sort of generalization of Q-systems which have interpretations, for example in terms of perfect matchings of graphs ([**14**]). Like Q-system relations, T-system relations can also be seen as cluster algebra mutations ([**3**]). We comment that equations (2) and (4) are independent of each other ((2) depends only on the k, ℓ, α parameters and equation (4) depends on the k, ℓ, β parameters,), so our tau-functions satisfy T-system relations in two different ways. We believe that the remaining two relations, (1) and (3), are independent of each other and are not implied by the T-system relations. We are currently working to find other situations in which equations (1) and (3) appear.

5. Orthogonal Polynomials from Connection Matrices

5.1. $\widehat{GL_2}$ **Case.** In the $\widehat{GL_2}$ case, we can use the fact that the connection matrices are nonnegative in z to obtain orthogonal polynomials. In the following, we will explain how to do this. (For more on orthogonal polynomials see, for example, [**7**]).

DEFINITION 5.1. [**1**] Define $S : \mathbb{C}[c_k]_{k\in\mathbb{Z}} \to \mathbb{C}[c_k]_{k\in\mathbb{Z}}$ to be the multiplicative map such that $S(1) = 0$ and $S(c_k) = c_{k+1}$ for all k. The shift fields, $S^\pm(z)$, are then given by $S^\pm(z) = (1 - \frac{S}{z})^\pm$, which also act multiplicatively.

EXAMPLE 5.2.

$$S^+(z)\tau_2^{(\alpha)} = S^+(z) \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+1} & c_{\alpha+2} \end{bmatrix} = (c_\alpha - c_{\alpha+1}/z)(c_{\alpha+2} - c_{\alpha+3}/z) - (c_{\alpha+1} - c_{\alpha+2}/z)^2 =$$

$$= \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+1} & c_{\alpha+2} \end{bmatrix} - \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+2} & c_{\alpha+3} \end{bmatrix} /z + \det \begin{bmatrix} c_{\alpha+1} & c_{\alpha+2} \\ c_{\alpha+2} & c_{\alpha+3} \end{bmatrix} /z^2 =$$

$$= \tau_2^{(\alpha)} - \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+2} & c_{\alpha+3} \end{bmatrix} /z + \tau_2^{(\alpha+1)}/z^2$$

We note that the positive shift field, $S^+(z)$ sends elements of $\mathbb{C}[c_k]_{k\in\mathbb{Z}}$ to polynomials in z^{-1} with coeffients in $\mathbb{C}[c_k]_{k\in\mathbb{Z}}$. In particular, as is illustrated in the

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