ELEMENTARY DIFFERENTIAL TOPOLOGY

BY

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PREFACE

Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under differentiable homeomorphisms. Problems in this field arise from the interplay between the topological, combinatorial, and differentiable structures of a manifold. They do not, however, involve such notions as connections, geodesics, curvature, and the like; in this way the subject may be distinguished from differential geometry.

One particular flowering of the subject took place in the 1930's, with work of H. Whitney, S. S. Cairns, and J. H. C. Whitehead. A second flowering has come more recently, with the exciting work of J. Milnor, R. Thom, S. Smale, M. Kervaire, and others. The later work depends on the earlier, of course, but differs from it in many ways, most particularly in the extent to which it uses the results and methods of algebraic topology. The earlier work is more exclusively geometric in nature, and is thus in some sense more elementary.

One may make an analogy with the discipline of Number Theory, in which a theorem is called <u>elementary</u> if its proof involves no use of the theory of functions of a complex variable — otherwise the proof is said to be <u>non-elementary</u>. As one is well aware, the terminology does not reflect the difficulty of the proof in question, the elementary proofs often being harder than the others.

It is in a similar sense that we speak of the elementary part of differential topology. This is the subject of the present set of notes.

Since our theorems and proofs (with one small exception) will involve no algebraic topology, the background we expect of the reader consists of a working knowledge of: the calculus of functions of several variables and the associated linear algebra, point-set topology, and, for Chapter II, the geometry (not the algebra) of simplicial complexes. Apart from these topics, the present notes endeavor to be self-contained.

The reader will not find them especially elegant, however. We are not hoping to write anything like the definitive work, even on the most elementary aspects of the subject. Rather our hope is to provide a set of notes from which the student may acquire a feeling for differential topology, at least in its geometric aspects. For this purpose, it is necessary that the student work diligently through the exercises and problems scattered throughout the notes; they were chosen with this object in mind.

The word problem is used to label an exercise for which either the result itself, or the proof, is of particular interest or difficulty. Even the best student will find some challenges in the set of problems. Those problems and exercises which are not essential to the logical continuity of the subject are marked with an asterisk.

A second object of these notes is to provide, in more accessible form than heretofore, proofs of a few of the basic often-used-but-seldom-proved facts about differentiable manifolds. Treated in the first chapter are the body of theorems which state, roughly speaking, that any result which holds for manifolds and maps which are infinitely differentiable holds also if lesser degrees of differentiability are assumed. Proofs of these theorems have been part of the "folk-literature" for some time; only recently has anyone written them down. ([8] and [9].) (The stronger theorems of Whitney, concerning analytic manifolds, require quite different proofs, which appear in his classical paper [15].)

In a sense these results are negative, for they declare that nothing really interesting occurs between manifolds of class C^1 and those of class C^∞. However, they are still worth proving, at least partly for the techniques involved.

The second chapter is devoted to proving the existence and uniqueness of a smooth triangulation of a differentiable manifold. In this, we follow J. H. C. Whitehead [14], with some modifications. The result itself is one of the most useful tools of differential topology, while the techniques involved are essential to anyone studying both combinatorial and differentiable structures on a manifold. The reader whose primary interest is in triangulations may omit §4, §5, and §6 with little loss of continuity.

We have made a conscious effort to avoid any more overlap with the lectures on differential topology [4] given by Milnor at Princeton in 1958 than was necessary. It is for this reason that we omit a proof of Whitney's imbedding theorem, contenting ourselves with a weaker one. We hope the reader will find our notes and Milnor's to be useful supplements to each other.

CONTENTS

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CHAPTER I.

DIFFERENTIABLE MANIFOLDS

§1. Introduction.

This section is devoted to defining such basic concepts as those of differentiable manifold, differentiable map, immersion, imbedding, and diffeomorphism, and to proving the implicit function theorem.

We consider the euclidean space R^m as the space of all infinite sequences of real numbers, $x = (x^1, x^2, \ldots)$, such that $x^i = 0$ for $i > m$; euclidean half-space H^m is the subset of R^m for which $x^m \geq 0$. Then $R^{m-1} \subset H^m \subset R^m$. We denote $\sqrt{((x^1)^2 + \ldots + (x^m)^2)}$ by $\|x\|$, and $\max |x^i|$, by $|x|$. The unit sphere S^{m-1} is the subset of R^m with $\|x\| = 1$; the unit ball B^m, the set with $\|x\| \leq 1$; and the r-cube $C^m(r)$ is the set with $|x| \leq r$. Often, we also consider R^m as simply the space of all m-tuples $(x^1, \ldots, x^m)$, where no confusion will arise.

1.1 Definition. A (topological) manifold M is a Hausdorff space with a countable basis, satisfying the following condition: There is an integer m such that each point of M has a neighborhood homeomorphic with an open subset of H^m or of R^m.

If $h : U \to H^m$ (or R^m) is a homeomorphism of the neighborhood U of x with an open set in H^m or R^m, the pair (U,h) is often called a coordinate neighborhood on M. If h(U) is open in H^m and h(x) lies in R^{m-1}, then x is called a boundary point of M, and the set of all such points is called the boundary of M, denoted by Bd M. If Bd M is empty, we say M is non-bounded. (In the literature, the word manifold is commonly used only when Bd M is empty; the more inclusive term then is manifold-with-boundary.) The set $M - \mathrm{Bd}\, M$ is called the interior of M, and is denoted by Int M. (If A is a subset of the topological space X, we also use Int A to mean $X - \mathrm{C}\ell(X-A)$, but this should cause no confusion.)

To justify these definitions, we must note that if $h_1 : U_1 \to H^m$ and $h_2 : U_2 \to H^m$ are homeomorphisms of neighborhoods of x with open sets in H^m, and if $h_1(x)$ lies in R^{m-1}, so does $h_2(x)$: For otherwise, the map $h_1 h_2^{-1}$ would give a homeomorphism of an open set in R^m with a neighborhood of the point $p = h_1(x)$ in H^m. The latter neighborhood is certainly not open in R^m, contradicting the Brouwer theorem on invariance of domain [3, p. 95].

One may also verify that the number m is uniquely determined by M; it is called the dimension of M, and M is called an m-manifold. This may be done either by using the Brouwer theorem on invariance of domain, or by applying the theorem of dimension theory which states that the topological dimension of M is m [3, p. 46]. Strictly speaking, to apply the latter theorem we need to know that M is a separable metrizable space; but this follows from a standard metrization theorem of point-set topology [2, p. 75].

It also follows from a standard theorem that M is paracompact [2, p. 79]. We remind the reader that this means that for any open covering $\mathfrak{A}$ of M, there is another such collection $\mathfrak{B}$ of open sets covering M such that

(1) The collection $\mathfrak{B}$ is a refinement of the first, i.e., every element of $\mathfrak{B}$ is contained in an element of $\mathfrak{A}$.

(2) The collection $\mathfrak{B}$ is locally-finite, i.e., every point of M has a neighborhood intersecting only finitely many elements of $\mathfrak{B}$.

In passing, let us note that because M has a countable basis, any locally-finite open covering of M must be countable.

(a) Exercise. If M is an m-manifold, show that Bd M is a non-bounded m-1 manifold or is empty.

(b) Exercise. Let M be an m-manifold with non-empty boundary. Let $M_0 = M \times 0$ and $M_1 = M \times 1$ be two copies of M. The double of M, denoted by $D(M)$, is the topological space obtained from $M_0 \cup M_1$ by identifying $(x,0)$ with $(x,1)$ for each x in Bd M. Prove that M is a non-bounded manifold of dimension m.

(c) Exercise. If M and N are manifolds of dimensions m and n, respectively, then $M \times N$ is a manifold of dimension $m + n$, and $\mathrm{Bd}(M \times N) = ((\mathrm{Bd}\ M) \times N) \cup (M \times (\mathrm{Bd}\ N))$.

1.2 <u>Definition</u>. If U is an open subset of R^m, then $f : U \to R^n$ is <u>differentiable</u> of class C^r if the partial derivatives of the component functions $f^1,\ldots,f^n$ through order r are continuous on U. If f is of class C^r for all finite r, it is said to be of class C^∞.

If A is any subset of R^m, then $f : A \to R^n$ is <u>differentiable</u> of class C^r $(1 \leq r \leq \infty)$ if f may be extended to a neighborhood U of A in R^m so that the extended function is of class C^r on U. In practice, we will apply this definition only (1) when A is an open subset of H^m, and (2) when A is a closed rectilinear simplex in R^m.

If $f : A \to R^n$ is differentiable, and x is in A, we use $Df(x)$ to denote the Jacobian matrix of f at x — the matrix whose general entry is $a_{ij} = \partial f^i/\partial x^j$. We also use the notation $\partial f^1,\ldots,f^n/\partial(x^1,\ldots,x^m)$ for this matrix. Now f must be extended to a neighborhood of A before these partials are defined; in the two cases of interest, the partials are independent of the choice of extension (see Exercise (b)).

We recall here the <u>chain rule</u> for derivatives, which states that $D(fg) = Df \cdot Dg$, where fg is the composite function, and the dot indicates matrix multiplication.

(a) Exercise. Check that differentiability is well-defined; i.e., that the differentiability of $f : A \to R^n$ does not depend on which "containing space" R^m for A is chosen.

(b) Exercise. Let A be open in H^m, or be a closed rectilinear m-simplex in R^m. If $f : A \to R^n$ is of class C^1, and x is in A, show that $Df(x)$ is independent of the extension of f to a neighborhood of A in R^m which is chosen.

(c) Exercise*. Find an open subset U of R^2 and a C^1 map $f : A \to R$ (where $A = \bar{U}$) such that the conclusion of the theorem in Exercise (b) fails.

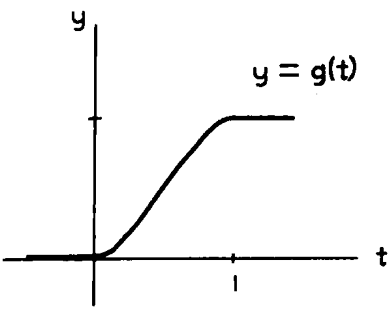
Remark. Let f map the subset A of R^m into R^n. If A is open in R^m, it is clear that f is differentiable if it is locally differentiable, i.e., if each point of A has a neighborhood V such that $f| V \cap A$ is differentiable. However, if A is not open in R^m, this is not nearly so clear; it needs verification, which is supplied by the following three lemmas.

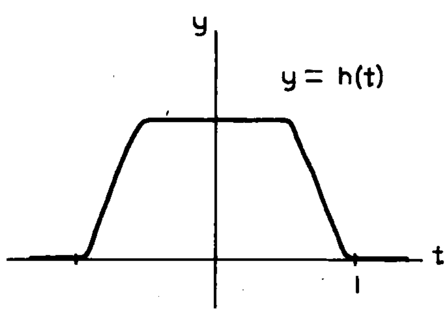
1.3 Lemma. There is a C^∞ function $\varphi : R^m \to R^1$ which equals 1 on $C(1/2)$, is positive on the interior of $C(1)$, and is zero outside $C(1)$.

Proof. Let $f(t) = e^{-1/t}$ for $t > 0$, and $f(t) = 0$ for $t \leq 0$. Then f is a C^∞ function which is positive for $t > 0$.

Let $g(t) = f(t)/(f(t) + f(1-t))$. Then g is a C^∞ function such that $g(t) = 0$ for $t \leq 0$, $g'(t) > 0$ for $0 < t < 1$, and $g(t) = 1$ for $t \geq 1$.

Let $h(t) = g(2t+2)\, g(-2t+2)$. Then h is a C^∞ function such that $h(t) = 0$ for $|t| \geq 1$, $h(t) > 0$ for $|t| \leq 1$, and $h(t) = 1$ for $|t| \leq 1/2$.





Let $\varphi(x^1,\ldots,x^m) = h(x^1) \cdot h(x^2) \cdots h(x^m)$.

(a) Exercise. Generalize the preceding lemma as follows: Let U be an open subset of R^m; let C be a compact subset of U. There is a C^r real-valued function ψ defined on R^m such that ψ is positive on C and is zero in a neighborhood of the complement of U.

Remark. Whenever an indexed collection $\{C_i\}$ of subsets of X is said to be locally-finite, we shall mean by this that every point of X has a neighborhood intersecting C_i for at most finitely many values of i. This convention is convenient, for otherwise a given set could appear in the sequence $C_1, C_2, \ldots$ infinitely many times.

1.4 Lemma. Let $\{U_i\}$ be a locally-finite open covering of the topological manifold M. There is a covering $\{C_i\}$ of M by closed sets such that $C_i \subset U_i$ for each i.

Proof. We construct this covering by induction. Let V_1 be an open set containing $M - (U_2 \cup U_3 \cup \ldots)$, whose closure is contained in U_1. (We use normality of M at this point.) Let $C_1 = \bar{V}_1$.

Suppose $V_1 \cup \ldots \cup V_{j-1} \cup U_j \cup \ldots = M$. Let V_j be an open set containing

$$M - (V_1 \cup \ldots \cup V_{j-1} \cup U_{j+1} \cup \ldots)$$

whose closure is contained in U_j. Let $C_j = \bar{V}_j$.

To prove that the collection $\{V_j\}$ covers M, note that any point x lies in only finitely many sets U_j. Hence for some j, x is not in $U_j \cup U_{j+1} \cup \ldots$. As a result, x must belong to $V_1 \cup \ldots \cup V_{j-1}$, by the induction hypothesis.

1.5 Lemma. Let A be a subset of R^m; let $f : A \to R^n$. Then f is of class C^r if it is locally of class C^r.

Proof. By hypothesis, for each point x of A, there is a neighborhood U_x of x such that $f|A \cap U_x$ may be extended to a function which is of class C^r on U_x. We choose $\bar{U}_x$ to be compact. Let M be the union of the sets U_x; it is an open subset of R^m. Let $\{V_i\}$ be a locally-finite open refinement of the covering $\{U_x\}$ of M. Let $\{C_i\}$ be a covering of M by closed sets such that $C_i \subset V_i$ for each i. Let ψ_i be a C^∞

function defined on R^m which is positive on C_i and equals zero in a neighborhood of the complement of V_i. Then $\Sigma \psi_j(x)$ is a C^∞ function on M, since it equals a finite sum in some neighborhood of any given point of M. Define $\varphi_i(x) = \psi_i(x)/\Sigma \psi_j(x)$; then $\Sigma \varphi_i(x) = 1$.

For each i, let f_i denote a C^r extension of $f|A \cap V_i$ to V_i; if $A \cap V_i$ is empty, let f_i be the zero function. Then $\varphi_i f_i$ may be extended to be of class C^r on M by letting it equal zero outside V_i. Define

$$\tilde{f}(x) = \Sigma_i \varphi_i(x) f_i(x) .$$

This is a finite sum in some neighborhood of any point x of M, and hence is of class C^r on M. Furthermore, if x is in A, then $f_i(x) = f(x)$ for every i, so that

$$\tilde{f}(x) = \Sigma \varphi_i(x) f(x) = f(x) .$$

Hence $\tilde{f}$ is the required C^r extension of f to the neighborhood M of A in R^m.

1.6 <u>Definition</u>. A <u>differentiable</u> <u>m-manifold</u> of class C^r is an m-manifold M and a differentiable structure $\mathfrak{D}$ of class C^r on M. A <u>differentiable</u> <u>structure</u> of class C^r on M, in turn, is a collection of coordinate neighborhoods (U,h) on M, satisfying three conditions:

(1) The coordinate neighborhoods in $\mathfrak{D}$ cover M.

(2) If (U_1,h_1) and (U_2,h_2) belong to $\mathfrak{D}$, then

$$h_1 h_2^{-1}: h_2(U_1 \cap U_2) \to R^m \text{ or } H^m$$

is differentiable of class C^r.

(3) The collection $\mathfrak{D}$ is maximal with respect to property (2); i.e., if any coordinate neighborhood not in $\mathfrak{D}$ is adjoined to the collection $\mathfrak{D}$, then property (2) fails.

The elements of $\mathfrak{D}$ are often called <u>coordinate</u> <u>systems</u> on the differentiable manifold M.

(a) Exercise. Let $\mathfrak{D}'$ be a collection of coordinate neighborhoods on M satisfying (1) and (2). Prove there is a unique differentiable structure $\mathfrak{D}$ of class C^r containing $\mathfrak{D}'$. (We call $\mathfrak{D}'$ a <u>basis</u> for $\mathfrak{D}$, by analogy with the relation between a basis for a topology and the topology.)

Hint: Let $\mathfrak{D}$ consist of all coordinate neighborhoods (U,h) on M which overlap every element of $\mathfrak{D}'$ differentiably with class C^r; this means that for each element (U_1,h_1) of $\mathfrak{D}'$,

$$h_1 h^{-1} : h(U \cap U_1) \to H^m \text{ or } R^m$$

and

$$h\, h_1^{-1} : h_1(U \cap U_1) \to H^m \text{ or } R^m$$

are differentiable of class C^r. To prove that $\mathfrak{D}$ is a differentiable structure, you will need Lemma 1.5.

(b) Exercise. Let M be a differentiable manifold of class C^r (we often suppress mention of the differentiable structure $\mathfrak{D}$, where no confusion will arise). Then M may also be considered to be a differentiable manifold of class C^{r-1}, in a natural way; one merely takes $\mathfrak{D}$ as a basis for a differentiable structure $\mathfrak{D}_1$ of class C^{r-1} on M. Verify that the inclusion $\mathfrak{D} \subset \mathfrak{D}_1$ is proper. This proves that the class C^r of a differentiable manifold is uniquely determined.

We see in this way that the class of a differentiable manifold M may be lowered as far as one likes merely by adding new coordinate systems to the differentiable structure. The reverse is also true, but it will require much work to prove.

(c) Exercise*. If M is a differentiable manifold, what are the difficulties involved in putting a differentiable structure on $D(M)$? ($D(M)$ was defined in Exercise (b) of 1.1.)

1.7 <u>Definition</u>. Let M and N be differentiable manifolds, of dimensions m and n, respectively, and of class at least C^r. Let A be a subset of M and let $f : A \to N$; then f is said to be <u>of class</u> C^r if for every pair (U,h) and (V,k) of coordinate systems of M and N, respectively, the composite

$$kfh^{-1} : h(A \cap U) \to R^n$$

is of class C^r. (Note that a map of class C^2 is also of class C^1, although a manifold of class C^2 is not one of class C^1 until the differentiable structure is changed.)

The <u>rank</u> of f at the point p of M is the rank of $D(kfh^{-1})$, where (U,h) and (V,k) are coordinate systems about p and $f(p)$, respectively. This number is well-defined, for if (U_1, h_1) and (V_1, k_1) were other such coordinate systems, we would have

$$D(k_1 f h_1^{-1}) = D(k_1 k^{-1}) \cdot D(k f h^{-1}) \cdot D(h h_1^{-1})$$

The requirements for a differentiable structure assure that $k_1 k^{-1}$ and $k k_1^{-1}$ are both differentiable, so that $D(k_1 k^{-1})$ is non-singular, having $D(k k_1^{-1})$ as its inverse. Similarly, $D(h h_1^{-1})$ is non-singular, so $D(k_1 f h_1^{-1})$ and $D(k f h^{-1})$ have the same rank.

(a) Exercise. The standard C^∞ differentiable structure on R^m is that having as basis the single coordinate system $i : R^m \to R^m$. Similarly for H^m. If one of the spaces M or N in the preceding definition is R^m or H^m, check that the definitions of differentiability given in 1.2 and 1.7 agree.

1.8 <u>Definition</u>. Let $f : M \to N$ be differentiable of class C^r; let M and N have dimensions m and n, respectively. If rank $f = m$ at each point p of M, f is said to be an <u>immersion</u>. If f is a homeomorphism (into) and is an immersion, it is called an <u>imbedding</u>. If f is a homeomorphism of M onto N and is an immersion, it is called <u>diffeomorphism</u>; of course, $m = n$ in this case.

(a) Exercise. Note that $\text{Bd}\, H^m = R^{m-1}$ and the inclusion $R^{m-1} \to H^m$ is an imbedding. Generalize this as follows: If M is a differentiable manifold of class C^r, then there is a unique differentiable structure of class C^r on Bd M such that the inclusion $\text{Bd}\, M \to M$ is a C^r imbedding.

(b) Exercise. Let M and N have class C^r; let M be non-bounded. Construct a C^r differentiable structure on $M \times N$ such that the natural inclusions of M and N into $M \times N$ are imbeddings. Why do we require M to be non-bounded?

(c) Exercise. Show that the composition of two immersions is an immersion.

(d) Exercise*. Construct a C^∞ immersion of S^1 into R^2 which