Undergraduate Texts in Mathematics

Serge Lang

Introduction to Linear Algebra

Second Edition

线性代数导论

第 2 版

Springer-Verlag
世界图书出版公司

www.wpcbj.com.cn

ISBN 7-5062-7186-9

9 787506 271868 >

ISBN 7-5062-7186-9/O · 504
WB7186 定价:49.00元

Serge Lang

Introduction to Linear Algebra

Second Edition

With 66 Illustrations

Springer
世界图书出版公司

书　　名：Introduction to Linear Algebra　Second Edition
作　　者：Serge Lang
中 译 名：线性代数导论 第 2 版
出 版 者：世界图书出版公司北京公司
印 刷 者：北京世图印刷厂
发　　行：世界图书出版公司北京公司 (北京朝内大街 137 号　100010)
联系电话：010-64015659, 64038347
电子信箱：kjsk@vip.sina.com
开　　本：24 开　　　印　张：13
出版年代：2004 年 11 月
书　　号：7-5062-7186-9/O・504
版权登记：图字:01-2004-5056
定　　价：49.00 元

Serge Lang
Department of Mathematics
Yale University
New Haven, CT 06520
U.S.A.

Mathematics Subjects Classifications (2000): 15-01

Library of Congress Cataloging in Publication Data
Lang, Serge, 1927–
Introduction to linear algebra.
(Undergraduate texts in mathematics)
Includes index.
1. Algebras, Linear. I. Title. II. Series.
QA184.L37 1986 512'.5 85-14758

Printed on acid-free paper.

The first edition of this book was published by Addison-Wesley Publishing Company, Inc., in 1970.

Springer-Verlag is a part of *Springer Science+Business Media*

springeronline.com

Preface

This book is meant as a short text in linear algebra for a one-term course. Except for an occasional example or exercise the text is logically independent of calculus, and could be taught early. In practice, I expect it to be used mostly for students who have had two or three terms of calculus. The course could also be given simultaneously with, or immediately after, the first course in calculus.

I have included some examples concerning vector spaces of functions, but these could be omitted throughout without impairing the understanding of the rest of the book, for those who wish to concentrate exclusively on euclidean space. Furthermore, the reader who does not like $n = n$ can always assume that $n = 1$, 2, or 3 and omit other interpretations. However, such a reader should note that using $n = n$ simplifies some formulas, say by making them shorter, and should get used to this as rapidly as possible. Furthermore, since one does want to cover both the case $n = 2$ and $n = 3$ at the very least, using n to denote either number avoids very tedious repetitions.

The first chapter is designed to serve several purposes. First, and most basically, it establishes the fundamental connection between linear algebra and geometric intuition. There are indeed two aspects (at least) to linear algebra: the formal manipulative aspect of computations with matrices, and the geometric interpretation. I do not wish to prejudice one in favor of the other, and I believe that grounding formal manipulations in geometric contexts gives a very valuable background for those who use linear algebra. Second, this first chapter gives immediately concrete examples, with coordinates, for linear combinations, perpendicularity, and other notions developed later in the book. In addition to the geometric context, discussion of these notions provides examples for subspaces, and also gives a fundamental interpretation for linear equations. Thus the first chapter gives a quick overview of many topics in the book. The content of the first chapter is also the most fundamental part of what is used in calculus courses concerning functions of several variables, which can do a lot of things without the more general matrices. If students have covered the material of Chapter I in another course, or if the instructor wishes to emphasize matrices right away, then the first chapter can be skipped, or can be used selectively for examples and motivation.

After this introductory chapter, we start with linear equations, matrices, and Gauss elimination. This chapter emphasizes computational aspects of linear algebra. Then we deal with vector spaces, linear maps and scalar products, and their relations to matrices. This mixes both the computational and theoretical aspects.

Determinants are treated much more briefly than in the first edition, and several proofs are omitted. Students interested in theory can refer to a more complete treatment in theoretical books on linear algebra.

I have included a chapter on eigenvalues and eigenvectors. This gives practice for notions studied previously, and leads into material which is used constantly in all parts of mathematics and its applications.

I am much indebted to Toby Orloff and Daniel Horn for their useful comments and corrections as they were teaching the course from a preliminary version of this book. I thank Allen Altman and Gimli Khazad for lists of corrections.

Contents

CHAPTER I

Vectors

The concept of a vector is basic for the study of functions of several variables. It provides geometric motivation for everything that follows. Hence the properties of vectors, both algebraic and geometric, will be discussed in full.

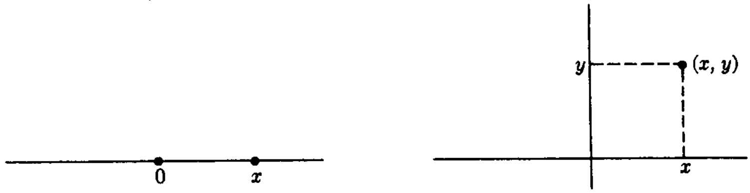
One significant feature of all the statements and proofs of this part is that they are neither easier nor harder to prove in 3-space than they are in 2-space.

I, §1. Definition of Points in Space

We know that a number can be used to represent a point on a line, once a unit length is selected.

A pair of numbers (i.e. a couple of numbers) (x, y) can be used to represent a point in the plane.

These can be pictured as follows:



(a) Point on a line

(b) Point in a plane

Figure 1

We now observe that a triple of numbers (x, y, z) can be used to represent a point in space, that is 3-dimensional space, or 3-space. We simply introduce one more axis. Figure 2 illustrates this.

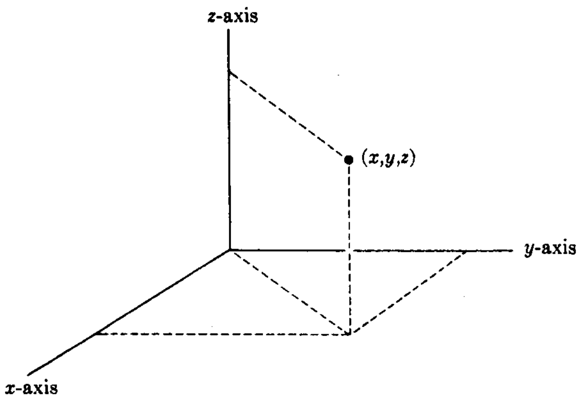


Figure 2

Instead of using x, y, z we could also use (x_1, x_2, x_3). The line could be called 1-space, and the plane could be called 2-space.

Thus we can say that a single number represents a point in 1-space. A couple represents a point in 2-space. A triple represents a point in 3-space.

Although we cannot draw a picture to go further, there is nothing to prevent us from considering a quadruple of numbers.

$$(x_1, x_2, x_3, x_4)$$

and decreeing that this is a point in 4-space. A quintuple would be a point in 5-space, then would come a sextuple, septuple, octuple,....

We let ourselves be carried away and **define a point in *n*-space** to be an n-tuple of numbers

$$(x_1, x_2, \ldots, x_n),$$

if n is a positive integer. We shall denote such an n-tuple by a capital letter X, and try to keep small letters for numbers and capital letters for points. We call the numbers $x_1, \ldots, x_n$ the **coordinates** of the point X. For example, in 3-space, 2 is the first coordinate of the point $(2, 3, -4)$, and -4 is its third coordinate. We denote n-space by $\mathbf{R}^n$.

Most of our examples will take place when $n = 2$ or $n = 3$. Thus the reader may visualize either of these two cases throughout the book. However, three comments must be made.

First, we have to handle $n = 2$ and $n = 3$, so that in order to avoid a lot of repetitions, it is useful to have a notation which covers both these cases simultaneously, even if we often repeat the formulation of certain results separately for both cases.

Second, no theorem or formula is simpler by making the assumption that $n = 2$ or 3.

Third, the case $n = 4$ does occur in physics.

Example 1. One classical example of 3-space is of course the space we live in. After we have selected an origin and a coordinate system, we can describe the position of a point (body, particle, etc.) by 3 coordinates. Furthermore, as was known long ago, it is convenient to extend this space to a 4-dimensional space, with the fourth coordinate as time, the time origin being selected, say, as the birth of Christ—although this is purely arbitrary (it might be more convenient to select the birth of the solar system, or the birth of the earth as the origin, if we could determine these accurately). Then a point with negative time coordinate is a BC point, and a point with positive time coordinate is an AD point.

Don't get the idea that "time is *the* fourth dimension", however. The above 4-dimensional space is only one possible example. In economics, for instance, one uses a very different space, taking for coordinates, say, the number of dollars expended in an industry. For instance, we could deal with a 7-dimensional space with coordinates corresponding to the following industries:

1. Steel	2. Auto	3. Farm products	4. Fish
5. Chemicals	6. Clothing	7. Transportation.	

We agree that a megabuck per year is the unit of measurement. Then a point

$$(1{,}000, 800, 550, 300, 700, 200, 900)$$

in this 7-space would mean that the steel industry spent one billion dollars in the given year, and that the chemical industry spent 700 million dollars in that year.

The idea of regarding time as a fourth dimension is an old one. Already in the *Encyclopédie* of Diderot, dating back to the eighteenth century, d'Alembert writes in his article on "dimension":

> Cette manière de considérer les quantités de plus de trois dimensions est aussi exacte que l'autre, car les lettres peuvent toujours être regardées comme représentant des nombres rationnels ou non. J'ai dit plus haut qu'il n'était pas possible de concevoir plus de trois dimensions. Un homme d'esprit de ma connaissance croit qu'on pourrait cependant regarder la durée comme une quatrième dimension, et que le produit temps par la solidité serait en quelque manière un produit de quatre dimensions; cette idée peut être contestée, mais elle a, ce me semble, quelque mérite, quand ce ne serait que celui de la nouveauté.
>
> *Encyclopédie*, Vol. 4 (1754), p. 1010

Translated, this means:

> This way of considering quantities having more than three dimensions is just as right as the other, because algebraic letters can always be viewed as representing numbers, whether rational or not. I said above that it was not possible to conceive more than three dimensions. A clever gentleman with whom I am acquainted believes that nevertheless, one could view duration as a fourth dimension, and that the product time by solidity would be somehow a product of four dimensions. This idea may be challenged, but it has, it seems to me, some merit, were it only that of being new.

Observe how d'Alembert refers to a "clever gentleman" when he apparently means himeself. He is being rather careful in proposing what must have been at the time a far out idea, which became more prevalent in the twentieth century.

D'Alembert also visualized clearly higher dimensional spaces as "products" of lower dimensional spaces. For instance, we can view 3-space as putting side by side the first two coordinates (x_1, x_2) and then the third x_3. Thus we write

$$\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}^1.$$

We use the product sign, which should not be confused with other "products", like the product of numbers. The word "product" is used in two contexts. Similarly, we can write

$$\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}^1.$$

There are other ways of expressing $\mathbf{R}^4$ as a product, namely

$$\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2.$$

This means that we view separately the first two coordinates (x_1, x_2) and the last two coordinates (x_3, x_4). We shall come back to such products later.

We shall now define how to add points. If A, B are two points, say in 3-space,

$$A = (a_1, a_2, a_3) \quad \text{and} \quad B = (b_1, b_2, b_3)$$

then we **define** $A + B$ to be the point whose coordinates are

$$A + B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3).$$

Example 2. In the plane, if $A = (1, 2)$ and $B = (-3, 5)$, then

$$A + B = (-2, 7).$$

In 3-space, if $A = (-1, \pi, 3)$ and $B = (\sqrt{2}, 7, -2)$, then

$$A + B = (\sqrt{2} - 1,\ \pi + 7,\ 1).$$

Using a neutral n to cover both the cases of 2-space and 3-space, the points would be written

$$A = (a_1, \ldots, a_n), \qquad B = (b_1, \ldots, b_n),$$

and we **define** $A + B$ to be the point whose coordinates are

$$(a_1 + b_1, \ldots, a_n + b_n).$$

We observe that the following rules are satisfied:

1. $(A + B) + C = A + (B + C)$.
2. $A + B = B + A$.
3. If we let

$$O = (0, 0, \ldots, 0)$$

be the point all of whose coordinates are 0, then

$$O + A = A + O = A$$

for all A.
4. Let $A = (a_1, \ldots, a_n)$ and let $-A = (-a_1, \ldots, -a_n)$. Then

$$A + (-A) = O.$$

All these properties are very simple, and are true because they are true for numbers, and addition of n-tuples is defined in terms of addition of their components, which are numbers.

Note. Do not confuse the number 0 and the n-tuple $(0, \ldots, 0)$. We usually denote this n-tuple by O, and also call it zero, because no difficulty can occur in practice.

We shall now interpret addition and multiplication by numbers geometrically in the plane (you can visualize simultaneously what happens in 3-space).

Example 3. Let $A = (2, 3)$ and $B = (-1, 1)$. Then

$$A + B = (1, 4).$$

The figure looks like a **parallelogram** (Fig. 3).

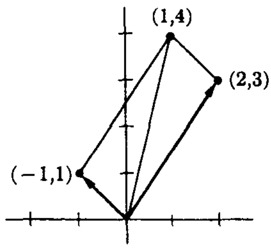


Figure 3

Example 4. Let $A = (3, 1)$ and $B = (1, 2)$. Then

$$A + B = (4, 3).$$

We see again that the geometric representation of our addition looks like a **parallelogram** (Fig. 4).

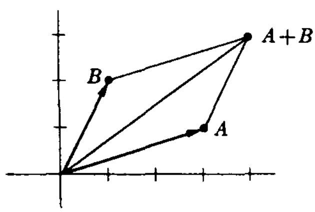


Figure 4

The reason why the figure looks like a **parallelogram** can be given in terms of plane geometry as follows. We obtain $B = (1, 2)$ by starting from the origin $O = (0, 0)$, and moving 1 unit to the right and 2 up. To get $A + B$, we start from A, and again move 1 unit to the right and 2 up. Thus the line segments between O and B, and between A and $A + B$ are the hypotenuses of right triangles whose corresponding legs are of the same length, and parallel. The above segments are therefore parallel and of the same length, as illustrated in Fig. 5.

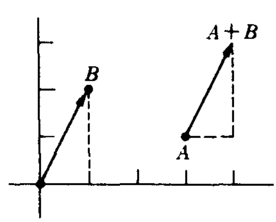


Figure 5

Example 5. If $A = (3, 1)$ again, then $-A = (-3, -1)$. If we plot this point, we see that $-A$ has opposite direction to A. We may view $-A$ as the reflection of A through the origin.

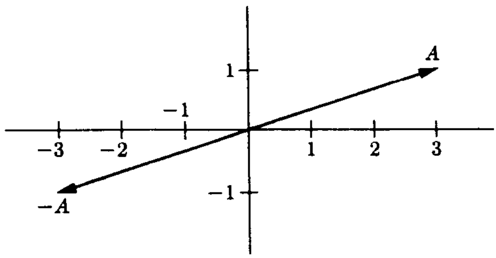


Figure 6

We shall now consider multiplication of A by a number. If c is any number, we **define** cA to be the point whose coordinates are

$$(ca_1, \dots, ca_n).$$

Example 6. If $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$.

It is easy to verify the rules:

5. $c(A + B) = cA + cB$.
6. If c_1, c_2 are numbers, then

$$(c_1 + c_2)A = c_1A + c_2A \qquad \text{and} \qquad (c_1c_2)A = c_1(c_2A).$$

Also note that

$$(-1)A = -A.$$

What is the geometric representation of multiplication by a number?

Example 7. Let $A = (1, 2)$ and $c = 3$. Then

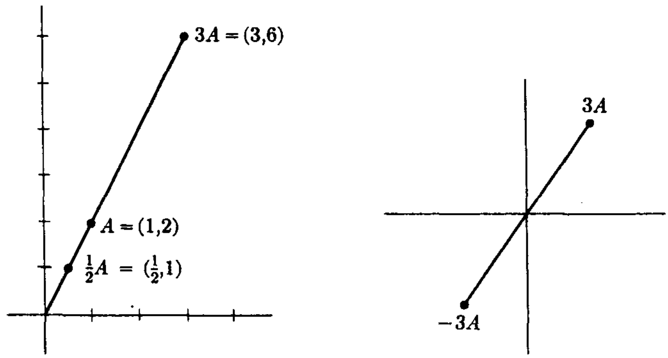
$$cA = (3, 6)$$

as in Fig. 7(a).

Multiplication by 3 amounts to stretching A by 3. Similarly, $\frac{1}{2}A$ amounts to stretching A by $\frac{1}{2}$, i.e. shrinking A to half its size. In general, if t is a number, $t > 0$, we interpret tA as a point in the same direction as A from the origin, but t times the distance. In fact, we define A and

B to have the **same direction** if there exists a number $c > 0$ such that $A = cB$. We emphasize that this means A and B have the same direction **with respect to the origin**. For simplicity of language, we omit the words "with respect to the origin".

Mulitiplication by a negative number reverses the direction. Thus $-3A$ would be represented as in Fig. 7(b).



(a) (b)

Figure 7

We define A, B (neither of which is zero) to have **opposite directions** if there is a number $c < 0$ such that $cA = B$. Thus when $B = -A$, then A, B have opposite direction.

Exercises I, §1

Find $A + B$, $A - B$, $3A$, $-2B$ in each of the following cases. Draw the points of Exercises 1 and 2 on a sheet of graph paper.

1. $A = (2, -1)$, $B = (-1, 1)$
2. $A = (-1, 3)$, $B = (0, 4)$
3. $A = (2, -1, 5)$, $B = (-1, 1, 1)$
4. $A = (-1, -2, 3)$, $B = (-1, 3, -4)$
5. $A = (\pi, 3, -1)$, $B = (2\pi, -3, 7)$
6. $A = (15, -2, 4)$, $B = (\pi, 3, -1)$
7. Let $A = (1, 2)$ and $B = (3, 1)$. Draw $A + B$, $A + 2B$, $A + 3B$, $A - B$, $A - 2B$, $A - 3B$ on a sheet of graph paper.
8. Let A, B be as in Exercise 1. Draw the points $A + 2B$, $A + 3B$, $A - 2B$, $A - 3B$, $A + \frac{1}{2}B$ on a sheet of graph paper.
9. Let A and B be as drawn in Fig. 8. Draw the point $A - B$.