London Mathematical Society
Lecture Note Series 441

Polynomials and the mod 2 Steenrod Algebra

Volume 1: The Peterson Hit Problem

Grant Walker and Reginald M. W. Wood

LONDON
MATHEMATICAL
SOCIETY
EST. 1865

CAMBRIDGE

LONDON MATHEMATICAL SOCIETY
LECTURE NOTE SERIES

Polynomials and the mod 2 Steenrod Algebra, Volume 1

Grant Walker and Reginald M. W. Wood

This is the first book to link the mod 2 Steenrod algebra, a classical object of study in algebraic topology, with modular representations of matrix groups over the field F of two elements. The link is provided through a detailed study of Peterson's 'hit problem' concerning the action of the Steenrod algebra on polynomials, which remains unsolved except in special cases. The topics range from decompositions of integers as sums of 'powers of 2 minus 1', to Hopf algebras and the Steinberg representation of GL(n, F).

Volume 1 develops the structure of the Steenrod algebra from an algebraic viewpoint and can be used as a graduate-level textbook.

Volume 2 broadens the discussion to include modular representations of matrix groups.

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-108-41448-7
9 781108 414487

For a list of books available in this series, see page i.

LMS 441

Walker and Wood

Polynomials and the mod 2 Steenrod Algebra
Volume 1

CAMBRIDGE

London Mathematical Society Lecture Note Series: 441

Polynomials and the mod 2 Steenrod Algebra

Volume 1: The Peterson Hit Problem

GRANT WALKER
University of Manchester

REGINALD M. W. WOOD
University of Manchester

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108414487
DOI: 10.1017/9781108333368

First published 2018

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

ISBN – 2 Volume Set 978-1-108-41406-7 Paperback
ISBN – Volume 1 978-1-108-41448-7 Paperback
ISBN – Volume 2 978-1-108-41445-6 Paperback

LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor M. Reid, Mathematics Institute,
University of Warwick, Coventry CV4 7AL, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press at
http://www.cambridge.org/mathematics

325 Lectures on the Ricci flow, P. TOPPING
326 Modular representations of finite groups of Lie type, J.E. HUMPHREYS
327 Surveys in combinatorics 2005, B.S. WEBB (ed)
328 Fundamentals of hyperbolic manifolds, R. CANARY, D. EPSTEIN & A. MARDEN (eds)
329 Spaces of Kleinian groups, Y. MINSKY, M. SAKUMA & C. SERIES (eds)
330 Noncommutative localization in algebra and topology, A. RANICKI (ed)
331 Foundations of computational mathematics, Santander 2005, L.M PARDO, A. PINKUS, E. SÜLI & M.J. TODD (eds)
332 Handbook of tilting theory, L. ANGELERI HÜGEL, D. HAPPEL & H. KRAUSE (eds)
333 Synthetic differential geometry (2nd Edition), A. KOCK
334 The Navier-Stokes equations, N. RILEY & P. DRAZIN
335 Lectures on the combinatorics of free probability, A. NICA & R. SPEICHER
336 Integral closure of ideals, rings, and modules, I. SWANSON & C. HUNEKE
337 Methods in Banach space theory, J.M.F. CASTILLO & W.B. JOHNSON (eds)
338 Surveys in geometry and number theory, N. YOUNG (ed)
339 Groups St Andrews 2005 &I, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
340 Groups St Andrews 2005 &II, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
341 Ranks of elliptic curves and random matrix theory, J.B. CONREY, D.W. FARMER, F. MEZZADRI & N.C. SNAITH (eds)
342 Elliptic cohomology, H.R. MILLER & D.C. RAVENEL (eds)
343 Algebraic cycles and motives I, J. NAGEL & C. PETERS (eds)
344 Algebraic cycles and motives II, J. NAGEL & C. PETERS (eds)
345 Algebraic and analytic geometry, A. NEEMAN
346 Surveys in combinatorics 2007, A. HILTON & J. TALBOT (eds)
347 Surveys in contemporary mathematics, N. YOUNG & Y. CHOI (eds)
348 Transcendental dynamics and complex analysis, P.J. RIPPON & G.M. STALLARD (eds)
349 Model theory with applications to algebra and analysis I, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
350 Model theory with applications to algebra and analysis II, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
351 Finite von Neumann algebras and masas, A.M. SINCLAIR & R.R. SMITH
352 Number theory and polynomials, J. MCKEE & C. SMYTH (eds)
353 Trends in stochastic analysis, J. BLATH, P. MÖRTERS & M. SCHEUTZOW (eds)
354 Groups and analysis, K. TENT (ed)
355 Non-equilibrium statistical mechanics and turbulence, J. CARDY, G. FALKOVICH & K. GAWEDZKI
356 Elliptic curves and big Galois representations, D. DELBOURGO
357 Algebraic theory of differential equations, M.A.H. MACCALLUM & A.V. MIKHAILOV (eds)
358 Geometric and cohomological methods in group theory, M.R. BRIDSON, P.H. KROPHOLLER & I.J. LEARY (eds)
359 Moduli spaces and vector bundles, L. BRAMBILA-PAZ, S.B. BRADLOW, O. GARCÍA-PRADA & S. RAMANAN (eds)
360 Zariski geometries, B. ZILBER
361 Words: Notes on verbal width in groups, D. SEGAL
362 Differential tensor algebras and their module categories, R. BAUTISTA, L. SALMERÓN & R. ZUAZUA
363 Foundations of computational mathematics, Hong Kong 2008, F. CUCKER, A. PINKUS & M.J. TODD (eds)
364 Partial differential equations and fluid mechanics, J.C. ROBINSON & J.L. RODRIGO (eds)
365 Surveys in combinatorics 2009, S. HUCZYNSKA, J.D. MITCHELL & C.M. RONEY-DOUGAL (eds)
366 Highly oscillatory problems, B. ENGQUIST, A. FOKAS, E. HAIRER & A. ISERLES (eds)
367 Random matrices: High dimensional phenomena, G. BLOWER
368 Geometry of Riemann surfaces, F.P. GARDINER, G. GONZÁLEZ-DIEZ & C. KOUROUNIOTIS (eds)
369 Epidemics and rumours in complex networks, M. DRAIEF & L. MASSOULIÉ
370 Theory of p-adic distributions, S. ALBEVERIO, A.YU. KHRENNIKOV & V.M. SHELKOVICH
371 Conformal fractals, F. PRZYTYCKI & M. URBAŃSKI
372 Moonshine: The first quarter century and beyond, J. LEPOWSKY, J. MCKAY & M.P. TUITE (eds)
373 Smoothness, regularity and complete intersection, J. MAJADAS & A. G. RODICIO
374 Geometric analysis of hyperbolic differential equations: An introduction, S. ALINHAC
375 Triangulated categories, T. HOLM, P. JØRGENSEN & R. ROUQUIER (eds)
376 Permutation patterns, S. LINTON, N. RUŠKUC & V. VATTER (eds)
377 An introduction to Galois cohomology and its applications, G. BERHUY
378 Probability and mathematical genetics, N. H. BINGHAM & C. M. GOLDIE (eds)
379 Finite and algorithmic model theory, J. ESPARZA, C. MICHAUX & C. STEINHORN (eds)
380 Real and complex singularities, M. MANOEL, M.C. ROMERO FUSTER & C.T.C WALL (eds)
381 Symmetries and integrability of difference equations, D. LEVI, P. OLVER, Z. THOMOVA & P. WINTERNITZ (eds)

382 Forcing with random variables and proof complexity, J. KRAJÍČEK
383 Motivic integration and its interactions with model theory and non-Archimedean geometry I, R. CLUCKERS, J. NICAISE & J. SEBAG (eds)
384 Motivic integration and its interactions with model theory and non-Archimedean geometry II, R. CLUCKERS, J. NICAISE & J. SEBAG (eds)
385 Entropy of hidden Markov processes and connections to dynamical systems, B. MARCUS, K. PETERSEN & T. WEISSMAN (eds)
386 Independence-friendly logic, A.L. MANN, G. SANDU & M. SEVENSTER
387 Groups St Andrews 2009 &in Bath I, C.M. CAMPBELL *et al* (eds)
388 Groups St Andrews 2009 &in Bath II, C.M. CAMPBELL *et al* (eds)
389 Random fields on the sphere, D. MARINUCCI & G. PECCATI
390 Localization in periodic potentials, D.E. PELINOVSKY
391 Fusion systems in algebra and topology, M. ASCHBACHER, R. KESSAR & B. OLIVER
392 Surveys in combinatorics 2011, R. CHAPMAN (ed)
393 Non-abelian fundamental groups and Iwasawa theory, J. COATES *et al* (eds)
394 Variational problems in differential geometry, R. BIELAWSKI, K. HOUSTON & M. SPEIGHT (eds)
395 How groups grow, A. MANN
396 Arithmetic differential operators over the p-adic integers, C.C. RALPH & S.R. SIMANCA
397 Hyperbolic geometry and applications in quantum chaos and cosmology, J. BOLTE & F. STEINER (eds)
398 Mathematical models in contact mechanics, M. SOFONEA & A. MATEI
399 Circuit double cover of graphs, C.-Q. ZHANG
400 Dense sphere packings: a blueprint for formal proofs, T. HALES
401 A double Hall algebra approach to affine quantum Schur-Weyl theory, B. DENG, J. DU & Q. FU
402 Mathematical aspects of fluid mechanics, J.C. ROBINSON, J.L. RODRIGO & W. SADOWSKI (eds)
403 Foundations of computational mathematics, Budapest 2011, F. CUCKER, T. KRICK, A. PINKUS & A. SZANTO (eds)
404 Operator methods for boundary value problems, S. HASSI, H.S.V. DE SNOO & F.H. SZAFRANIEC (eds)
405 Torsors, étale homotopy and applications to rational points, A.N. SKOROBOGATOV (ed)
406 Appalachian set theory, J. CUMMINGS & E. SCHIMMERLING (eds)
407 The maximal subgroups of the low-dimensional finite classical groups, J.N. BRAY, D.F. HOLT & C.M. RONEY-DOUGAL
408 Complexity science: the Warwick master's course, R. BALL, V. KOLOKOLTSOV & R.S. MACKAY (eds)
409 Surveys in combinatorics 2013, S.R. BLACKBURN, S. GERKE & M. WILDON (eds)
410 Representation theory and harmonic analysis of wreath products of finite groups, T. CECCHERINI-SILBERSTEIN, F. SCARABOTTI & F. TOLLI
411 Moduli spaces, L. BRAMBILA-PAZ, O. GARCÍA-PRADA, P. NEWSTEAD & R.P. THOMAS (eds)
412 Automorphisms and equivalence relations in topological dynamics, D.B. ELLIS & R. ELLIS
413 Optimal transportation, Y. OLLIVIER, H. PAJOT & C. VILLANI (eds)
414 Automorphic forms and Galois representations I, F. DIAMOND, P.L. KASSAEI & M. KIM (eds)
415 Automorphic forms and Galois representations II, F. DIAMOND, P.L. KASSAEI & M. KIM (eds)
416 Reversibility in dynamics and group theory, A.G. O'FARRELL & I. SHORT
417 Recent advances in algebraic geometry, C.D. HACON, M. MUSTAŢĂ & M. POPA (eds)
418 The Bloch-Kato conjecture for the Riemann zeta function, J. COATES, A. RAGHURAM, A. SAIKIA & R. SUJATHA (eds)
419 The Cauchy problem for non-Lipschitz semi-linear parabolic partial differential equations, J.C. MEYER & D.J. NEEDHAM
420 Arithmetic and geometry, L. DIEULEFAIT *et al* (eds)
421 O-minimality and Diophantine geometry, G.O. JONES & A.J. WILKIE (eds)
422 Groups St Andrews 2013, C.M. CAMPBELL *et al* (eds)
423 Inequalities for graph eigenvalues, Z. STANIĆ
424 Surveys in combinatorics 2015, A. CZUMAJ *et al* (eds)
425 Geometry, topology and dynamics in negative curvature, C.S. ARAVINDA, F.T. FARRELL & J.-F. LAFONT (eds)
426 Lectures on the theory of water waves, T. BRIDGES, M. GROVES & D. NICHOLLS (eds)
427 Recent advances in Hodge theory, M. KERR & G. PEARLSTEIN (eds)
428 Geometry in a Fréchet context, C. T. J. DODSON, G. GALANIS & E. VASSILIOU
429 Sheaves and functions modulo p, L. TAELMAN
430 Recent progress in the theory of the Euler and Navier-Stokes equations, J.C. ROBINSON, J.L. RODRIGO, W. SADOWSKI & A. VIDAL-LÓPEZ (eds)
431 Harmonic and subharmonic function theory on the real hyperbolic ball, M. STOLL
432 Topics in graph automorphisms and reconstruction (2nd Edition), J. LAURI & R. SCAPELLATO
433 Regular and irregular holonomic D-modules, M. KASHIWARA & P. SCHAPIRA
434 Analytic semigroups and semilinear initial boundary value problems (2nd Edition), K. TAIRA
435 Graded rings and graded Grothendieck groups, R. HAZRAT
436 Groups, graphs and random walks, T. CECCHERINI-SILBERSTEIN, M. SALVATORI & E. SAVA-HUSS (eds)
437 Dynamics and analytic number theory, D. BADZIAHIN, A. GORODNIK & N. PEYERIMHOFF (eds)
438 Random walks and heat kernels on graphs, M.T. BARLOW
439 Evolution equations, K. AMMARI & S. GERBI (eds)
440 Surveys in combinatorics 2017, A. CLAESSON *et al* (eds)
441 Polynomials and the mod 2 Steenrod algebra I, G. WALKER & R.M.W. WOOD
442 Polynomials and the mod 2 Steenrod algebra II, G. WALKER & R.M.W. WOOD
443 Asymptotic analysis in general relativity, T. DAUDÉ, D. HÄFNER & J.-P. NICOLAS (eds)
444 Geometric and cohomological group theory, P.H. KROPHOLLER, I.J. LEARY, C. MARTÍNEZ-PÉREZ & B.E.A. NUCINKIS (eds)

Preface

This book is about the mod 2 Steenrod algebra A_2 and its action on the polynomial algebra $\mathsf{P}(n) = \mathbb{F}_2[x_1,\ldots,x_n]$ in n variables, where $\mathbb{F}_2$ is the field of two elements. Polynomials are graded by degree, so that $\mathsf{P}^d(n)$ is the set of homogeneous polynomials of degree d. Although our subject has its origin in the work of Norman E. Steenrod in algebraic topology, we have taken an algebraic point of view. We have tried as far as possible to provide a self-contained treatment based on linear algebra and representations of finite matrix groups. In other words, the reader does not require knowledge of algebraic topology, although the subject has been developed by topologists and is motivated by problems in topology.

There are many bonuses for working with the prime $p = 2$. There are no coefficients to worry about, so that every polynomial can be written simply as a sum of monomials. We use a matrix-like array of 0s and 1s, which we call a 'block', to represent a monomial in $\mathsf{P}(n)$, where the rows of the block are formed by the reverse binary expansions of its exponents. Thus a polynomial is a set of blocks, and the sum of two polynomials is the symmetric difference of the corresponding sets. Using block notation, the action of A_2 on $\mathsf{P}(n)$ can be encoded in computer algebra programs using standard routines on sets, lists and arrays. In addition, much of the literature on the Steenrod algebra and its applications in topology concentrates on the case $p = 2$. Often a result for $p = 2$ has later been extended to all primes, but there are some results where no odd prime analogue is known.

We begin in Chapter 1 with the algebra map $\mathsf{Sq} : \mathsf{P}(n) \to \mathsf{P}(n)$ defined on the generators by $\mathsf{Sq}(x_i) = x_i + x_i^2$. The map Sq is the total Steenrod squaring operation, and the Steenrod squares $Sq^k : \mathsf{P}^d(n) \to \mathsf{P}^{d+k}(n)$ are its graded parts. The linear operations Sq^k can be calculated using induction on degree and the Cartan formula $Sq^k(fg) = \sum_{i+j=k} Sq^i(f)Sq^j(g)$, which is equivalent to the multiplicative property of Sq. A general Steenrod operation is a sum of compositions of Steenrod squares.

The multiplicative monoid $\mathsf{M}(n)$ of $n \times n$ matrices over $\mathbb{F}_2$ acts on the right of $\mathsf{P}(n)$ by linear substitution of the variables. Thus $\mathsf{P}^d(n)$ gives a representation over $\mathbb{F}_2$ of $\mathsf{M}(n)$ and of $\mathsf{GL}(n)$, the general linear group of invertible matrices. This matrix action commutes with the action of the Steenrod squares, and the interplay between the two gives rise to a host of interesting algebraic problems.

One of these, the 'hit' problem, is a constant theme here. A polynomial f is 'hit' if there are polynomials f_k such that $f = \sum_{k>0} Sq^k(f_k)$. The hit polynomials form a graded subspace $\mathsf{H}(n)$ of $\mathsf{P}(n)$, and the basic problem is to find the dimension of the quotient space $\mathsf{Q}^d(n) = \mathsf{P}^d(n)/\mathsf{H}^d(n)$. We call $\mathsf{Q}(n)$ the space of 'cohits'. Since $\mathsf{P}(n)$ is spanned by monomials, $\mathsf{Q}(n)$ is spanned by their equivalence classes, which we refer to simply as 'monomials in $\mathsf{Q}(n)$'. A monomial whose exponents are integers of the form $2^j - 1$ is called a 'spike', and cannot appear as a term in a hit polynomial. It follows that a monomial basis for $\mathsf{Q}(n)$ must include all the spikes.

At a deeper level, the hit problem concerns the structure of $\mathsf{Q}(n)$ as a representation of $\mathsf{GL}(n)$ or $\mathsf{M}(n)$. We develop the tools needed to answer this in the 1- and 2-variable cases in Chapter 1. These include the 2-variable version of the maps introduced by Masaki Kameko to solve the 3-variable case, and a map which we call the duplication map. We hope that this opening chapter is accessible to graduate students and mathematicians with little or no background in algebraic topology, and that it will serve as an appetizer for the rest of the book.

Chapter 2 introduces a second family of Steenrod operations, the conjugate Steenrod squares $Xq^k : \mathsf{P}^d(n) \to \mathsf{P}^{d+k}(n)$. These are useful in the hit problem because of a device known as the 'χ-trick'. This states that the product of f and $Xq^k(g)$ is hit if and only if the product of $Xq^k(f)$ and g is hit. We use the χ-trick to prove that $\mathsf{Q}^d(n) = 0$ if and only if $\mu(d) > n$, where $\mu(d)$ is the smallest number of integers of the form $2^j - 1$ (with repetitions allowed) whose sum is d. This establishes the 1986 conjecture of Franklin P. Peterson which first stimulated interest in the hit problem.

Here is a rough guide to the structure of the rest of the book, in terms of three main themes: the Steenrod algebra A_2, the Peterson hit problem, and matrix representations. Volume 1 contains Chapters 1 to 15, and Volume 2 contains Chapters 16 to 30.

Chapters 3 to 5 develop A_2 from an algebraic viewpoint.

Chapters 6 to 10 provide general results on the hit problem, together with a detailed solution for the 3-variable case.

Chapters 11 to 14 introduce the Hopf algebra structure of A_2 and study its structure in greater depth.

Chapter 15 introduces the theme of modular representations by relating the hit problem to invariants and the Dickson algebra.

Chapters 16 to 20 develop the representation theory of $GL(n, \mathbb{F}_2)$ via its action on 'flags', or increasing sequences of subspaces, in an n-dimensional vector space $V(n)$ over $\mathbb{F}_2$.

Chapter 21 explores a fundamental relation between linear maps $V(m) \to V(n)$ and Steenrod operations, leading to a maximal splitting of $P(n)$ as a direct sum of A_2-modules.

Chapter 22 studies the A_2-summands of $P(n)$ corresponding to the Steinberg representation of $GL(n, \mathbb{F}_2)$.

Chapters 23 and 24 develop the relation between flag modules and the dual hit problem.

Chapters 25 and 26 study the hit problem for symmetric polynomials over $\mathbb{F}_2$.

Chapters 27 and 28 study the splitting of $P(n)$ as an A_2-module obtained using a cyclic subgroup of order $2^n - 1$ in $GL(n, \mathbb{F}_2)$.

Chapters 29 and 30 return to Peterson's original problem, with a partial solution of the 4-variable case.

The contents of Chapters 3 to 30 are summarized below in more detail.

In Chapter 3 we interpret the operations Sq^k as generators of a graded algebra A_2, subject to a set of relations called the Adem relations. The algebra A_2 is the mod 2 Steenrod algebra, and the operations Sq^k of Chapter 1 provide $P(n)$ with the structure of a left A_2-module. If $f = \sum_{k>0} Sq^k(f_k)$, then f can be reduced to a set of polynomials of lower degree modulo the action of the positively graded part A_2^+ of A_2. A monomial basis for $Q(n)$ gives a minimal generating set for $P(n)$ as an A_2-module. Thus the hit problem is an example of the general question of finding a minimal generating set for a module over a ring. The structure of A_2 itself is completely determined by its action on polynomials, in the sense that two expressions in the generators Sq^k are equal in A_2 if and only if the corresponding operations on $P(n)$ are equal for all n. For example, the results $Sq^1 Sq^{2k}(f) = Sq^{2k+1}(f)$ and $Sq^1 Sq^{2k+1}(f) = 0$ of Chapter 1 imply the Adem relations $Sq^1 Sq^{2k} = Sq^{2k+1}$ and $Sq^1 Sq^{2k+1} = 0$.

In Chapter 3 we also establish the two most important bases for A_2 as a vector space over $\mathbb{F}_2$. These are the admissible monomials in the generators Sq^k, due to Henri Cartan and Jean-Pierre Serre, and the basis introduced by John W. Milnor by treating A_2 as a Hopf algebra. As mentioned above, we represent a monomial by a 'block' whose rows are the reversed binary expansions of its exponents, and whose entries are integers 0 or 1. We use blocks to keep track of Steenrod operations on monomials. This 'block technology' and 'digital engineering' works well for the prime 2, and greatly facilitates our understanding of techniques which can appear opaque when expressed in more standard notation.

Chapter 4 begins with the multiplication formula for elements of the Milnor basis. This combinatorial formula helps to explain the ubiquity of the Milnor basis in the literature, as a product formula is not available for other bases of A_2. We also discuss the compact formulation of the Adem relations due to Shaun R. Bullett and Ian G. Macdonald. We use this to construct the conjugation χ of A_2, which interchanges Sq^k and Xq^k.

Chapter 5 provides combinatorial background for the algebra A_2, the hit problem and the representation theory of $\mathsf{GL}(n)$ over $\mathbb{F}_2$. Sequences of non-negative integers appear in various forms, and we distinguish 'finite sequences' from 'sequences'. A 'finite sequence' has a fixed number of entries, called its 'size', while a 'sequence' is an infinite sequence $R = (r_1, r_2, \dots)$ with only a finite number of nonzero terms, whose 'length' is the largest ℓ for which $r_\ell > 0$. However, a sequence R is usually written as a finite sequence $(r_1, \dots, r_n)$, where $n \geq \ell$, by suppressing some or all of the trailing 0s. The modulus of a sequence or a finite sequence is the sum of its terms. For example, the degree of a monomial is the modulus of its sequence of exponents. The set of all sequences indexes the Milnor basis of A_2.

For brevity, we call a sequence R 'decreasing' if $r_i \geq r_{i+1}$ for all i, i.e. if it is non-increasing or weakly decreasing. Thus a decreasing sequence of modulus d is a partition of d. Such a partition can alternatively be regarded as a multiset of positive integers with sum d. We discuss two special types of partition; 'binary' partitions, whose parts are integers of the form 2^j, and 'spike' partitions, whose parts are integers of the form $2^j - 1$.

We introduce two total order relations on sequences, the left (lexicographic) order and the right (reversed lexicographic) order, and two partial order relations, dominance and 2-dominance. The ω-sequence $\omega(f) = (\omega_1, \omega_2, \dots, \omega_k)$ of a monomial f is defined by writing f as a product $f_1 f_2^2 \cdots f_k^{2^{k-1}}$, where f_i is a product of ω_i distinct variables. In terms of blocks, $\omega(f)$ is the sequence of column sums of the block representing f, and the degree of f is $\omega_1 + 2\omega_2 + 4\omega_3 + \cdots + 2^{k-1}\omega_k$, the '2-degree' of $\omega(f)$. The set of decreasing sequences of 2-degree d has a minimum element $\omega^{\min}(d)$, which plays an important part in the hit problem, and is the same for the left, right and 2-dominance orders. We end Chapter 5 by relating this combinatorial material to the admissible and Milnor bases of A_2.

In Chapter 6 we return to the hit problem and introduce 'local' cohit spaces $\mathsf{Q}^\omega(n)$. A total order relation on ω-sequences of monomials gives a filtration on $\mathsf{P}^d(n)$ with quotients $\mathsf{P}^\omega(n)$. For the left and right orders, this passes to a filtration on $\mathsf{Q}^d(n)$ with quotients $\mathsf{Q}^\omega(n)$. A polynomial f in $\mathsf{P}^\omega(n)$ is 'left reducible' if it is the sum of a hit polynomial and monomials with ω-sequences $< \omega$ in the left order, and similarly for the right order. We prove the theorem of William M. Singer that $\mathsf{Q}^\omega(n) = 0$ if $\omega < \omega^{\min}(d)$ in the left order. We introduce the 'splicing' technique for manufacturing hit equations, extend the Kameko and duplication maps of Chapter 1 to the n-variable case, and determine $\mathsf{Q}^\omega(n)$ for 'head' sequences $\omega = (n-1,\dots,n-1)$ and 'tail' sequences $\omega = (1,\dots,1)$.

We begin Chapter 7 by proving that $\dim \mathsf{Q}^d(n)$ is bounded by a function of n independent of d. Thus only finitely many isomorphism classes of $\mathbb{F}_2\mathsf{GL}(n)$-modules can be realized as cohit modules $\mathsf{Q}^d(n)$. We extend splicing techniques and show that $\mathsf{Q}^\omega(n) = 0$ if ω is greater than every decreasing ω-sequence in the left order. A correspondence between blocks with decreasing ω-sequences and Young tableaux is used to define 'semi-standard' blocks (or monomials), and we show that $\mathsf{Q}^\omega(n)$ is spanned by such blocks when $\omega = \omega^{\min}(d)$.

In Chapter 8, we obtain reduction theorems for $\mathsf{Q}^\omega(n)$ when the sequence ω has a 'head' of length $\geq n-1$ or a 'tail' of length $\geq n$. It follows that $\dim \mathsf{Q}^d(n) = \prod_{i=1}^n (2^i - 1)$ for degrees $d = \sum_{i=1}^n (2^{a_i} - 1)$, when $a_i - a_{i+1} \geq i+1$ for $i < n$ and when $a_i - a_{i+1} \geq n-i+1$ for $i < n$. We complete a solution of the 3-variable hit problem by giving bases for $\mathsf{Q}^\omega(3)$ in the remaining cases.

The techniques introduced so far are useful for obtaining upper bounds for $\dim \mathsf{Q}^d(n)$, but are less efficient for obtaining lower bounds, where we may wish to prove that no linear combination of a certain set of monomials is hit. Chapter 9 introduces the dual problem of finding $\mathsf{K}^d(n)$, the simultaneous kernel of the linear operations $Sq_k : \mathsf{DP}^d(n) \to \mathsf{DP}^{d-k}(n)$ dual to Sq^k for $k > 0$. Here $\mathsf{DP}(n)$ is a 'divided power algebra' over $\mathbb{F}_2$, whose elements are sums of dual or 'd-monomials' $v_1^{(d_1)} \cdots v_n^{(d_n)}$. As a $\mathbb{F}_2\mathsf{GL}(n)$-module, $\mathsf{K}^d(n)$ is the dual of $\mathsf{Q}^d(n)$ defined by matrix transposition, and so $\dim \mathsf{K}^d(n) = \dim \mathsf{Q}^d(n)$. Thus we aim to find upper bounds for $\dim \mathsf{Q}^d(n)$ by using spanning sets in $\mathsf{Q}^d(n)$, and lower bounds by using linearly independent elements in $\mathsf{K}^d(n)$.

An advantage of working in the dual situation is that $\mathsf{K}(n)$ is a subalgebra of $\mathsf{DP}(n)$. Since the dual spikes are in $\mathsf{K}(n)$, they generate a subalgebra $\mathsf{J}(n)$ of $\mathsf{K}(n)$ which is amenable to calculation. In the cases $n = 1$ and 2, $\mathsf{J}(n) = \mathsf{K}(n)$, and when $n = 3$, $\mathsf{K}^d(n)/\mathsf{J}^d(n)$ has dimension 0 or 1. We explain how to construct the dual $\mathsf{K}^\omega(n)$ of $\mathsf{Q}^\omega(n)$ with respect to an order relation. We study the duals of the Kameko and duplication maps, and solve the dual hit problem for $n \leq 3$. In Chapter 10 we extend these results by determining $\mathsf{K}^d(3)$ and

$Q^d(3)$ as modules over $\mathbb{F}_2\mathsf{GL}(3)$. Here the flag module $\mathsf{FL}(3)$, given by the permutation action of $\mathsf{GL}(3)$ on subspaces of the defining module $\mathsf{V}(3)$, plays an important part. We describe tail and head modules in terms of the exterior powers of $\mathsf{V}(3)$.

Hopf algebras are introduced in Chapter 11. A Hopf algebra A has a 'coproduct' $A \to A \otimes A$ compatible with the product $A \otimes A \to A$, and an 'antipode' $A \to A$. We show that the coproduct $Sq^k \mapsto \sum_{i+j=k} Sq^i \otimes Sq^j$ and the conjugation χ provide the mod 2 Steenrod algebra A_2 with the structure of a Hopf algebra. For a graded Hopf algebra A of finite dimension in each degree, the graded dual A^* is also a Hopf algebra. In this sense, the divided power algebra $\mathsf{DP}(n)$ is dual to the polynomial algebra $\mathsf{P}(n)$. We show that the graded dual A_2^* of A_2 is a polynomial algebra on generators ξ_j of degree $2^j - 1$ for $j \geq 1$, and determine its structure maps. We conclude this chapter with the formula of Zaiqing Li for conjugation in A_2, which complements Milnor's product formula of Chapter 4.

Chapters 12 and 13 give more detail on the internal structure of A_2. In Chapter 12 we focus on two important families of Hopf subalgebras of A_2, namely the subalgebras of 'Steenrod qth powers' A_q, where q is a power of 2, and the finite subalgebras $\mathsf{A}_2(n)$ generated by Sq^k for $k < 2^{n+1}$. We also introduce some more additive bases of A_2. We continue in Chapter 13 by introducing a 'cap product' action of the dual algebra A_2^* on A_2, which can be used to obtain relations in A_2 by a process which we call 'stripping'. We use the 'halving' map (or Verschiebung) of A_2 to explain why its action on $\mathsf{P}(n)$ reproduces itself by doubling exponents of monomials Sq^A and squaring polynomials. This map sends Sq^k to 0 if k is odd and to $Sq^{k/2}$ if k is even. Since it is the union of the finite subalgebras $\mathsf{A}_2(n)$, the algebra A_2 is nilpotent. We apply the stripping technique to obtain the nilpotence order of certain elements of A_2.

Chapter 14 is devoted to a proof of the 2-dominance theorem of Judith H. Silverman and Dagmar M. Meyer. This deep result states that a monomial f in $\mathsf{P}^d(n)$ is hit if $\omega(f)$ is not greater than $\omega^{\min}(d)$ in the 2-dominance order. This strengthens the Peterson conjecture of Chapter 2 and the theorem of Singer from Chapter 6. One consequence is the Silverman–Singer criterion, which states that if g and h are homogeneous polynomials such that $\deg g < (2^k - 1)\mu(\deg h)$, where μ is the numerical function of Chapter 2, then $f = gh^{2^k}$ is hit.

In Chapter 15, we consider the Dickson algebra $\mathsf{D}(n)$ of $\mathsf{GL}(n)$-invariants in $\mathsf{P}(n)$. Following Nguyen H. V. Hung and Tran Ngoc Nam, we show that all Dickson invariants of positive degree are hit in $\mathsf{P}(n)$ when $n \geq 3$. There is a large class of similar problems: given a subgroup G of $\mathsf{GL}(n)$, the subalgebra of G-invariant polynomials $\mathsf{P}(n)^G$ is an A_2-module, and the 'relative' hit problem asks for the elements of $\mathsf{P}(n)^G$ which are hit in $\mathsf{P}(n)$. The corresponding 'absolute' hit problem asks for a minimal generating set for $\mathsf{P}(n)^G$. We consider the absolute problem for the Weyl subgroup $G = \mathsf{W}(n)$ of permutation matrices in $\mathsf{GL}(n)$ in Chapter 25.

In the chapters which follow, we shift attention to the representation theory of $\mathsf{GL}(n)$ over $\mathbb{F}_2$. We begin in Chapter 16 by studying the flag module $\mathsf{FL}(n)$, which is defined by the permutation action of $\mathsf{GL}(n)$ on the right cosets of the Borel subgroup $\mathsf{B}(n)$ of lower triangular matrices. This module is isomorphic to $\mathsf{Q}^d(n)$ when the degree d is 'generic' in the sense of Chapter 8. The Bruhat decomposition $A = BWB'$ of a matrix A in $\mathsf{GL}(n)$ is used to define certain subspaces of $\mathsf{FL}(n)$ which we call 'Schubert cells'. Here $B, B' \in \mathsf{B}(n)$ are lower triangular matrices and $W \in \mathsf{W}(n)$ is a permutation matrix. We show that $\mathsf{FL}(n)$ is the direct sum of 2^{n-1} submodules $\mathsf{FL}_I(n)$, where $I \subseteq \{1, 2, \ldots, n-1\}$ is the set of dimensions of the subspaces in the 'partial' flags given by right cosets of parabolic subgroups of $\mathsf{GL}(n)$.

The main aim of Chapter 17 is to construct a full set of 2^{n-1} irreducible $\mathbb{F}_2\mathsf{GL}(n)$-modules $L(\lambda)$. Following C. W. Curtis, $L(\lambda)$ is defined as the head of the summand $\mathsf{FL}_I(n)$ of $\mathsf{FL}(n)$, where λ is the 'column 2-regular' partition corresponding to I, i.e. $\lambda_i - \lambda_{i+1} = 1$ if $i \in I$, 0 if $i \notin I$. The summand of $\mathsf{FL}(n)$ corresponding to complete flags is the Steinberg module $\mathsf{St}(n)$. We use the Hecke algebra $\mathsf{H}_0(n)$ of endomorphisms of $\mathsf{FL}(n)$ which commute with the action of $\mathsf{GL}(n)$, and follow the methods of R. W. Carter and G. Lusztig.

In Chapter 18, we review the background from modular representation theory that we use to study $\mathsf{P}(n)$ and $\mathsf{DP}(n)$ as $\mathbb{F}_2\mathsf{GL}(n)$-modules. We explain the role of idempotents in obtaining direct sum decompositions, and introduce the Steinberg idempotent $e(n) = \overline{\mathsf{B}}(n)\overline{\mathsf{W}}(n) \in \mathbb{F}_2\mathsf{GL}(n)$, the sum of all products BW, where $B \in \mathsf{B}(n)$ and $W \in \mathsf{W}(n)$. We study $e(n)$ and the conjugate idempotent $e'(n) = \overline{\mathsf{W}}(n)\overline{\mathsf{B}}(n)$ by means of an embedding of $\mathsf{H}_0(n)$ in the group algebra $\mathbb{F}_2\mathsf{GL}(n)$ due to N. J. Kuhn. We also discuss Brauer characters and the representation ring $R_2(\mathsf{GL}(n))$.

In Chapter 19 we use idempotents in $\mathbb{F}_2\mathsf{GL}(n)$ to split $\mathsf{P}(n)$ as a direct sum of A_2-submodules $\mathsf{P}(n,\lambda)$, each occurring $\dim L(\lambda)$ times. We discuss the problem of determining the number of factors isomorphic to $L(\lambda)$ in a composition series for $\mathsf{P}^d(n)$. Following Ton That Tri, we use the Mui algebra of $\mathsf{B}(n)$-invariants in $\mathsf{P}(n)$ to determine the minimum degree d in which $\mathsf{P}^d(n)$ has a submodule isomorphic to $L(\lambda)$.

As Weyl modules and their duals are central topics of modular representation theory, it is no surprise that they appear here also. As these modules are defined over infinite coefficient fields, we begin Chapter 20 by reviewing some results on modular representations of the algebraic group $\overline{\mathsf{G}}(n)$ of nonsingular $n \times n$ matrices over $\overline{\mathbb{F}}_2$, the algebraic closure of $\mathbb{F}_2$. We then introduce the 'restricted' Weyl module $\Delta(\lambda, n)$ over $\mathbb{F}_2$ and its transpose dual $\nabla(\lambda, n)$, and show that if λ is column 2-regular and if the ordering on ω-sequences is suitably chosen, then $\Delta(\lambda, n) \cong \mathsf{K}^{\omega}(n)$ and $\nabla(\lambda, n) \cong \mathsf{Q}^{\omega}(n)$, where ω is the partition conjugate to λ. We use the theory of polynomial $\overline{\mathsf{G}}(n)$-modules to determine the minimum degree d in which $L(\lambda)$ occurs as a composition factor of $\mathsf{P}^d(n)$.

Chapter 21 gives a self-contained proof of an important result of J. F. Adams, J. Gunawardena and H. R. Miller. This states that every degree-preserving A_2-module map $\mathsf{P}(m) \to \mathsf{P}(n)$ is given by a sum of linear substitutions given by the action of $m \times n$ matrices over $\mathbb{F}_2$. It follows that the A_2-summands in a maximal splitting of $\mathsf{P}(n)$ obtained using idempotents in $\mathbb{F}_2\mathsf{M}(n)$ rather than $\mathbb{F}_2\mathsf{GL}(n)$ are indecomposable. Hence such a splitting is a maximal direct sum decomposition of $\mathsf{P}(n)$ as an A_2-module.

Chapter 22 is concerned with the A_2-summands of $\mathsf{P}(n)$ corresponding to the Steinberg representation $\mathsf{St}(n)$ of $\mathsf{GL}(n)$. We discuss the 'internal' model $\mathsf{MP}(n)$ of the A_2-module $\mathsf{P}(\mathsf{St}(n))$ defined by Stephen A. Mitchell and Stewart B. Priddy using admissible monomials of length n in A_2 itself. Although the hit problem for $\mathsf{P}(n)$ can be split into a corresponding problem for $\mathsf{P}(n, \lambda)$ for each λ, the Steinberg summand is the only case where this problem has been solved for all n, and we give minimal generating sets for the summands given by the idempotents $e(n)$ and $e'(n)$ of Chapter 18.

In Chapter 23, we identify the module $\mathsf{J}^d(n)$ of $\mathsf{K}^d(n)$ generated by the dual spikes in degrees $d = \sum_{i=1}^{n}(2^{a_i} - 1)$, where $a_1 > a_2 > \cdots > a_n$, in terms of the flag module $\mathsf{FL}(n)$. We show that $\mathsf{Q}^d(n) \cong \mathsf{FL}(n)$ in 'generic' degrees d, and extend the method in Chapter 24 to obtain results of Tran Ngoc Nam on $\mathsf{J}(n)$ relating cohit modules to partial flag modules. Following Nguyen Sum, we give counterexamples for $n \geq 5$ to Kameko's conjecture that $\dim \mathsf{Q}^d(n) \leq \dim \mathsf{FL}(n)$ for all d.

In Chapters 25 and 26 we discuss the hit problem for the action of A_2 on the algebra of symmetric polynomials $\mathsf{S}(n)$, the invariants in $\mathsf{P}(n)$ of the group $\mathsf{W}(n)$ of permutations of the variables. More generally, we discuss the 'absolute' hit problem for any subgroup G of $\mathsf{W}(n)$, and show that the Peterson conjecture, the Kameko map and Singer's minimal spike theorem have analogues for $\mathsf{P}(n)^G$. We solve the symmetric hit problem for $n \leq 3$, using the dual problem to obtain the lower bound in the case $n = 3$. Following Singer, we introduce the 'bigraded Steenrod algebra' $\widetilde{\mathsf{A}}_2$, which is obtained by omitting the relation $Sq^0 = 1$ in the definition of A_2, and apply $\widetilde{\mathsf{A}}_2$ to the dual problem.

In Chapters 27 and 28 we consider the cyclic subgroup $\mathsf{C}(n)$ of order $2^n - 1$ in $\mathsf{GL}(n)$. This is obtained by regarding $\mathsf{P}^1(n)$ as the underlying vector space of the Galois field $\mathbb{F}_{2^n}$. The action of $\mathsf{C}(n)$ on $\mathsf{P}(n)$ is diagonalized over $\mathbb{F}_{2^n}$ by a change of variables which 'twists' the action of A_2, in the sense that $Sq^1(t_i) = t_{i-1}$ in the new variables, which are indexed mod n. Following H. E. A. Campbell and P. S. Selick, we show that the polynomial algebra $\widetilde{\mathsf{P}}(n) = \mathbb{F}_2[t_1,\ldots,t_n]$ splits as the direct sum of $2^n - 1$ A_2-modules $\widetilde{\mathsf{P}}(n,j)$ corresponding to the 1-dimensional representations of $\mathsf{C}(n)$. In particular, $\widetilde{\mathsf{P}}(n,0)$ can be identified with the ring of $\mathsf{C}(n)$-invariants of $\mathsf{P}(n)$. In Chapter 28, we solve the dual cyclic hit problem for $n = 3$ by using the twisted analogue $\widetilde{\mathsf{J}}(n)$ of the d-spike module $\mathsf{J}(n)$.

In Chapters 29 and 30, we collect some results on the hit problem in the 4-variable case as further illustration of our methods. Nguyen Sum has extended the method introduced by Kameko to find a monomial basis for $\mathsf{Q}^d(4)$ for all d. We include without proof some of the results of Sum, and also some results which we have verified only by computer using MAPLE. Thus there remain some challenging aspects of the hit problem even in the case $n = 4$.

The Steenrod algebra was originally defined for all primes p, but we have restricted attention to the case $p = 2$. All the problems have analogues for odd primes, but in general much less is known, and a number of difficulties arise in trying to extend our techniques to the odd prime case. The 2-variable hit problem for the action of A_p on the polynomial algebra $\mathbb{F}_p[x,y]$ has been solved by Martin D. Crossley, but little appears to be known even for the 3-variable case. In common with many authors on the Steenrod algebra, we have therefore confined ourselves to the prime 2.

There are several good textbooks on topology which include material on the Steenrod algebra and its applications, such as those by Brayton I. Gray [70] and by Robert E. Mosher and Martin C. Tangora [147], in addition to the classic Annals of Mathematics Study [196], based on lectures by Steenrod himself, and the Cartan seminars [33]. A treatment of the Steenrod algebra from an algebraic viewpoint, including Steenrod operations over an arbitrary finite field, is given in Larry Smith's book [190] on invariant theory. The book of Harvey R. Margolis [129] treats the general theory of modules over the Steenrod algebra. Still other approaches to the Steenrod algebra are possible. The survey article [233] treats Steenrod operations as linear differential operators with polynomial coefficients, and is a precursor for this book.

We sometimes introduce definitions and constructions for a small number of variables and extend them to the general case in later chapters. Although this can involve a certain amount of repetition, it has the advantage of leading to interesting results at an early stage by elementary methods. We hope that our approach will appeal to readers whose main interests are in algebra, especially in the modular representation theory of linear groups, or in the combinatorics related to symmetric polynomials and to the invariant theory of finite groups.

In order to avoid interruptions to the text, citations and background material are collected in the 'Remarks' sections at the end of each chapter. The occasional reference to topology may occur in these, but we have not tried to explain the topology. We have also omitted important topics such as the Singer transfer map and its applications to the homotopy groups of spheres through the Adams spectral sequence. These would require another volume, which we are not qualified to write. For similar reasons, we do not treat the theory of analytic functors and the category $\mathcal{U}/\mathcal{N}il$ of unstable A_2-modules modulo nilpotent objects due to Hans-Werner Henn, Jean Lannes and Lionel Schwartz. Finally, in a subject which crosses several disciplines, notation presents a problem because the traditional symbols of one area may be in conflict with those of another. A list of symbols for the main ingredients of our subject appears at the end of the book, together with an index of the main terms defined in the text.

We should like to offer our sincere thanks to the School of Mathematics of the University of Manchester for providing us with office space and computing facilities during work on this text. We should also like to thank several colleagues at Manchester for mathematical help and support, and in particular Peter Eccles, Nige Ray, Peter Rowley and Bob Sandling. Of colleagues farther afield, we should like to thank Nguyen H. V. Hung, Ali S. Janfada, Bill Singer and Larry Smith for their interest in our project. The first author would also like to thank Stephen R. Doty for teaching him something about modular representation theory. Finally we should like to thank Roger Astley and his colleagues at Cambridge University Press for their encouragement, support and patience. We set out with the modest aim of providing a beginning graduate student in topology and algebra with a basic primer on the Steenrod algebra, illustrated by our favourite application to a problem proposed by Frank Peterson, but the project has expanded substantially in scope over the past eight years.