Merit
Students
Encyclopedia

4

VOLUME 4

MERIT STUDENTS ENCYCLOPEDIA

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ALPHABETICAL ARRANGEMENT OF ENTRIES

The entries in the *Merit Students Encyclopedia* are arranged in a simple alphabetical order. The method of arrangement combines elements of the system used in most dictionaries with that used in telephone directories. Each entry begins with a heading in dark type. Some of these headings contain a comma; others do not. The basic principles of arrangement are listed below, including rules for placement of identical headings.

The alphabetical sequence is letter by letter.

air
air conditioning
aircraft
aircraft carrier
aircraft landing system
airedale terrier
airfoil

When headings contain words out of their usual order, a comma is used to indicate the change of order, as in

Alaska, University of
Alba, Duke of
Alger, Horatio

Such entry headings are arranged in alphabetical sequence only up to the comma.

Bryansk
Bryant, William Cullen
Bryant College

When words preceding a comma are the same in two or more consecutive entries, the order is determined by the arrangement of the letters following the comma.

Brooks, Phillips
Brooks, Van Wyck

When two or more entries have the same heading, the entries are placed in the following order: persons, places, things.

Hannibal (person)	**Hercules** (person)	**Phoenix** (place)
Hannibal (place)	**Hercules** (constellation)	**phoenix** (bird)

Rulers with identical names are listed alphabetically by the name of the territory ruled. Rulers with the same name and same realm are listed according to dates of reign.

Frederick IX (of Denmark)
Frederick I (of Holy Roman Empire)
Frederick II (of Holy Roman Empire)
Frederick II (of Prussia)

Popes are listed by dates of reign, and they precede rulers of the same name.

Paul VI (Pope)
Paul I (Emperor of Russia)

Other persons with identical names are listed according to date of birth.

Butler, Samuel (born 1612)
Butler, Samuel (born 1835)

Places with identical names are listed according to the importance of the political unit, in descending order.

New Brunswick (Canadian province)
New Brunswick (U.S. city)

When places of the same political unit have identical names, they are arranged alphabetically by location. Cities in the United States and Canada are always located in reference to states or provinces. Cities elsewhere are usually located in reference to countries.

Abilene (Kansas)	**Abydos** (Egypt)
Abilene (Texas)	**Abydos** (Turkey)

Things with identical names are arranged alphabetically according to the subject in which they are classified.

aberration, in astronomy
aberration, in optics

GUIDE TO PRONUNCIATION

Pronunciations in *Merit Students Encyclopedia* appear in parentheses following entry headings. Heavy and light stress marks are used after syllables to indicate primary and secondary accents. A heavy stress mark is used in words that contain one primary accent, such as **comet** (kom′it). Both heavy and light stress marks are used in words that have secondary as well as primary accents, as in **communication** (kə mū′ nə kā′ shən). When two or more entries have exactly the same pronunciation, as with Paris the mythological hero and Paris the French city, the pronunciation is given only with the entry that appears first. Where possible, letters of the standard alphabet are used as symbols in the pronunciation system in preference to less familiar symbols. The symbols used are shown below with some words in which their sounds appear.

a	hat, cap
ā	age, face
ã	care, air
ä	father, far
b	bad, rob
ch	child, much
d	did, red
e	let, best
ē	equal, see
ėr	term, learn
f	fat, if
g	go, bag
h	he, how
i	it, pin
ī	ice, five
j	jam, enjoy
k	kind, seek
l	land, coal
m	me, am
n	no, in
ng	long, bring
o	hot, rock
ō	open, go
ô	order, all
oi	oil, voice
ou	house, out
p	paper, cup
r	run, try
s	say, yes
sh	she, rush
t	tell, it
th	thin, both
ŦH	then, smooth
u	cup, butter
u̇	full, put
ü	rule, move
ū	use, music
v	very, save
w	will, woman
y	young, yet
z	zero, breeze
zh	measure, seizure

ə represents:
- a in about
- e in taken
- i in April
- o in lemon
- u in circus

In pronunciations for entries describing foreign persons and places it is sometimes necessary to represent sounds that are not used in English. Such foreign sounds are represented by four special symbols, which are listed below. Each symbol is accompanied by a brief indication of how the sound it represents is produced.

Y as in French *du*. Pronounce ē with the lips rounded as for English ü in rule.

œ as in French *peu*. Pronounce ā with the lips rounded as for ō.

N as in French *bon*. The N is not pronounced but shows that the vowel before it is nasal.

H as in German *ach*. Pronounce k without closing the breath passage.

CAISSON to CLOISONNÉ

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caisson (kā′sən), a large, strong, boxlike or cylindrical structure used for laying foundations underwater. Caissons are made of steel, concrete, or wood. They act as molds for concrete foundations or as protective shells within which workers can excavate and lay foundations. The term "caisson" is sometimes also applied to open-ended tubes used for laying foundations on land. The tubes are sunk into the ground, and the soil within them is removed, creating deep pits. The pits are then filled with concrete to form foundations.

For such underwater structures as bridge piers, the foundations of seawalls, breakwaters, wharves, jetties, and other similar structures, three types of caissons are used: the box, the open, and the pneumatic caisson.

Box Caisson. A box caisson, sometimes called a floating caisson, is a large watertight box, open at the top. It is usually floated to the construction site where it is sunk by being partly loaded with concrete. Once the caisson has sunk into position, it is filled with concrete to form a pier or foundation. The caisson remains as the outer shell of the structure. The box caisson is used only on solid bottoms that have been prepared to receive it and give it firm footing.

Open Caisson. Open caissons are used in soft muddy or sandy water beds. They are sunk through the mud or sand to the underlying rock. An open caisson is a box that is open at the top and bottom and is braced inside with many intersecting walls. The intersecting walls have vertical open spaces, called dredging wells, between them. Another kind of open caisson usually consists of a solid block of reinforced concrete pierced by many vertical tunnels, also called dredging wells.

The sharp bottom rim of the caisson is designed to cut into the water bed and is often faced with structural steel. As the caisson sinks into the water bed, the mud and sand gouged loose by the bottom rim are pumped out through the dredging wells or are lifted out by hoisting devices. When the caisson has sunk to the solid rock, the exposed surface of the rock is cleaned and concrete is deposited on it, forming a sealed bottom for the caisson. Water is then pumped out of the wells, and they are filled with concrete.

Pneumatic Caisson. When the caisson must be sunk through hard, rocky material, a pneumatic caisson is used. It consists of a frame or shell with an airtight roof, or bulkhead, about 7 feet above the bottom cutting rim. A box caisson can sometimes be converted into a pneumatic caisson by the addition of a bulkhead. Air is pumped under pressure into the space under the bulkhead, forcing out the water and creating a chamber for workers. The workers enter this pressurized chamber through a tube, and the rock and

The top sections of pier caissons are precast before being sunk to form foundations.

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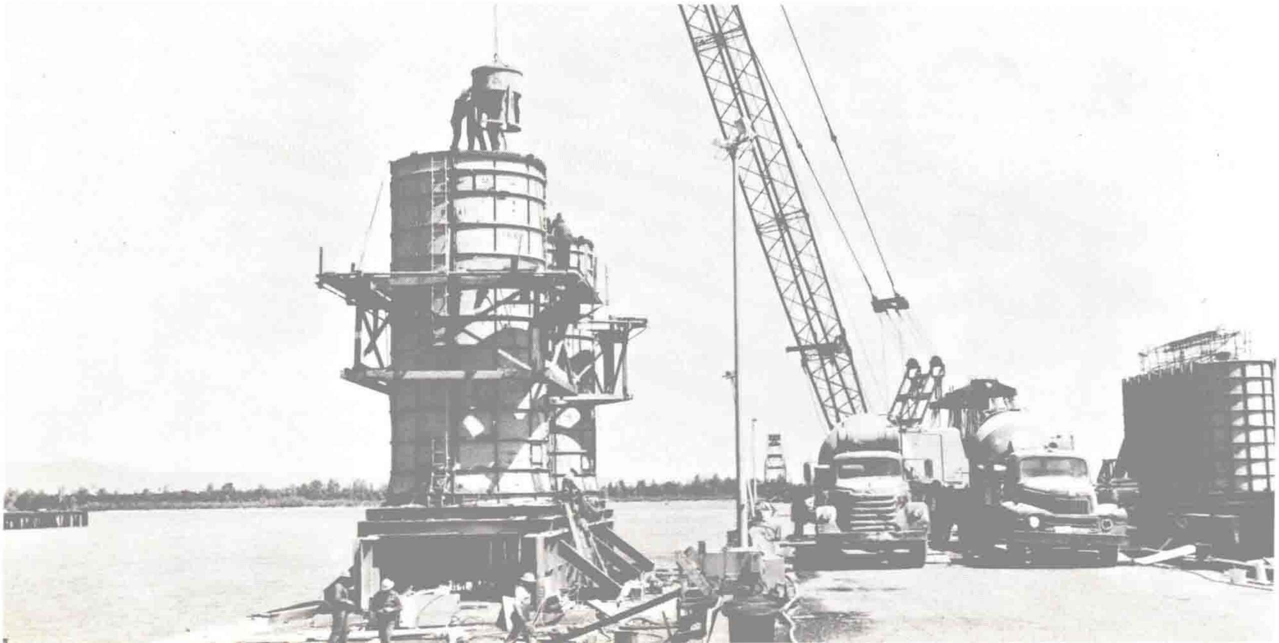
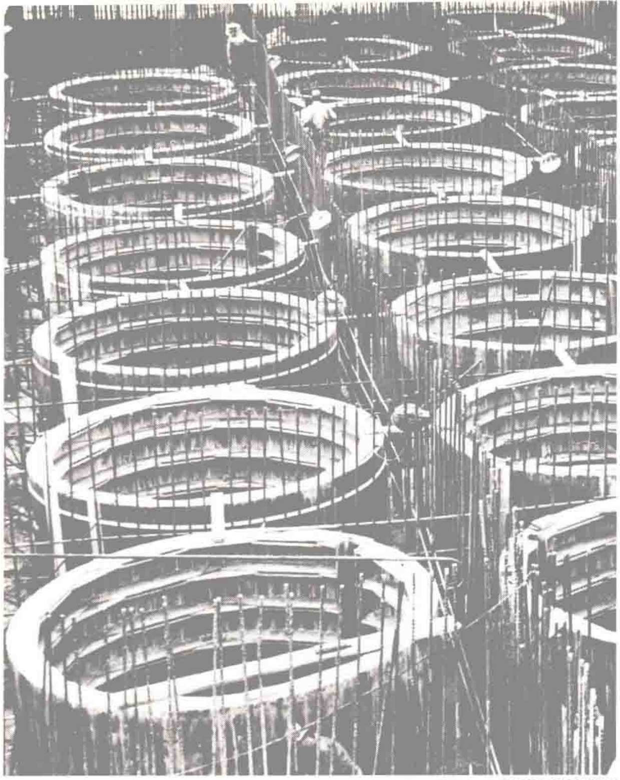




EWING GALLOWAY

Steel caissons are filled with concrete to form part of the foundations of buildings or of underwater structures.

soil they dig up is removed through another tube. Both tubes are equipped with air locks so that men and materials can enter and leave the chamber.

An air lock has two airtight doors. The outer door, which opens to the outside air, is hinged to swing into the air lock and cannot be opened when the air lock is pressurized. The inner door, which leads to the pressurized working chamber, swings into the chamber and can be opened only when the air pressure in the air lock has been raised to equal the air pressure in the chamber.

Entering the working chamber is done in several stages. First the workers go into the air lock and shut the outer door behind them. Then air is pumped into the air lock. Finally, when the air pressure in the air lock equals the air pressure in the working chamber, the workers open the inner door and enter the working chamber. To leave the chamber, the workers enter the air lock and shut the door behind them. The pressure is then reduced, and when it is equal to the outside air pressure, the outer door can be opened, permitting the men to leave the caisson.

The pressure in the air lock must be lowered slowly to prevent workers from suffering attacks of caisson disease, also called the bends. The disease is a painful and sometimes fatal condition in which nitrogen bubbles form in the blood and body tissues.

William K. Fallon

caisson disease. *See* BENDS.

Caithness-shire (kāth′nes shər), a former county in northeastern Scotland, separated from the Orkney Islands by the Pentland Firth, with the North Sea on its east coast and the Atlantic Ocean to its north. John O'Groats, in Caithness-shire, is often said to be the northernmost point on the British mainland, but Dunnett Head on a nearby promontory extends further north. Caithness-shire, or Caithness, is mountainous in the south and west, and slopes to a low, treeless plain in the north. Its coastline on the Atlantic Ocean is rocky, but on the North Sea it is sandy. The area is drained by the rivers Thurso and Wick.

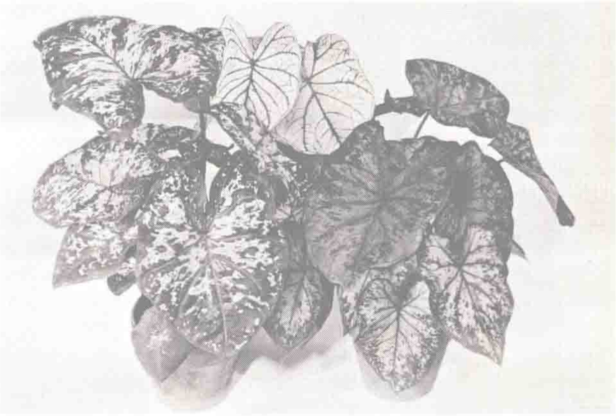
Sheep, oats, and turnips are raised in Caithness, and stone is quarried there. Fishing is carried on along the coast. Thurso and Wick, the former county town, are the largest communities. In 1975, when the Scottish local government was reorganized, Caithness became part of the region of Highland.

Cajuns. *See under* ACADIA.

Calabria (kə lā′bri ə), *a region in southern Italy; bordering the Ionian and Tyrrhenian seas and the Strait of Messina. Area about 5,821 square miles (15,076 sq km). Pop. (1974 est.) 2,023,000.*

Calabria is made up of the three provinces of Catanzaro, Cosenza, and Reggio di Calabria and forms the southernmost part of the Italian mainland. It is a rugged mountainous area where the Apennines rise to elevations of more than 7,300 feet (2,200 meters).

Farming is the principal economic activity in Calabria. Reggio di Calabria is the largest city and the regional capital. Calabria became part of the kingdom of Naples in the late 13th century. In 1861 it was incorporated into Italy. **Norman J. G. Pounds*



A. B. MORSE

The caladium is a small tropical American plant.

caladium (kə lā′di əm), any of a group of small tropical American plants with beautiful foliage. Caladiums may grow 24 inches (61 cm) tall and have heart-shaped or spade-shaped leaves about 8 inches (20 cm) long. The leaves are various shades of green and have red, pink, violet, or yellow patterns. Tiny greenish blossoms grow on spikes, which are enclosed by leaflike spathes. Many caladiums are cultivated in pots and window boxes, and in mild North American climates two species (*Caladium bicolor* and *C. picturatum*) are frequently cultivated as garden plants.

Caladiums are classified as genus *Caladium* of the family Araceae (arum). Perennial. **Reed C. Rollins*

Calais (ka lā′; British, kal′ā or kal′is), *a city in northern France; in the region of Artois; on the Strait of Dover. Pop. (1975) 78,820.*

Calais is a seaport and a terminus for ferryboats that operate across the strait to Dover and Folkestone in England. Calais is closer to England than any other city on the European continent and is only 26 miles (42 km) from Dover. The chief exports of

Calais include woven goods, glassware, and metalware. Petroleum, timber, and raw wool are imported. Fishing is a major occupation of the inhabitants of the city. Fishing boats, telephone cables, chemicals, and textiles, notably lace and tulle, are made there.

Originally a small fishing village, Calais was fortified by the counts of Boulogne in the 13th century because of its strategic location as the nearest point in France to England. The English captured Calais in 1347 and held the city until 1558, when it was retaken by the French. The harbor facilities and much of the city were severely damaged in World War II.

**Norman J. G. Pounds*

calamine. *See* HEMIMORPHITE.

Calamity Jane (kə lam′ə ti), *American frontierswoman. Born Martha Jane Canary, at Princeton, Mo., about 1852. Died Deadwood, S.D., Aug. 1, 1903.*

Calamity Jane has become a legend in the history of the American frontier. Regarding herself as the equal of any man, she refused to live under the 19th-century code of acceptable feminine behavior. She dressed in men's clothing and became an excellent shot and horsewoman. Many events in Calamity Jane's life are either unknown or fictionalized. Her nickname is believed to have originated from her threats that calamity would befall any man who offended her.

Reared in frontier mining towns in Utah, Wyoming, and Montana, Calamity Jane went to the Dakota territory in 1876, during the Black Hills gold rush. She may have been a mail carrier for the U.S. Post Office while living in Deadwood. According to her own account, she was also a pony express rider and a scout for generals George Custer and Nelson Miles.

Calamity Jane considered herself the equal of any man.

STATE HISTORICAL SOCIETY OF COLORADO

Toward the end of her life, Calamity Jane lived in poverty. To earn money, she toured the West with a performing troupe and sold her autobiography and self-portraits. Calamity Jane is buried in Deadwood, next to the grave of Wild Bill Hickok.

**Robert V. Remini*

calamus. *See* SWEET FLAG.

calcite (kal′sīt), *a calcium mineral. Formula* $CaCO_3$. *Glassy to earthy luster. White or colorless, sometimes tinted gray, green, red, blue, or yellow. Transparent to translucent. Hardness 3. Specific gravity 2.72.*

Calcite, a form of calcium carbonate, is one of the most common minerals. It is the chief constituent of limestone, marble, and chalk. Calcite is often deposited by spring water as travertine and Mexican onyx, and forms the stalagmites and stalactites found in caves. Iceland spar is a pure form of calcite.

Calcite, in the form of limestone, is used to make cement, quicklime, and mortar. Marble and some limestones are used as building materials. In the form of chalk, calcite is used in the manufacture of blackboard crayons, putty, and rubber goods.

**Cornelius S. Hurlbut, Jr.*

calcium (kal′si əm), *a chemical element. Symbol Ca. First prepared in pure form in 1808 by Sir Humphry Davy (British). Melting point 842–848° C. (1550–1560° F.). Boiling point 1487° C. (2709° F.). Oxidation number +2. Atomic weight 40.08. Atomic number 20.*

Calcium is a malleable, ductile, silvery-white metal. Although it never occurs uncombined in nature, it is the fifth most abundant element and the third most abundant metal, accounting for 3.6 percent of the earth's crust. It is obtained by passing an electric current through molten calcium chloride.

Calcium is essential for the growth of bones and teeth, as well as for plant growth. It is used in bearing-metal alloys, for making steel, and for refining chromium, uranium, and other metals. Because it absorbs gases, it is used to remove the last trace of gas from vacuum tubes.

Calcium compounds are numerous, relatively inexpensive, and commercially important. Calcium carbonate, $CaCO_3$, the most abundant calcium compound, is found in limestone, marble, gypsum, Iceland spar, and other minerals. It is used to make portland cement and various kinds of plaster and mortar.

Other calcium compounds are used in drying agents, paint pigments, disinfectants, photographic chemicals, textile dying and printing solutions, baking powders, bleaches, and other products. **Alfred B. Garrett*

calcium chloride (klô′rīd), a chemical compound of calcium and chlorine. Formula $CaCl_2$. It is a colorless, crystalline solid at room temperature, and it is highly soluble in water. In addition to the anhydrous form, calcium chloride also exists in several hydrous forms: $CaCl_2 \cdot H_2O$, $CaCl_2 \cdot 2H_2O$, and $CaCl_2 \cdot 6H_2O$.

Calcium chloride is deliquescent, which means that it absorbs water from the air to such an extent that it dissolves in the water it collects. Because of this property, calcium chloride is sprinkled on unpaved roads and in mines, where the water it absorbs helps to keep down dust. It is also used as a drying agent in chemical analysis and as an antifreeze. **Alfred B. Garrett*

calculus (kal′kū ləs), a branch of mathematics that deals with continuously changing quantities, such as the position of a point as it traces out a mathematical curve. The calculus can be applied to innumerable areas of physics and engineering. Its development in the 17th century allowed the solution of many problems that had been insoluble by the methods of arithmetic, algebra, and geometry. Among these problems were the determination of the laws of motion and the theory of electromagnetism.

The calculus consists of two main branches: differential calculus and integral calculus. Differential calculus deals with the rates at which quantities change. Integral calculus develops methods for finding the areas enclosed by curved boundaries, a problem that has a wide variety of applications in other branches of science.

In both the main divisions of calculus, two concepts are fundamental: function and limit. Functions and limits must be thoroughly understood before the calculus can be attempted.

Functions

A function is an association between the elements of two sets of numbers. For example, for every value of x in the equation $y = 2x$, there is a corresponding value of y. That is, if $x = 1$, $y = 2$. If $x = 2$, $y = 4$. If $x = 2\pi r$, then $y = 4\pi r$. All the possible values that x may take on are members of a set called the domain. All the possible values that y may take on are members of a set called the range. Since y always equals twice as much as x, every element in the range has twice the value of its corresponding element in the domain. There is thus a definite association between the elements of these two sets, and that association is called a function. In mathematical terminology, this is written

$$y = f(x),$$

which is read: "y equals f of x."

The expression $f(x)$ is a general expression, showing that y is a function of x but not saying exactly what that function is. For example, y may equal $2x$, $x^3 + 5x - 3$, or any other algebraic expression containing x.

Since x and y may take on the value of any element in their sets, they are called variables. In general, a variable assumes a number of values in a given problem. For example, if a car travels 50 miles an hour, the distance it travels changes constantly. Distance is thus a variable. The time of travel also changes, and thus time is another variable. Since the distance covered depends on the time of travel, distance is called the dependent variable and time the independent variable. Distance is thus a function of time.

Many other relationships in nature and in mathematics may be expressed by functions and variables. A boy grows larger with the years. Both his height and his weight may therefore be expressed as functions of time. A coal mine becomes hotter as one descends. Its temperature may therefore be expressed as a function of depth. A mathematical curve may take on new values of y as x changes. Thus y is a function of x.

All of the functions mentioned may be represented visually on a graph. Fig. 1 shows a graph of the function $y = x^2$. It is customary to graph the domain

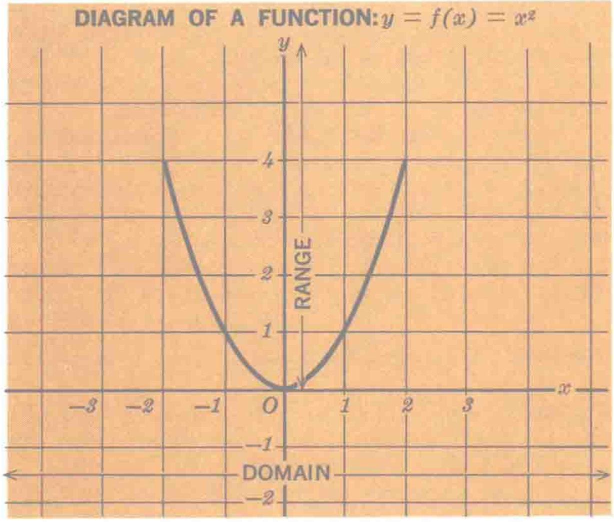


Fig. 1. Graphs of functions are often able to illustrate properties of the function that are not immediately evident from the equation alone. For example, a graph of the parabola $y = f(x) = x^2$ shows at once that y can never take on a negative value.

of the function (the values taken on by x) along a horizontal axis, while the range is graphed along a vertical axis. The domain and the range are marked in the figure. Note that the domain consists of the set of all real numbers, but that the range is the set of all nonnegative real numbers.

Within the area covered by this graph, every point whose x and y values satisfy this function ($y = x^2$) lies on the curve. Also, every point on the curve has x and y coordinates that satisfy the function. The curve is thus an accurate "picture" of the function, showing some of its properties at a glance.

Limits

The idea of a limit is the second fundamental concept of the calculus. An example of a limit is the sum of the following series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdot\cdot\cdot$$

No restriction is placed on the number of terms in the series. Any number of terms may be added so long as each term is one-half the preceding term. Yet the sum of this unending series has a limit, as indicated by the following table:

Number of terms	Sum of terms
1	0.5
2	0.75
3	0.875
4	0.9375
5	0.96875
6	0.984375
7	0.9921875
8	0.99609375
9	0.998046875
10	0.9990234375

The sum seems to be approaching the number 1 as a limit. That the limit is indeed exactly 1 may be seen by considering the sequence of terms obtained by starting out with a pie and repeatedly giving away

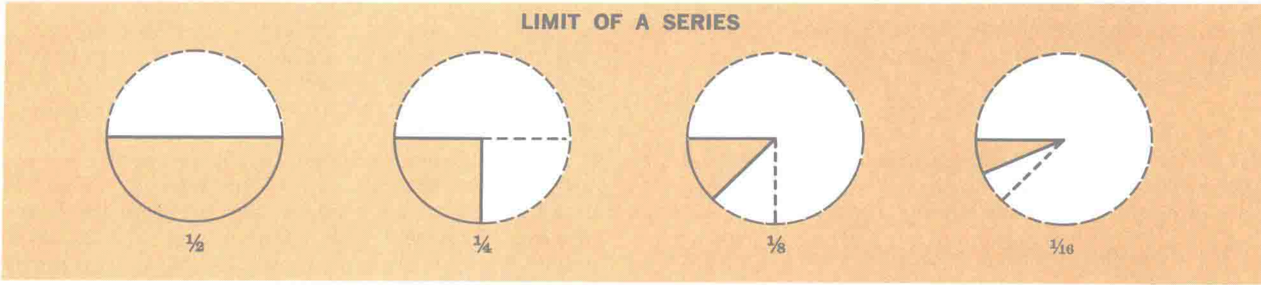


Fig. 2. If a boy eats half a pie (the unshaded area in first circle), then eats half of the other half, and so on, he can never finish it. The most he can eat is the mathematical limit of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2n}$, where n is the number of times the boy cuts the pie.

half of what is left. (See Fig. 2.) Half of the pie is $\frac{1}{2}$; half of the remaining half is $\frac{1}{4}$; half of the remaining quarter is $\frac{1}{8}$; and, in general, the successive terms of this sequence are identical with the terms of the first series. Thus the two series are the same. In the case of the sequence generated by cutting up the pie, it is obvious that the sum of the amounts given away cannot exceed 1, since there were no extra pies or fractions of a pie to start with. Therefore, the sum of the first series cannot exceed 1.

It can also be shown that the sum never actually reaches the value of 1. No matter how small a piece of the pie is left, giving away half of it will still leave a tiny segment that is not included in the sum. However, the sum can be made to approach the limit of 1 as closely as is desired, simply by taking a sufficiently large number of terms.

Limit of a Function. In the above series there is a definite relationship between the number of terms to be added and the sum of these terms. If the number of terms is 2, then their sum is 0.75; if the number of terms is 5, then their sum is 0.96875. Thus for any selected number of terms, there is associated exactly one number corresponding to their sum. This relationship satisfies the definition of function. The sum of the terms in the series is therefore a function of the number of terms. Thus, if S represents the sum and n is the number of terms, the function may be written

$$S = f(n).$$

As we have seen, if n, the number of terms, is increased without bound, the sum S approaches a limiting value of 1. This may be expressed mathematically as follows:

$$\lim_{n \to \infty} f(n) = 1,$$

which is read "The limit of f of n as n approaches infinity is 1." The symbol ∞ is used to indicate the absence of any upper bound on the value of n. This absence of a bound is indicated by the name "infinity."

In the series above, the sum S may take on a number of values depending on the value of n. S may thus be regarded as the dependent variable and n as the independent variable. The limit of the function $S = f(n)$ is therefore the value that the function approaches as n, the independent variable, approaches infinity.

In one respect, this function is unlike the majority of functions treated by the calculus: it "jumps" from one value to the next. Such a function is called *discontinuous*, and its graph, in this case, consists of a set of unconnected points. The calculus usually treats *continuous* functions, whose graphs are unbroken curves. However, for both types of functions, the concept of the limit is essentially the same.

For example, consider the limit of the continuous function $f(x) = \frac{1}{x}$, as x approaches infinity. (The domain of this function is the positive real numbers.) As x increases, the value of the function decreases continuously. (See Fig. 3.) In fact, no matter how small a positive number is chosen, the value of the function can be made still smaller, simply by choosing x large enough. Thus the limit of this function is zero:

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

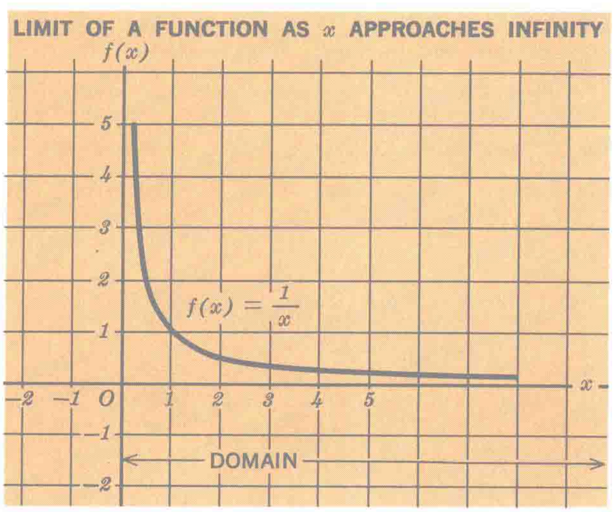


Fig. 3. The graph helps in showing that the limit of the function $f(x) = \frac{1}{x}$, as x increases without bound, is zero.

In both of the examples above, the independent variables were allowed to increase without bound, or to approach infinity. However, in other problems the independent variable may approach some value other than infinity, such as zero or some constant value. In general, then, *the limit of any function* f(x) *is the value that the function approaches as* x *approaches some selected value* a. This may be expressed mathematically as follows:

$$\lim_{x \to a} f(x) = L,$$

which is read "The limit of f of x, as x approaches a, is L." An example of a function that has a limit when x approaches a constant value is the following:

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

This function is not defined when $x = 1$, because its denominator is zero at that point, and division by zero has no meaning. However, the function may be factored into the following expression: $\frac{(x+1)(x-1)}{x-1}$. Now as long as x is not equal to 1, the expression $(x - 1)$ is not equal to zero. Moreover, since any nonzero number divided by itself equals 1, the function equals $x + 1$ for all values of x not equal to 1.

As x comes closer and closer to 1, the function comes closer and closer to a value of 2. In fact, the function can be made to come arbitrarily close to the value of 2, simply by choosing a value of x equally close to 1. This answers the definition of the limit:

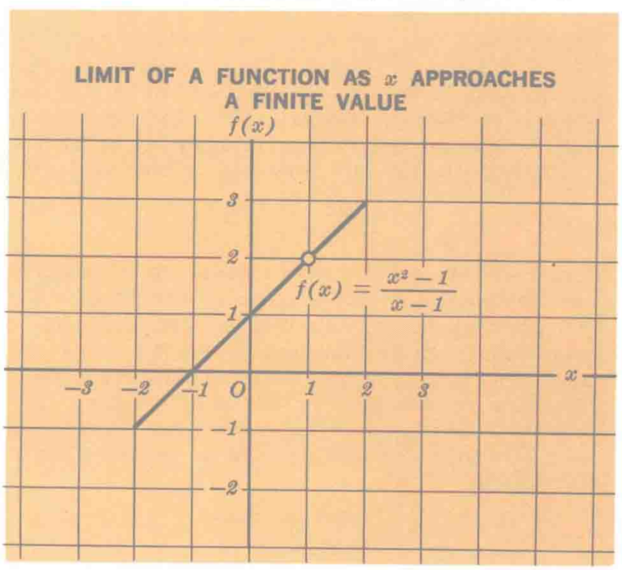
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Fig. 4 illustrates this function, with an empty circle at the point where the function is undefined. The distance of the circle, measured from the x-axis, is equal to the limit of the function.

Differential Calculus

Rate of Change. The differential calculus is largely concerned with finding the rate of change of functions. Rates of change are important in many problems of physics and mathematics. The velocity of a moving object, for example, is the rate of change of the object's distance with respect to time. If the function relating distance to time is known, it is usually a simple matter to determine the velocity at any time by means of differential calculus. However, velocity is not the only kind of rate of change. Acceleration, for example, is the rate of change of velocity with respect to time, and mathematicians and physicists find it necessary to study countless other rates of change. One of the most interesting rates of change is the slope, or steepness, of a curve, which makes it possible to visualize the rate of change of any continuous function.

Fig. 4. The function is not defined at the point $x = 1$ because its equation would reduce to $\frac{0}{0}$. The empty circle indicates that the function has no value at that point. However, the graph helps to show that the limit of the function, as x approaches 1, is 2.



Average Slope of a Curve. In Fig. 5a, as the boy travels from the first tree (T_1) to the next (T_2), there is a large change in his vertical distance from the base of the hill, but only a small change in his horizontal distance. Between these points the hill is steep and is said to have a large average slope. In other words, there is a high rate of change of vertical distance with respect to horizontal distance.

As the boy travels along the nearly level portion between the third and fourth trees (T_3 and T_4), however, he goes a long distance forward but only a short distance upward. Thus, there is a low rate of change in vertical distance with respect to horizontal distance, and the average slope of the hill between these points is said to be small.

In a similar way the slope of a mathematical curve may be imagined as a measure of the steepness of the curve. However, the mathematical definition of slope is more precise than the everyday usage of the term. The average slope between any two points T_1 and T_2 is defined as the ratio of the change in y (the vertical

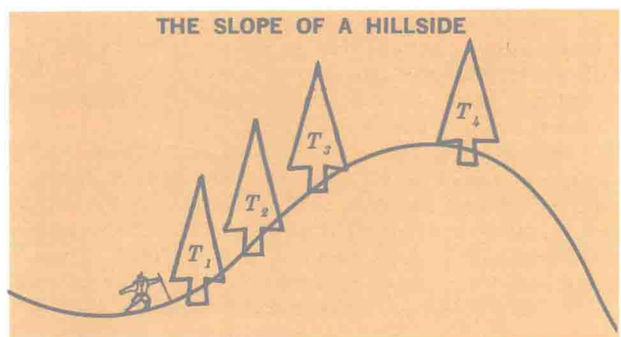
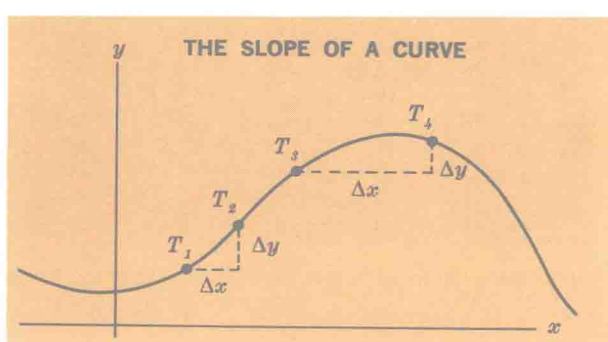


Fig. 5a. As the boy travels from the first tree to the second, he gains altitude rapidly. The hill is said to have a large slope between these points. But near the top the boy's altitude changes slowly, because the slope is small. Thus, there is a relation between slope and the rate of change of altitude.

Fig. 5b. In mathematics, the average slope of a curve is defined more accurately as the change in y divided by the change in x, or $\frac{\Delta y}{\Delta x}$. When the average slope is large, as between points T_1 and T_2, the ratio $\frac{\Delta y}{\Delta x}$ is also large. When the average slope is small, as between points T_3 and T_4, the ratio $\frac{\Delta y}{\Delta x}$ is also small.



distance between the two points) to the change in x (the horizontal distance between them). In Fig 5b these distances are symbolized as Δy and Δx, respectively. The symbol Δ is the Greek letter delta and stands for "the change in." The change in y may be found by subtracting the old value from the new one, and a similar process determines the change in x. Thus in mathematical terms the average slope between T_1 and T_2 is $\frac{\Delta y}{\Delta x}$. This mathematical definition of slope corresponds closely to the physical meaning of the term. That is, the ratio $\frac{\Delta y}{\Delta x}$ is large between points T_1 and T_2, corresponding to the steep part of the graph, while the ratio $\frac{\Delta y}{\Delta x}$ is small between points T_3 and T_4, corresponding to the nearly level part of the graph.

Exact Slope of a Curve. In the sections above, the *average* slope of a curve between two points was found. This procedure, however, tells little about the actual behavior of the curve, which could have taken any conceivable path between the two points. The following discussion shows how to determine the *exact* slope of a curve at any point, by using the concept of the limit.

The curve in Fig. 6 is the graph of a function $y = f(x)$, and the problem is to find the slope of the curve at the point P. If only one point is considered, the procedure discussed above for finding average slope is impossible to carry out: there is no Δy or Δx, because there is no second point from which to determine these distances. However, suppose a second point Q is moved closer and closer to the fixed point P. The line PQ cuts the curve in two points. But as the distance between the two points approaches zero, the line connecting them approaches a limiting position in which it touches the curve only at the point P. Such a line is called a tangent. In other words, the tangent to the curve at P is the limit approached by the line PQ as the point Q approaches the point P.

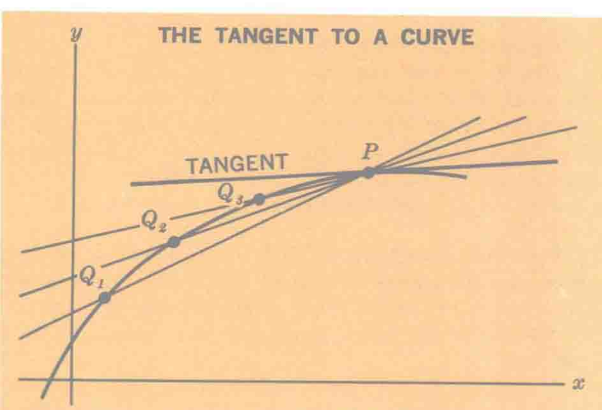
It has already been shown that the average slope of the curve between points P and Q is $\frac{\Delta y}{\Delta x}$. But as Q moves closer and closer to P, Δx becomes smaller and smaller. In the limit as Q approaches P, Δx approaches zero, and the ratio $\frac{\Delta y}{\Delta x}$ approaches the exact slope of the curve at the point P. In other words, the exact slope of the curve at the point P is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

This expression is also the slope of the tangent at P.

At the point P, the value of y is $f(x)$, while at the point Q, the value of y is $f(x + \Delta x)$. (See Fig. 7.)

Fig. 6. As the point Q moves along the curve and approaches the fixed point P, the line connecting P and Q tends toward a limiting position that is the tangent to the curve at point P.



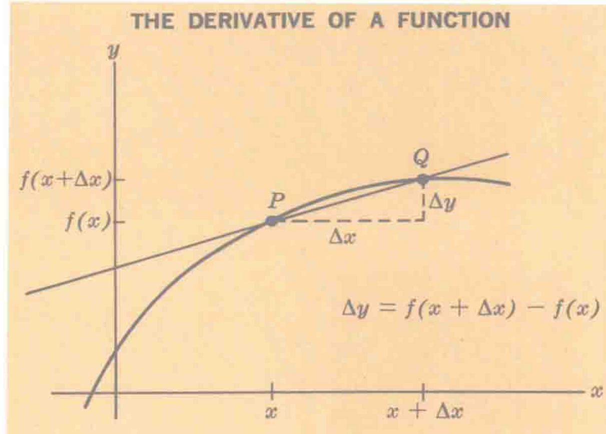


Fig. 7. The slope of the line PQ is defined as $\frac{\Delta y}{\Delta x}$. When the limit of this expression is taken as Q moves toward P (as Δx approaches zero), this limit is the slope of the tangent at P, and it is called the derivative of the function at the point P.

Therefore Δy is equal to $f(x + \Delta x) - f(x)$, and the ratio $\frac{\Delta y}{\Delta x}$ may be written

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

It has already been stated that the limit of this ratio as Δx approaches zero is the exact slope of the curve at any point P. Thus,

$$\text{slope of curve} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Derivative. The preceding expression for the slope of a curve is called the derivative of the function graphed by the curve. It is the basic concept of differential calculus and one of the most useful mathematical tools. The derivative gives the rate of change of the function $f(x)$ with respect to the independent variable x. If the original function is represented by $f(x)$, the derivative is often represented by $f'(x)$. Thus,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

If the derivative of a function is positive at a particular point, the curve will slope upward to the right, while if the derivative is negative, the curve will slope downward to the right. If the derivative is zero, the curve will have a horizontal direction. (See Fig. 8.)

The derivative of a function is itself a function: For every value of x there is just one value of the derivative. When the graph of $f(x)$ is drawn as in Fig. 8,

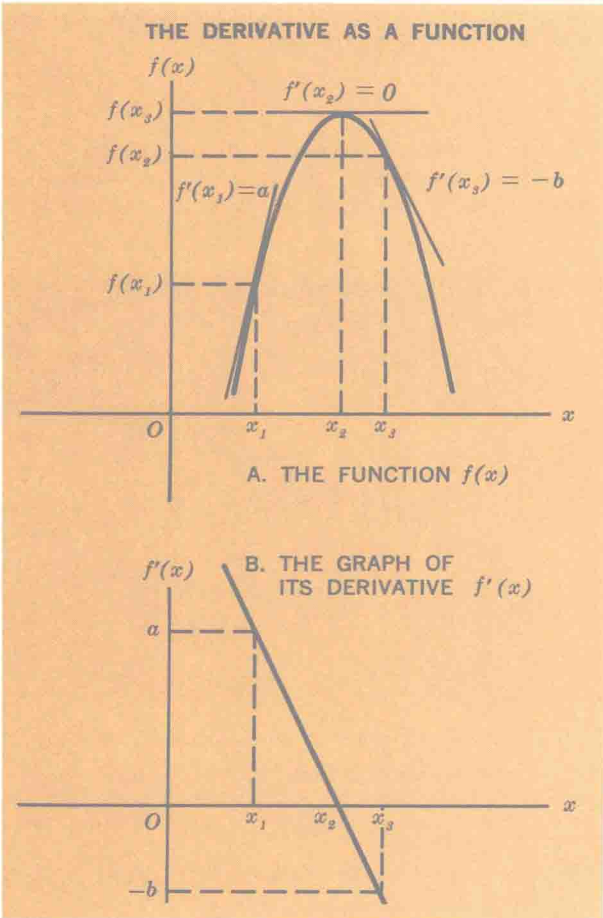


Fig. 8. In part A, tangents to the curve have been drawn at three points. The slope of the rising tangent is positive, that of the level tangent is zero, and that of the falling tangent is negative. Thus, a graph of the derivative (part B) includes a positive value at x_1, a value of zero at x_2, and a negative value at x_3, as shown. The derivative of a function is thus itself a function.

$f(x)$ = height of curve above point x on the x-axis
$f'(x)$ = slope of curve above point x.

Finding a Derivative. The operation of finding a derivative is called differentiation. The operation receives its name from the fact that the derivative $f'(x)$ is the limit of the difference $f(x_2) - f(x_1)$ divided by the difference $x_2 - x_1$.

Example. The problem is to find the derivative of the function

$$f(x) = x^2 + x$$

at a given point, say $x = k$. The value of this function at $x = k$ is

$$f(k) = k^2 + k.$$

The value of the function at a nearby point, for example $(k + h)$, is

$$\begin{aligned} f(k+h) &= (k+h)^2 + (k+h) \\ &= k^2 + 2hk + h^2 + k + h. \end{aligned}$$

Subtracting one value from another gives

$$\begin{aligned} f(k+h) - f(k) &= (k^2 + 2hk + h^2 + k + h) - (k^2 + k) \\ &= 2hk + h^2 + h. \end{aligned}$$

Dividing by h results in the following:

$$\frac{2hk + h^2 + h}{h} = 2k + 1 + h.$$

Letting h approach zero gives

$$f'(k) = \lim_{h \to 0} (2k + 1 + h) = 2k + 1.$$

The symbol $f'(k)$ stands for the derivative of $f(k)$ and is read "f prime of k." Thus the derivatives of $x^2 + x$, at the point $x = k$, is $2k + 1$. Since k could be any value of x, the derivative of $x^2 + x$, for any value of x, is $2x + 1$.

Second Derivatives. In many cases it is necessary to find the rate of change of a rate of change. For example, the acceleration of an object is the rate of change, or derivative, of its velocity. But the velocity is the rate of change of the distance the body has covered, and thus the body's acceleration is the derivative of a derivative. Such a derivative is called a second derivative and is written $f''(x)$, which is read "f double prime of x." For example, if

$$f(x) = x^2 + x,$$

it has just been shown that the derivative is

$$f'(x) = 2x + 1.$$

By taking the derivative of this expression and using the same method as before, the second derivative is found:

$$f''(x) = 2.$$

Notation. Derivatives may be written in several different ways. The derivative of the function $y = f(x)$ may be written y' (read: "y prime") or, as above, $f'(x)$. Some other ways of writing the derivative of this function are

$$\frac{dy}{dx},\ Df(x),\ D_x y.$$

The symbol $\frac{dy}{dx}$ is not a quotient, but a single symbol, although the differentials dx and dy may occur singly.

Ways of writing the second derivative in these three systems follow:

$$\frac{d^2y}{dx^2},\ D^2f(x),\ D_x^2 y.$$

Rules for Finding Derivatives. Mathematicians have developed rules that enable them to differentiate many functions at a glance. Some of the principal rules are given in the following paragraphs.

Constant. If $f(x) = C$, where C is a constant, then the derivative of $f(x)$ is 0. This becomes obvious when the graph of $f(x)$ is drawn, since the graph must be a horizontal line, which has zero slope.

Powers of the Variable. If a function of x consists of x raised to some power n, then its derivative is equal to n multiplied by x raised to a power smaller by one. That is, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. For example, the derivative of x^3 is $3x^2$; the derivative of x^4 is $4x^3$. Since $x = x^1$, the derivative of x is equal to x^{1-1}, or $x^0 = 1$.

Constant Coefficients. If C stands for a constant, then the derivative of $Cf(x)$ is $Cf'(x)$. Since the derivative of x^3 is $3x^2$, for example, the derivative of $5x^3$ is $5 \cdot 3x^2$, or $15x^2$. Similarly, the derivative of $2x^5$ is $10x^4$.

Fractional Exponents. Both of the rules stated above hold for fractional exponents. That is, the derivative of $x^{\frac{4}{3}}$ is $\frac{4}{3}x^{\frac{1}{3}}$, and the derivative of $x^{\frac{1}{2}}$ is $\frac{1}{2}x^{-\frac{1}{2}}$.

Trigonometric Functions. The derivatives of three important trigonometric functions are listed below:

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$, or $\frac{1}{\cos^2 x}$

Sums of Functions. The derivative of the sum of two functions is equal to the sum of the derivatives of each function. That is, if the two functions are $f(x)$ and $g(x)$, their sum is $f(x) + g(x)$, and the derivative of their sum is $f'(x) + g'(x)$. For example, if $f(x) = x^3$ and $g(x) = x^4$, then the derivative of their sum is $3x^2 + 4x^3$.

Differences of Functions. The derivative of the difference of two functions is equal to the difference of the derivatives of the two functions. Thus, the derivative of $f(x) - g(x)$ is simply $f'(x) - g'(x)$. For example, the derivative of $x^3 - \sin x$ is $3x^2 - \cos x$.

Products of Functions. If $h(x) = f(x) \cdot g(x)$, then the derivative $h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$. For example, if $h(x) = x^3 \cdot \sin x$, then $h'(x) = x^3 \cdot \cos x + 3x^2 \cdot \sin x$.

Quotients of Functions. If $h(x) = \frac{f(x)}{g(x)}$, then

$$h'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}.$$

For example, if $h(x) = \frac{\sin x}{x^2}$, then

$$h'(x) = \frac{x^2 \cdot \cos x - \sin x \cdot 2x}{x^4}.$$

Maxima and Minima. If a function has a greater value at a certain point than at any neighboring point, it is said to have a maximum at that point. Similarly, if it has a smaller value at a certain point than at any neighboring point, it is said to have a minimum at that point. In Figure 9 the function $f(x) = x^3 - 3x$ is graphed. This function has a maximum at the point marked A and a minimum at the point marked B.

It has been shown above that a function is increasing at any point where its first derivative is positive and decreasing where its first derivative is negative. A point at which the first derivative is equal to zero is called a critical point. If the second derivative at this point is negative, then the point marks the position of a maximum. If the second derivative is positive, the point marks a minimum. If the second derivative is equal to zero, then this critical point may mark a point of inflection, where the curve is neither a maximum nor a minimum. Point F in Fig. 9, for example, is a point of inflection. Here the curve changes from convex in one direction to convex in the other.

Thus the first and second derivatives of a function may be used to discover where the function reaches its maximum and minimum values. For example, if the function is $x^3 - 3x$, then its derivative is $f'(x) = 3x^2 - 3$. Since the first derivative is zero at critical points, this expression is set equal to zero and solved for x:

$$3x^2 - 3 = 0;$$
$$x^2 = \frac{3}{3}, \text{ or } 1;$$
$$x = \pm\sqrt{1} = +1 \text{ or } -1.$$

Now these values of x are substituted in the expression obtained for the second derivative:

$$f''(x) = 6x;$$
$$f''(+1) = 6;$$
$$f''(-1) = -6.$$

Since the second derivative is positive at the point $x = +1$, the curve is a minimum at that point. Similarly, since the second derivative is negative at $x = -1$, the curve reaches a maximum at that point. Tangents drawn to the curve at these points are horizontal.

Solving Problems With the Differential Calculus

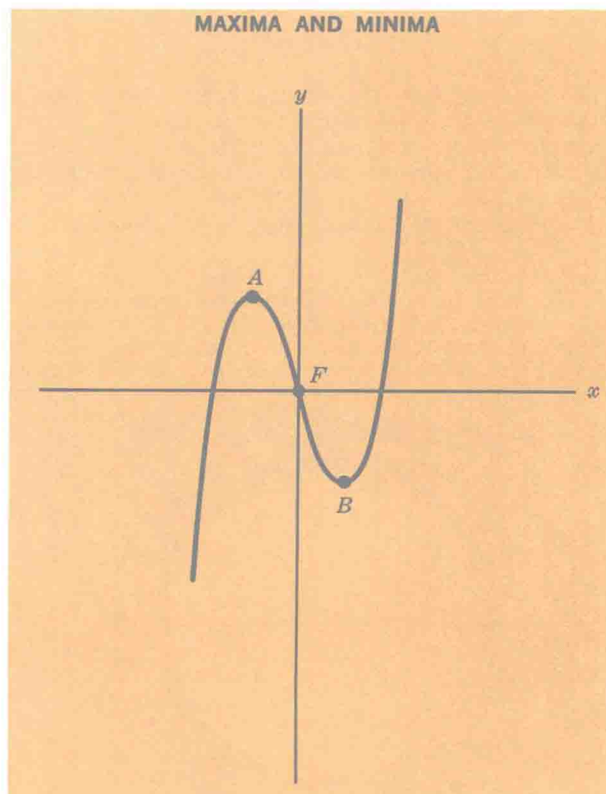
The methods for finding derivatives and maxima and minima can be used to solve many different types of problems. Some examples are given below.

Example 1. When a ball falls freely, the distance it travels is given by the formula $s = 16t^2$, where t is the time of travel in seconds and s is the distance in feet. As stated above, the derivative of any function is its rate of change with respect to the independent variable. Therefore, the derivative of this function is the rate of change of distance with respect to time. But this is the definition of velocity, and thus the very important conclusion is reached that *the derivative of the distance a body travels as a function of time is the body's velocity.*

Thus, the velocity of a freely falling ball is

$$v = s' = 32t.$$

Fig. 9. Point *A* is called a maximum because here the function reaches a greater value than it has at any other point in the immediate neighborhood. Point *B* is called a minimum because here the function has a lesser value than it has at any other point in the immediate neighborhood. Point *F*, where the curve changes the direction of its convexity, is an inflection point.



Similarly, since acceleration is defined as rate of change of velocity, the derivative of a body's velocity is its acceleration:

$$a = v' = s'' = 32.$$

That is, the speed of the ball increases at the rate of 32 feet per second for each second that the ball is falling. In fact, if air friction is neglected, all freely falling objects on the earth's surface have the same acceleration of 32 ft/sec^2, and this acceleration is called the acceleration due to gravity.

Example 2. The problem is to find the maximum height that a ball will reach if it is thrown upward at a velocity of 100 feet per second. From physics, the height is given as a function of the time t by the following formula:

$$h(t) = 100\,t - 16t^2.$$

Differentiating, $h'(t) = 100 - 32t.$

Differentiating a second time,

$$h''(t) = -32.$$

Since the second derivative is negative for all values of t, any point for which $h'(t) = 0$ will be a maximum. Setting the first derivative equal to zero,

$$100 - 32t = 0;$$
$$32t = 100;$$
$$t = \frac{100}{32}, \text{ or } 3\frac{1}{8}.$$

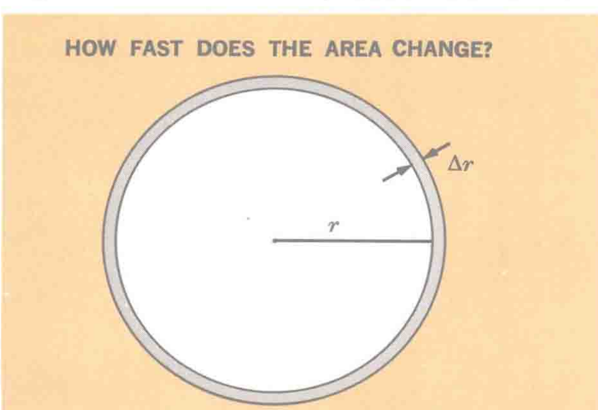
Thus the ball will reach a maximum height just $3\frac{1}{8}$ seconds after it is thrown. The maximum height can be found by substituting this value of t into the expression for the height of the ball:

$$h\left(3\frac{1}{8}\right) = 100\left(3\frac{1}{8}\right) - 16\left(3\frac{1}{8}\right)^2$$
$$= 156\frac{1}{4} \text{ feet.}$$

Example 3. A pebble dropped in the water sends out a circular ring whose radius increases at the constant rate of 1 foot per second. The problem is to determine how fast the area of the ring is growing when the radius is 5 feet.

The area of the circle is given by the function

Fig. 10. As the circle expands from radius r to radius $(r + \Delta r)$, its area grows by an amount equal to the shaded region. The shaded area is approximately $2\pi r \Delta r$. Division by Δr shows that the rate of change of the area of the circle with respect to the change in its radius is numerically equal to its circumference.



$$A = \pi r^2.$$

The instantaneous rate of change of this function is simply its derivative:

$$A' = 2\pi r.$$

When r equals 5,

$$A' = 10\pi = 10 \cdot 3.14 = 31.4,$$

and thus the area is growing at the rate of 31.4 square feet per second. The rate of change of the area of the circle is equal to the circumference of the circle, $2\pi r$. This should be intuitively clear, since as the circle expands a very tiny distance (the shaded section of Fig. 10), its area grows by an amount approximately equal to the area of an imaginary string looped around the circle. The string could be straightened out into a rectangle of length $2\pi r$ and width Δr. The area of this rectangle would then be $2\pi r \Delta r$. The rate of change of the area would then be $\frac{2\pi r \Delta r}{\Delta r}$, or $2\pi r$.

Integral Calculus

The integral calculus, like the differential calculus, is based on the concept of the limit. Quantities called integrals are each defined as the limit of a sum, and the process of taking this limit is called integration. Integration may be used to find the area enclosed by curved lines or the volume enclosed by curved surfaces.

Area Under a Curve. The area of any figure bounded by straight lines can be found by elementary geometry. The figure is simply divided into rectangles and triangles, the areas of which are easily found and can be added together. (See Fig. 11.) The area of a figure bounded by a curve, however, cannot be found in this way. Such a problem can be solved by applying the concept of the limit.

For example, in the case of the curve in Fig. 12, the height above the x-axis is given by the function $f(x)$. The desired area is that bounded by the curve, the x-axis, and the two vertical lines $x = a$ and $x = b$. If this area is divided into a number of rectangles, as in Fig. 13, it is clear that the areas of the rectangles will approximate the desired area. If there are n rectangles, the x-axis will be divided into n intervals. The length of the first interval may be called h_1, the length of the second h_2, and so on up to h_n. Somewhere in the first region a point x_1 is selected. The height of the curve at this point is $f(x_1)$, and the rectangle is given this value as its height. The area of the first rectangle is its height times its width, or

$$f(x_1)h_1.$$

Similarly, the area of the second rectangle is $f(x_2)h_2$, and the area of the nth rectangle is $f(x_n)h_n$. The sum S of these areas will approximate the total area A of the figure:

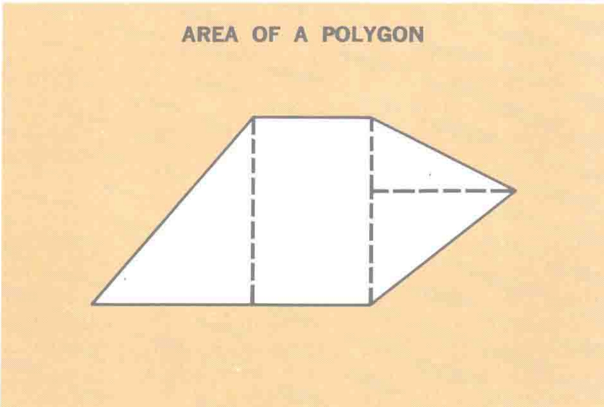
$$S = f(x_1)h_1 + f(x_2)h_2 + \ldots + f(x_n)h_n.$$

If the figure is divided into a larger number of rectangles, as in Fig. 14, the approximation given by the formula above will be more exact. In other words, as the number of rectangles (n) gets larger and larger (that is, as n approaches infinity), and as the widths of the rectangles get smaller and smaller (that is, as the largest h approaches zero), the sum S will get closer and closer to the exact area of the figure. If the sum S approaches a limit (as n approaches infinity and the largest h approaches zero), then this limit is called the definite integral of the function $f(x)$, and is written

$$\int_a^b f(x)\,dx.$$

This notation may be read "the definite integral of $f(x)$ between the limits a and b." The symbol $\int$ is an

Fig. 11. The area of any polygon can be found by dividing it into triangles and rectangles and summing the separate areas of the simpler figures, whose areas can be found easily.



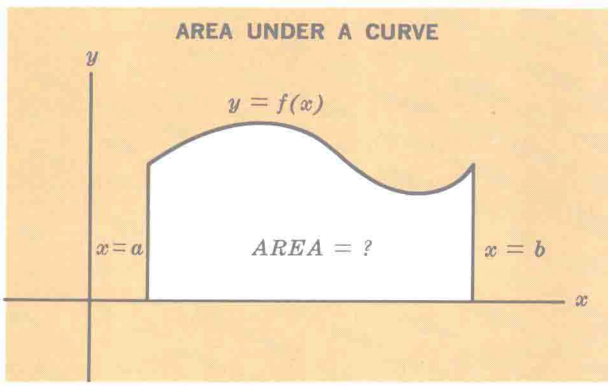
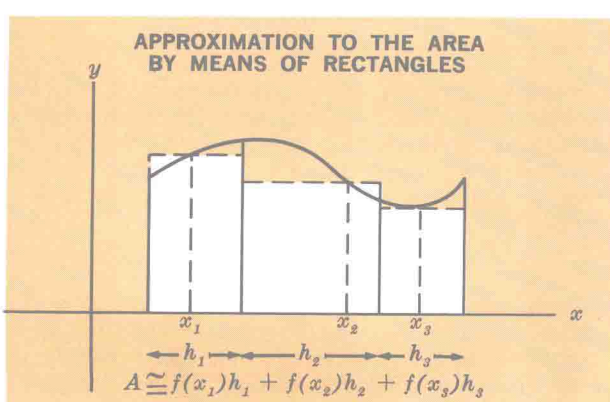


Fig. 12. The area bounded by a figure having at least one curved boundary can usually be found only by integration.

Fig. 13. If the area under the curve is divided into rectangles, and their separate areas summed, an approximation to the true area under the curve will be obtained. The area of the first rectangle, for example, is its height, $f(x_1)$, times its width, h_1.



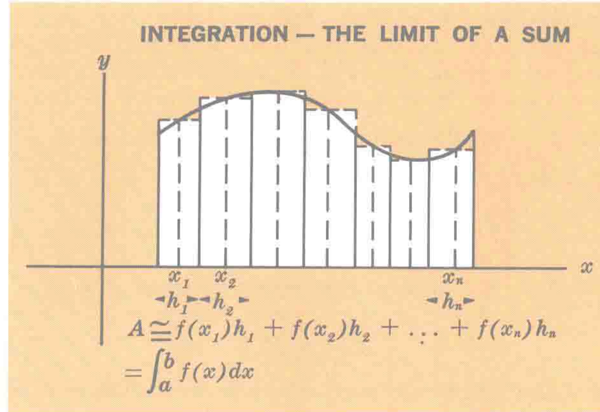
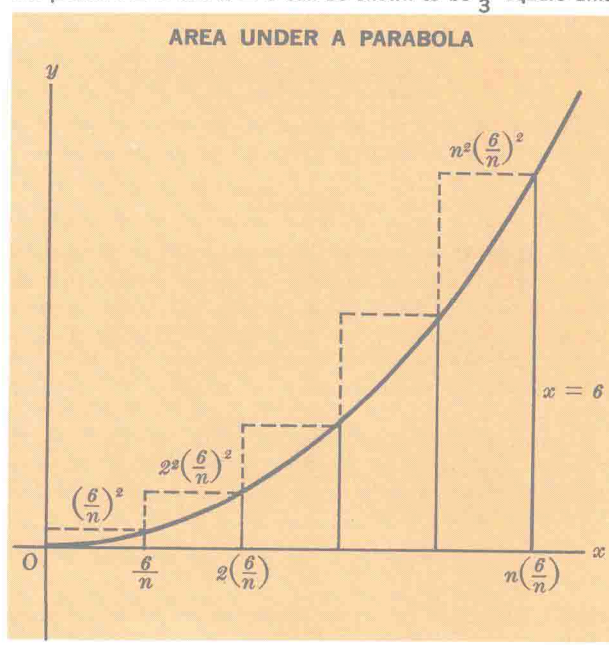


Fig. 14. If the number of rectangles is greatly increased, a closer approximation can be made. As the number approaches infinity, and as the width of the largest rectangle approaches zero, the limit of the sum of their areas is equal to the true area. The process of taking the limit is called integration.

elongated form of the letter S, and the symbol dx indicates that the differences between consecutive values of x have been allowed to approach zero.

Finding the Value of a Definite Integral. The definition above may now be applied to a specific example. In Fig. 15, a parabola given by the function $f(x) = x^2$ is graphed. What is the area under this parabola between the points $x = 0$ and $x = 6$? If the area is divided into n rectangles of the same width, the width of each will be given by the expression $\frac{6}{n}$. Then the value of x_1 will be $\frac{6}{n}$, the value of x_2 will be $2\left(\frac{6}{n}\right)$, the value of x_3 will be $3\left(\frac{6}{n}\right)$, and so on, up to the value of x_n, which is $n\left(\frac{6}{n}\right)$. Thus, $f(x_1) = \left(\frac{6}{n}\right)^2$, $f(x_2) = 2^2\left(\frac{6}{n}\right)^2$, . . ., and $f(x_n) = n^2\left(\frac{6}{n}\right)^2$. But the height of the curve at x_1

Fig. 15. By integration the area under this parabola between the points $x = 0$ and $x = 6$ can be shown to be $\frac{6^3}{3}$ square units.



is also the height of the first rectangle, the height of the curve at x_2 is the height of the second rectangle, and so on. Thus the height of each rectangle is known and may be used to calculate its area. For example. the area of the first rectangle equals its width $\left(\frac{6}{n}\right)$ multiplied by its height $\left(\frac{6}{n}\right)^2$:

$$\frac{6}{n}\left(\frac{6}{n}\right)^2.$$

Similarly, the area of the second rectangle equals

$$\frac{6}{n}\left[2^2\left(\frac{6}{n}\right)^2\right],$$

and the sum of the areas of all the rectangles is given by

$$S=\frac{6}{n}\left(\frac{6}{n}\right)^2+\frac{6}{n}\left[2^2\left(\frac{6}{n}\right)^2\right]+\ldots+\frac{6}{n}\left[n^2\left(\frac{6}{n}\right)^2\right].$$

Since the terms $\frac{6}{n}$ and $\left(\frac{6}{n}\right)^2$ appear in every term of the expression above, they may be factored out:

$$S=\frac{6}{n}\left(\frac{6}{n}\right)^2\left[1^2+2^2+\ldots+n^2\right].$$

It can be shown that the sum indicated by the expression in brackets is given by

$$\frac{n(n+1)(2n+1)}{6}.$$

The sum of the areas of the rectangles is thus

$$S=\frac{6^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]=\frac{6^3}{6}\left[\frac{n}{n}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$
$$=\frac{6^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

This sum approaches a limit as n approaches infinity, because the terms $\frac{1}{n}$ approach zero:

$$\lim_{n\to\infty} S=\frac{6^3}{6}(1)(2)=\frac{6^3}{3}.$$

Thus the definite integral of x^2 as x ranges from 0 to 6 is

$$\int_0^6 x^2dx=\frac{6^3}{3}=72.$$

Further, it is easy to see that the area under x^2 as x ranges from 0 to any other number a would be

$$\int_0^a x^2dx=\frac{a^3}{3}.$$

Fundamental Theorem. The value of any definite integral of a continuous function can be found by methods similar to those used in the preceding section. However, these methods are often extremely complicated and difficult. For this reason, one of the great breakthroughs in the history of mathematics was the discovery that definite integrals could be evaluated by reversing the rules of differentiation, which are comparatively simple. In a somewhat simplified form, the theorem states that *if $F(x)$ is any function whose derivative is $f(x)$, then*

$$\int_a^b f(x)dx=F(b)-F(a).$$

Any function $F(x)$ whose derivative is $f(x)$ is called a primitive function of $f(x)$. Thus, *to find the value of a definite integral $\int_a^b f(x)dx$, simply find a primitive function $F(x)$ such that $F'(x)=f(x)$, and then subtract $F(a)$ from $F(b)$.*

Example 1. How this rule simplifies the process of finding the value of definite integrals can be demonstrated by using it to solve the problem discussed in the preceding section. The problem was to find the area under the curve $f(x)=x^2$ between $x=0$ and $x=6$. Applying the rule for evaluating definite integrals,

$$\int_0^6 x^2dx=F(6)-F(0),$$

where $F(x)$ is a primitive function of x^2.

The first step is to find a primitive function of x^2. This can be done by means of the rule of differential calculus that the derivative of x^m is mx^{m-1}. That is,

$$\text{if } f(x)=x^m,$$
$$\text{then } f'(x)=mx^{m-1}.$$

Thus, to find the derivative of a power of x, first the coefficient of x is multiplied by the exponent of x, and then 1 is subtracted from the exponent. A primitive function of a power of x can thus be found by reversing these steps: First, 1 is added to the exponent of x and then the coefficient of x is divided by the new exponent. This procedure is applied to x^2 as follows:

$$\text{Step 1: } x^{2+1}=x^3$$
$$\text{Step 2: } \frac{1x^3}{3}=\frac{x^3}{3}.$$

(Since the coefficient of x^2 is 1, it need not be written.) By differentiating $\frac{x^3}{3}$, it can be shown that $\frac{x^3}{3}$ is in fact a primitive function of x^2. Substituting this function for $F(x)$ in the above equation,

$$\int_0^6 x^2\,dx=\frac{6^3}{3}-\frac{0^3}{3}$$
$$=\frac{6^3}{3}=72.$$

Example 2. To find the area under the curve $f(x)=x^3$ between $x=2$ and $x=3$, a similar procedure is followed. Using the rule given in example 1, the primitive function $\frac{x^4}{4}$ is found. Thus,

$$\int_2^3 x^3dx=\frac{(3)^4}{4}-\frac{(2)^4}{4}$$
$$=\frac{81}{4}-\frac{16}{4}$$
$$=\frac{65}{4}, \text{ or } 16\frac{1}{4}.$$

Indefinite Integrals. The examples above showed how to start with a function $f(x)$ and reverse the rules of differentiation to arrive at another function $F(x)$. The function $F(x)$, which we have called a primitive function, is often called an *indefinite integral,* and is written

$$F(x)=\int f(x)dx.$$

Note that the indefinite integral, unlike the definite integral, is not evaluated between limits. Any given function may thus have many indefinite integrals. For example, one indefinite integral of $3x^2$ is x^3, another is x^3+5, and still another is x^3-2. Since two indefinite