Cauchy Problem for Quasilinear Hyperbolic Systems

Liu Fagui

黄河水利出版社
Yellow River Conservancy Press

Cauchy Problem for Quasilinear Hyperbolic Systems

Liu Fagui

黄 河 水 利 出 版 社
Yellow River Conservancy Press

图书在版编目(CIP)数据

拟线性双曲型方程组Cauchy问题=Cauchy Problem for Quasilinear Hyperbolic Systems / 刘法贵著. —郑州：黄河水利出版社，2006.1
ISBN 7-80734-020-7

Ⅰ.拟…　Ⅱ.刘…　Ⅲ.拟线性双曲型方程-英文
Ⅳ.0175.27

中国版本图书馆 CIP 数据核字(2005)第 152567 号

策划组稿：王路平　☎ 0371-66022212　E-mail：wlp@yrcp.com

出 版 社：黄河水利出版社
地址：河南省郑州市金水路 11 号　邮政编码：450003
发行单位：黄河水利出版社
发行部电话：0371-66026940　传真：66022620
E-mail:yrcp@public.zz.ha.cn
承印单位：黄河水利委员会印刷厂
开本：787 mm × 1 092 mm　1/16
印张：11.75
字数：300 千字　印数：1—1 300
版次：2006 年 1 月第 1 版　印次：2006 年 1 月第 1 次印刷

书号：ISBN 7-80734-020-7/O・16　定价：32.00 元

Preface

Quasilinear hyperbolic systems describe many physical phenomena. Particularly, important examples occur in gas dynamics, shallow water theory, plasma physics, combustion theory, nonlinear elasticity, acoustics, classical or relativistic fluid dynamics and petroleum reservoir engineering. For linear hyperbolic system with suitably smooth coefficients, it is well-known that Cauchy problem always admits a unique global classical solution on the whole domain, provided that the initial data are smooth enough. For nonlinear hyperbolic system, however, the situation is quite different. Generally speaking, in this case, the classical solutions to Cauchy problem exist only locally in time and singularities may occur in a finite time, even if the initial data are sufficiently smooth and small.

This book is concerned with the classical solution to quasilinear hyperbolic system. The greatest part of the book is the fruit academic research on the part of the author. Some of what is contained in the book has been published for the first time, and what was previously published in the form of separate papers has also been revised and upgraded.

The whole approach to the problems under discussion is primarily based on the theory on the local solution. For more comprehensive information, the reader may refer to the book by Li Tatsien and Yu Wenci: *Boundary Value Problems for Quasilinear Hyperbolic Systems* (Duke University Mathematics Series V, 1985).

It must be pointed out that the local existence and uniqueness of the classical solution serves as an important basis for getting the global existence and uniqueness of the classical solutions. The method employed in this book is the *extension method of local solution.* This method requires us: first, establish the local classical solution theory, then derive some uniform *a priori* estimates on the solution. Using these uniform a priori estimates, we can draw the final conclusions. This method can be expressed simply as follows:

Local classical solution theory

$+$

Uniform a priori estimates on solution

$\Downarrow$

Final results (Global existence or Breakdown)

Because the local classical solution theory has been established well, the key point of this method is how to establish some uniform *a priori* estimates on the solution.

The author hereby takes this opportunity to express his hearty thanks to Professor Li Tatsien and Professor Li Caizhong for their sustained supported, guidance and encouragement. Their inspired academic thinking and informed approach to academic research have proved to be an inexhaustible source of wisdom to the author.

This work was partially supported by the Foundation of key subject of North China Institute of Water Conservancy and Hydroelectric Power, the National Natrual Foundation of Henan Province and the National Natural Foundation of Educational Department of Henan Province.

Liu Fagui
October, 2005
Zhengzhou, China

Abstract

This book is concerned with Cauchy problem for the first order quasilinear hyperbolic systems, some basic concepts of quasilinear hyperbolic systems and basic methods for studying classical solutions are given. In this book, we discussed single quasilinear hyperbolic equation, classical solutions to reducible quasilinear hyperbolic systems, dissipation and relaxation problem, singularities caused by the eigenvectors, and quasilinear hyperbolic systems in linearly degenerate type.

This book can be used for students and postgraduates of mathematical speciality, teachers, scientists and engineers.

内 容 提 要

本书主要研究一阶拟线性双曲型方程组 Cauchy 问题，介绍了一阶拟线性双曲型方程组的基本概念及研究经典解的基本方法。全书讨论了单个拟线性双曲型方程、可约化拟线性双曲型方程组、拟线性双曲型方程的耗散和张弛问题、由特征向量引发的奇性和具线性退化特征双曲型方程组。

本书供高等院校数学专业本科生、研究生、教师、科研人员阅读参考。

Contents

Preface

Chapter 1 Introduction .... (1)

1.1 Intention and Significances .... (1)

1.2 Basic Concepts .... (6)

1.3 Preliminaries .... (12)

Chapter 2 The Single Quasilinear Hyperbolic Systems .... (17)

2.1 Homogeneous Quasilinear Hyperbolic Equation .... (17)

2.2 Inhomogeneous Quasilinear Hyperbolic Equation .... (26)

2.3 Periodic Solutions to Quasilinear Hyperbolic Equation .... (28)

2.4 Quasilinear Hyperbolic Equation with Higher Dissipation .... (32)

Chapter 3 Quasilinear Hyperbolic Systems in Diagonal Form .... (36)

3.1 Reducible Quasilinear Hyperbolic Systems .... (36)

3.2 Life Span of C^1 Solutions to Reducible Quasilinear Hyperbolic Systems .... (40)

3.3 Quasilinear Hyperbolic Systems in Diagonal Form .... (44)

3.4 Hyperbolic Systems with Discontinuous Coefficients .... (50)

Chapter 4 Discontinuous Initial Value Problem for Quasilinear Hyperbolic Systems .... (59)

4.1 Discontinuous Initial Value Problem .... (59)

4.2 A Uniform *a Priori* Estimate .... (62)

4.3 Typical Boundary Value Problem .... (65)

4.4 Typical Free Boundary Value Problem on a Fan-shaped Domain ... (69)

4.5 Discontinuous Initial Value Problem for Inentropic Folw with Dissipative Term .... (72)

4.6 Typical Free Value Problem for Quasilinear Hyperbolic Systems with Dissipation .... (79)

Chapter 5 Singularities Caused by the Eigenvectors .... (87)

5.1 Introduction .... (87)

5.2 Completely Reducible Systems .... (91)
5.3 2-Step Completely Reducible Systems with Constant Eigenvalues .. (94)
5.4 $m\,(m > 2)$-Step Completely Reducible Systems with Constant Eigenvalues .... (102)
5.5 Non-Completely Reducible Systems .... (108)
5.6 Examples .... (109)

Chapter 6 Quasilinear Hyperbolic Systems with Dissipative Term .... (116)

6.1 Introduction .... (116)
6.2 Reducible Qualinear Hyperbolic Systems with Dissipation .... (118)
6.3 Formation of Singularities .... (122)
6.4 Applications .... (127)

Chapter 7 Quasilinear Hyperbolic Systems with Relaxation .... (141)

7.1 Introduction .... (141)
7.2 Global Classical Solutions .... (143)
7.3 Applications .... (148)
7.4 Convergence of Approximate Solutions .... (151)

Chapter 8 Cauchy Problem for Quasilinear Hyperbolic Systems in Linearly Degenerate Type .... (159)

8.1 Introduction .... (159)
8.2 Preliminaries .... (161)
8.3 Proof of Theorem 8.2.1 .... (164)
8.4 Applications .... (173)

Bibliography .... (176)

Chapter 1

Introduction

In this chapter we give some basic concepts of quasilinear hyperbolic system: genuinely nonlinear, linearly degenerate, weak linear degenerate, matching condition etc.

1.1 Intention and Significances

For the following quasilinear hyperbolic system with inhomogeneous terms:

$$u_t + A(u)u_x = B(u) \tag{1.1.1}$$

where $u = (u_1, u_2, \cdots, u_n)^{\mathrm{T}}$ is unknown vector function, $B(u) \in C^1(R^n)$ is known vector function with $B(u) = (b_1(u), \cdots, b_n(u))^{\mathrm{T}}$, and $A(u) = (a_{ij}(u))_{n\times n}(a_{ij}(u) \in C^1(R^n),\ i,j = 1,2,\cdots,n)$ is known matrix function, it is well-known that system (1.1.1) may be arisen in many physics, such as nonlinear wave phenomena, gas dynamics system, elastic dynamics, the kinetic theory and multiphase flow. These equations play an important role in both science (such as physics, mechanics, biology, etc.) and technology.

We have known that, for linear hyperbolic system, there are many very well results, however, for quasilinear hyperbolic system (1.1.1), the results are imperfections.

Generally speaking, the classical solutions to system (1.1.1) exists only locally in time and singularities may occur in a finite time, even if the initial data are sufficiently smooth or sufficiently small. To illustrate this, we give two simple examples.

Example 1.1.1 Consider the following Cauchy problem of Burger's equation with inhomenoeous term:

$$\begin{cases} u_t + uu_x = u^2 \\ t = 0 : u = u_0(x) \end{cases} \tag{1.1.2}$$

where $u_0(x) \in C_0^2([a,b]), u_0(x)$ exists maximum value at the point $\beta_0 \in (a,b)$, and

$$u_0(\beta_0) > 0, \quad u_0''(\beta_0) \neq 0$$

On the existence domain $\{(t,x)|0 \leqslant t \leqslant T_0, x \in R\}$ of the classical solution to Cauchy problem (1.1.2), let

$$x = \phi(t,\beta), \quad \phi(0,\beta) = \beta$$

be characteristics, and

$$v(t,\beta) = u(t,\phi(t,\beta))$$

then, (ϕ, v) satisfies

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = v^2, \quad \phi(0,\beta) = \beta, \quad v(0,\beta) = u_0(\beta) \tag{1.1.3}$$

It follows from (1.1.3) that

$$u(t,x) = v(t,\beta) = \frac{u_0(\beta)}{1 - tu_0(\beta)} \tag{1.1.4}$$

Obviously, the life span $\bar{T}$ for $v(t,\beta)$ satisfies

$$\bar{T} \triangleq \frac{1}{\max u_0(\beta)}$$

Moreover, we have

$$\phi(t,\beta) = \beta - \ln(1 - tu_0(\beta))$$

Hence,

$$\frac{\partial \phi}{\partial \beta} = 1 + \frac{tu_0'(\beta)}{1 - tu_0(\beta)} \tag{1.1.5}$$

Suppose that $\partial_x u$ blows up at $t = T^* > 0$. Since

$$\frac{\partial u}{\partial x} \to \infty \quad \frac{\partial \phi}{\partial \beta} \to 0$$

as $t \to T^*$. Thus, we obtain

$$T^* = h(\beta) \triangleq \frac{1}{u_0(\beta) - u_0'(\beta)}$$

By $u_0'(\beta_0) = 0, u_0''(\beta_0) \neq 0$, we have

$$h'(\beta_0) \neq 0$$

Noting the continuity of $h(\beta)$,there exists a neighborhood domain $D(\beta_0)$ of β_0, such that

$$h'(\beta) \neq 0, \quad \beta \in D(\beta_0)$$

Without loss of generality, we suppose that

$$h'(\beta) > 0, \quad \beta \in D(\beta_0)$$

Then, there exists $\beta_* \in D(\beta_0)$, such that

$$h(\beta_*) < h(\beta_0)$$

that is,

$$T^* < \bar{T} \tag{1.1.6}$$

(1.1.6) shows that we can choose suitably $u_0(x)$ such that $u_x(t,x)$ first blows up in a finite time.

On the other hand, by (1.1.4) and (1.1.5), if $u_0(x) \in C^1(R)$, and

$$u_0(x) \leqslant 0, \quad u_0'(x) \geqslant 0, \quad \forall x \in R$$

then Cauchy problem (1.1.2) admits a unique global classical solution on $t \geqslant 0$.

Example 1.1.2 Consider quasilinear hyperbolic system with dissipation:

$$\begin{cases} u_t + uu_x = -\alpha u \\ t = 0 : u = \phi(x) \end{cases} \tag{1.1.7}$$

where α $(\alpha > 0)$ is a constant,$\phi(x) \in C^1(R)$ with bounded C^1 norm.

Suppose that $x = x(t,\beta)$ $(x(0,\beta) = \beta)$ is characteristics, then, we have

$$u(t,x) = \phi(\beta)\exp(-\alpha t)$$

$$u_x(t,x) = \frac{\phi'(\beta)\exp(-\alpha t)}{1 + \alpha^{-1}\phi'(\beta)(1 - \exp(-\alpha t))} \tag{1.1.8}$$

By (1.1.8), if α $(\alpha > 0)$ is suitably large, then $\partial_x u(t,x)$ admits uniform *a priori* estimate, and then, Cauchy problem (1.1.7) admits a unique global classical solution on $t \geqslant 0$. If α $(\alpha > 0)$ is suitably small, then there exists $T_0 > 0$ (depending on β and α), such that

$$u_x(t,x) \to \infty$$

as $t \to T_0^-$. Hence, the classical solution to Cauchy problem (1.1.7) must blow up in a finite time.

There is considerable practical interest in obtaining numerical approximations of solution to system (1.1.1). Knowing that the solution is smooth and allows one to take advantage of efficient high-order schemes which may be in appropriate for solutions with discontinuity. In fact, the global existence of the approximate finite element solution shows that the approximate solution is always in a neighborhood of a classical solution to system (1.1.1).

Therefore, for the first order quasilinear hyperbolic system (1.1.1), it is of great important in both theory and application to study the following three problems.

(1) *Under what conditions, does the problem under consideration (Cauchy problem, Boundary value problem, Generalized Riemann problem etc.) for the first order*

quasilinear hyperbolic system admit a unique global classical solution on $t \geqslant 0$? Basing on this problem, we can further study the regularity and the global behavior of the solution, particularly the asymptotic behavior of the solution as $t \to +\infty$.

(2) *Under what conditions, does the classical solution to the problem under consideration blow up in a finite time? When and where does the solution blow up? Which quantities will blow up? Can we further investigate the behavior or mechanisms of the blow-up phenomenon?*

Even if the solution blows up in a finite time, physical phenomenon still exists with singularities. Therefore one wants to understand further.

(3) *How do the singularities, in particular, shocks grow out of nothing? What is the structure of the singularities? What about the stability of the singularities?*

For a single quasilinear equation, these problems have been solved completely by the method of characteristics and the Whitney's theory of singularities of mapping of the plane into the plane.

A systematic theory on the global existence and the breakdown of the classical solutions to quasilinear reducible hyperbolic system has been established. In the system case, most of studies are concentrated on the reducible homogeneous 2×2 quasilinear hyperbolic system:

$$\begin{cases} r_t + \lambda(r,s) r_x = 0 \\ s_t + \mu(r,x) s_x = 0 \end{cases} \tag{1.1.9}$$

and the following simple and important case:

$$u_t + A(u) u_x = 0 \tag{1.1.10}$$

Suppose that system (1.1.10) is strictly hyperbolic and genuinely nonlinear (see (1.2.3) and Definition 1.2.1).

Consider Cauchy problem of system (1.1.10) with the following initial data:

$$t = 0 : u = \phi(x) \tag{1.1.11}$$

F.John[14] proved that if $A(u), \phi(x) \in C^2$, $\text{supp}\phi(x) \subseteq [\alpha_0, \beta_0]$, and

$$\theta = (\beta_0 - \alpha_0)^2 \sup_x |\phi''(x)| > 0$$

is small enough, then the first order derivatives of C^2 solution $u = u(t,x)$ to Cauchy problem (1.1.10)-(1.1.11) must blow up in a finite time. Liu Taiping[62] generalized F. John's result to the case that a part of eigenvalues is genuinely nonlinear, while the other part of eignvalues is linearly degenerate (see Definition 1.2.1). In this situation he showed that for a quite large class of small initial data, the first order derivatives of the classical solution still blows up in a finite time. Hörmander[11] improved F. John's result, by a self-contained and somewhat simplified exposition of the method. Moreover, by determining the time of blow-up asymptotically, he gave a sharp estimate on the life span of the solution.

Employing the nonlinear geometrical optics, S. Alinhac[1] reconsidered the result presented by Hörrmander and gave a more precise estimate on the life span.

Here, we point out the work obtained by Li Tatsien *et al* ([25 and therein]). They introduce some new concepts— **null condition** and **weak linear degeneracy**, gave a quite complete result on the global existence and the life span of C^1 solution to Cauchy problem (1.1.10)-(1.1.11), where the eigenvalues of system (1.1.10) might be neither genuinely nonlinear nor linearly degenerate, and $\phi(x)$ is small in the following sense: there exists a constant μ ($\mu > 0$) such that

$$\theta \equiv \sup_{x\in R}\{(1+|x|)^{1+\mu}(|\phi(x)|+|\phi'(x)|)\} \tag{1.1.12}$$

is small. Kong Dexing[19] generalized these results. Moreover, he also discussed the non-strictly hyperbolic system.

The discontinuous initial value problem is advanced by Li Tatsien *et al.*

Consider system (1.1.10) with the following discontinuous initial data

$$t=0: u=\begin{cases} u_0^-(x), & x\leqslant 0 \\ u_0^+(x), & x\geqslant 0 \end{cases} \tag{1.1.13}$$

where $u_0^-(x)$ and $u_0^+(x)$ are bounded C^1 functions on $x\leqslant 0$ and $x\geqslant 0$ respectively, and

$$u_0^-(0)\neq u_0^+(0) \tag{1.1.14}$$

If the corresponding Riemann problem for system (1.1.10) with the following initial data:

$$t=0: u=\begin{cases} u_0^-(0), & x\leqslant 0 \\ u_0^+(0), & x\geqslant 0 \end{cases} \tag{1.1.15}$$

possesses $n+1$ constant states and n shock waves or n contact discontinuities with small decay initial data, then, discontinuous initial value problem (1.1.10) and (1.1.13) admits a unique global classical discontinuous solution $u=u(t,x)$ only containing n shocks or n contact discontinuities. This solution possesses a global structure similar to the similarity solution to the corresponding Riemann problem. The result shows that the similarity solution possesses a global nonlinear structure stability. At the same time, Li Tatsien simplified the proof on existence of shocks, given by P.D.Lax[22] and J.Smoller[69], and some concise inequalities are obtained.

For the case that $B(u)\not\equiv 0$, if $B(u)$ is linear vector value function, $B(0)=0$, and

$$A=-L(0)\nabla B(0)L^{-1}(0) \tag{1.1.16}$$

is weak row-diagonally dominant, where $L(u) = (l_{ij}(u))$ is composed by the left eigenvectors, $L^{-1}(0)$ is the inverse of $L(0)$, $\|u_0(x)\|_{C^1}$ is sufficiently small, then, Cauchy problem for system (1.1.1) admits a unique global classical solution on $t \geqslant 0$. If $B(u)$ is nonlinear vector value function, $B(0) = 0$, and A is strictly row-diagonally dominant, $\|u_0(x)\|_{C^1}$ is sufficiently small, then, Cauchy problem for system (1.1.1) admits a unique global classical solution on $t \geqslant 0$[25,37].

Kong Dexing[17~19], Yu Jinguo and Zhao Yanchun[73],Zhang Weiguo[76], Liu Fagui and Yang Zejiang[61] improved above results.

1.2 Basic Concepts

1.2.1 Types of Eigenvalues

System (1.1.1) is called **hyperbolic** on the domain under consideration, if

(1) $A(u)$ has n real eigenvalues $\lambda_i(u)$ $(i = 1, 2, \cdots, n)$;

(2) $A(u)$ is diagonalizable, i.e., there exists a complete set of left (resp. right) eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{1i}(u), \cdots, r_{ni}(u))^{\mathrm{T}}$) corresponding to $\lambda_i(u)$ $(i = 1, 2, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp.} \quad A(u)r_i(u) = \lambda_i(u)) \tag{1.2.1}$$

we have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{resp.} \quad \det|r_{ij}(u)| \neq 0) \tag{1.2.2}$$

System (1.1.1) is called **strictly hyperbolic** on a certain domain, if $A(u)$ admits n real and distinct eigenvalues $\lambda_i(u)$ $(i = 1, 2, \cdots, n)$. Without loss of generality, we suppose that

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \tag{1.2.3}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, 2, \cdots, n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i, j = 1, 2, \cdots, n)$.

Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, 2, \cdots, n) \tag{1.2.4}$$

and

$$r_i^{\mathrm{T}}(u)r_i(u) \equiv 1 \quad (i = 1, 2, \cdots, n) \tag{1.2.5}$$

where δ_{ij} stands for the Kronecker's symbol.

Definition 1.2.1 The eigenvalue $\lambda_i(u)$ is **genuinely nonlinear** (denoted by GNL) in the sense of P.D. Lax, if

$$\nabla\lambda_i(u)r_i(u) \neq 0, \quad \forall u \in R^n \tag{1.2.6}$$

While $\lambda_i(u)$ is **linearly degenerate** (denoted by LD), if

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \quad \forall u \in R^n \tag{1.2.7}$$

System (1.1.1) is GNL (resp. LD), if all eigenvalues are GNL (resp. LD).

Example 1.2.1 The following 2×2 quasilinear hyperbolic system in diagonal form

$$\begin{cases} r_t + \lambda(r,s)r_x = 0 \\ s_t + \mu(r,x)s_x = 0 \end{cases} \tag{1.2.8}$$

is GNL system if and only if

$$\frac{\partial\lambda(r,s)}{\partial r} \neq 0, \quad \frac{\partial\mu(r,s)}{\partial s} \neq 0, \quad \forall (r,s) \in R^2 \tag{1.2.9}$$

System (1.2.8) is LD system if and only if

$$\frac{\partial\lambda(r,s)}{\partial r} \equiv 0, \quad \frac{\partial\mu(r,s)}{\partial s} \equiv 0, \quad \forall (r,s) \in R^2 \tag{1.2.10}$$

that is

$$\lambda(r,s) \equiv \lambda(s), \quad \mu(r,s) \equiv \mu(r) \tag{1.2.11}$$

Definition 1.2.2 System (1.1.10) satisfies the **null condition**, if each small plane wave solution $u = u(s)$ ($u(0) = 0$), where $s = ax + bt$ (a, b are constants), to system

$$u_t + A(0)u_x = 0$$

is always a solution to system (1.1.10).

Definition 1.2.3 The i-th ($1 \leqslant i \leqslant n$) eigenvalue $\lambda_i(u)$ is **weak linear degenerate** (denoted by WLD), if

$$\nabla\lambda_i(u)r_i(u) \equiv 0 \quad (\forall |u| \quad \text{small})$$

holds along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\begin{cases} \frac{du}{ds} = r_i(u(s)) \\ u(0) = 0 \end{cases} \tag{1.2.12}$$

Therefore, if $\lambda_i(u)$ is WLD, then,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0)$$

If all eigenvalues are WLD, system (1.1.1) is called the WLD. Obviously, if, in a neighborhood domain of $u = 0$, the i-th eigenvalue $\lambda_i(u)$ is LD in the sense of P.D. Lax, then $\lambda_i(u)$ is WLD.

For any C^1 solution $u = u(t,x)$ to system (1.1.1)

$$\frac{dx}{dt} = \lambda_i(u(t,x)) \tag{1.2.13}$$

is called the ***i*-th characteristic direction**, its integral curve is said to be the ***i*-th characteristics.**

Let

$$\frac{d}{d_i t} \equiv \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}$$

then, along the i-th characteristic direction,

$$\frac{du}{d_i t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + \lambda_i(u)u_x$$

Multiplying (1.1.1) by $l_i(u)$ from the left side, and noting (1.2.1), system (1.1.1) equivalently reduces to the following system of **characteristic form**

$$l_i(u)\frac{du}{d_i t} = l_i(u)(u_t + \lambda_i(u)u_x) = l_i(u)B(u) \quad (i = 1, 2, \cdots, n) \tag{1.2.14}$$

or

$$\sum_{j=1}^{n} l_{ij}(u)(\frac{\partial u_j}{\partial t} + \lambda_i(u)\frac{\partial u_j}{\partial x}) = \sum_{j=1}^{n} l_{ij}(u)b_j(u) \quad (i = 1, 2, \cdots, n) \tag{1.2.15}$$

in which the i-th equation only contains the directional derivatives of all the unknown functions along the i-th characteristic direction.

For the case that $n = 2$, it is well-known that at least in a local domain of u there exist integral factors $\pi_i(u) \neq 0$ $(i = 1, 2)$, such that

$$\pi_i(u)l_i(u)du = \pi_i(u)(l_{i1}(u)du_1 + l_{i2}(u)du_2) \quad (i = 1, 2)$$

is a total differential dU_i $(i = 1, 2)$. Hence, taking U_1 and U_2 as new unknown functions, (1.2.14) reduces to a system of **diagonal form**

$$\begin{cases} \partial_t U_1 + \lambda_1 \partial_x U_1 = f_1 \\ \partial_t U_2 + \lambda_2 \partial_x U_2 = f_2 \end{cases} \tag{1.2.16}$$

in which U_1, U_2 are called the **Riemann invariants.**

Remark 1.2.1 Generally speaking, the preceding procedure for reducing system (1.1.1) (with $n = 2$) to system (1.2.16) is only valid in a local domain of u. However, we can see in the sequel that this procedure is actually globally valid in some important practical case. In the case that $n > 2$, system (1.1.1), in general, can not be reduced to the diagonal form.

Example 1.2.2 Consider system of isentropic flow with dissipation

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x + 2\alpha u = 0, \quad \alpha > 0 \end{cases} \tag{1.2.17}$$

where $v = \rho^{-1}$ is the specific volume, u is the velocity and p is the pressure with satisfying

$$p'(v) < 0, \quad p''(v) > 0$$

For polytropic gases,

$$p = p(v) = Av^{-\gamma} \quad (0 < \gamma < 3, A > 0)$$

It is easily to show that on any finite domain of $v > 0$, (1.2.17) is strictly hyperbolic with two eigenvalues λ and μ:

$$\lambda = -\sqrt{-p'(v)} < 0 < \mu = \sqrt{-p'(v)}$$

Moreover, introducing Riemann invariants (r, s):

$$2r = u - \frac{2\sqrt{A}\gamma}{\gamma - 1} v^{-\frac{\gamma-1}{2}}, \quad 2s = u + \frac{2\sqrt{A}\gamma}{\gamma - 1} v^{-\frac{\gamma-1}{2}}$$

then, system (1.2.17) reduces to

$$\begin{cases} r_t + \lambda r_x = -\alpha(r+s) \\ s_t + \mu s_x = -\alpha(r+s) \end{cases} \tag{1.2.18}$$

in the global sense, and we can easily check that the system (1.2.17) is GNL.

Example 1.2.3 Consider the nonlinear vibration string equation

$$u_{tt} - (K(u_x))_x = 0 \tag{1.2.19}$$

where $K = K(v) \in C^1$, and

$$K(0) = 0, \quad K'(v) > 0$$

Let

$$v = u_x, \quad w = u_t \tag{1.2.20}$$

then, (1.2.19) can be rewritten as the following first order quasilinear system

$$\begin{cases} v_t - w_x = 0 \\ w_t - K(v)_x = 0 \end{cases} \tag{1.2.21}$$

in which (1.2.21) is strictly hyperbolic system with two distinct real eigenvalues

$$\lambda = -\sqrt{K'(v)} < 0 < \mu = \sqrt{K'(v)}$$

Introducing Riemann invariants (r, s)

$$2r = w + \int \sqrt{K'(v)} \mathrm{d}v, \quad 2s = u - \int \sqrt{K'(v)} \mathrm{d}v$$