国外数学名著系列
（影印版）24

Jürgen Neukirch

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Foreword

It is a very sad moment for me to write this "Geleitwort" to the English translation of Jürgen Neukirch's book on Algebraic Number Theory. It would have been so much better, if he could have done this himself.

But it is also very difficult for me to write this "Geleitwort": The book contains Neukirch's Preface to the German edition. There he himself speaks about his intentions, the content of the book and his personal view of the subject. What else can be said?

It becomes clear from his Preface that Number Theory was Neukirch's favorite subject in mathematics. He was enthusiastic about it, and he was also able to implant this enthusiasm into the minds of his students.

He attracted them, they gathered around him in Regensburg. He told them that the subject and its beauty justified the highest effort and so they were always eager and motivated to discuss and to learn the newest developments in number theory and arithmetic algebraic geometry. I remember very well the many occasions when this equipe showed up in the meetings of the "Oberwolfach Arbeitsgemeinschaft" and demonstrated their strength (mathematically and on the soccer field).

During the meetings of the "Oberwolfach Arbeitsgemeinschaft" people come together to learn a subject which is not necessarily their own speciality. Always at the end, when the most difficult talks had to be delivered, the Regensburg crew took over. In the meantime many members of this team teach at German universities.

We find this charisma of Jürgen Neukirch in the book. It will be a motivating source for young students to study Algebraic Number Theory, and I am sure that it will attract many of them.

At Neukirch's funeral his daughter Christiane recited the poem which she often heard from her father: *Herr von Ribbeck auf Ribbeck im Havelland* by Theodor Fontane. It tells the story of a nobleman who always generously gives away the pears from his garden to the children. When he dies he asks for a pear to be put in his grave, so that later the children can pick the pears from the growing tree.

This is – I believe – a good way of thinking of Neukirch's book: There are seeds in it for a tree to grow from which the "children" can pick fruits in the time to come.

G. Harder

Translator's Note

When I first accepted Jürgen Neukirch's request to translate his *Algebraische Zahlentheorie*, back in 1991, no-one imagined that he would not live to see the English edition. He did see the raw version of the translation (I gave him the last chapters in the Fall of 1996), and he still had time to go carefully through the first four chapters of it.

The bulk of the text consists of detailed technical mathematical prose and was thus straightforward to translate, even though the author's desire to integrate involved arguments and displayed formulae into comprehensive sentences could not simply be copied into English. However, Jürgen Neukirch had peppered his book with more meditative paragraphs which make rather serious use of the German language. When I started to work on the translation, he warned me that in every one of these passages, he was not seeking poetic beauty, but only the precisely adequate expression of an idea. It is for the reader to judge whether I managed to render his ideas faithfully.

There is one neologism that I propose in this translation, with Jürgen Neukirch's blessing: I call *replete* divisor, ideal, etc., what is usually called Arakelov divisor, etc. (a terminology that Neukirch had avoided in the German edition). Time will deliver its verdict.

I am much indebted to Frazer Jarvis for going through my entire manuscript, thus saving the English language from various infractions. But needless to say, I alone am responsible for all deficiencies that remain.

After Jürgen Neukirch's untimely death early in 1997, it was Ms Eva-Maria Strobel who took it upon herself to finish as best she could what Jürgen Neukirch had to leave undone. She had already applied her infinite care and patience to the original German book, and she had assisted Jürgen Neukirch in proofreading the first four chapters of the translation. Without her knowledge, responsibility, and energy, this book would not be what it is. In particular, a fair number of small corrections and modifications of the German original that had been accumulated thanks to attentive readers, were taken into account for this English edition. Kay Wingberg graciously helped to check a few of them. We sincerely hope that the book published here would have made its author happy.

Hearty thanks go to Raymond Seroul, Strasbourg, for applying his wonderful expertise of TeX to the final preparation of the camera-ready manuscript.

Thanks go to the Springer staff for seeing this project through until it was finally completed. Among them I want to thank especially Joachim Heinze for interfering rarely, but effectively, over the years, with the realization of this translation.

Strasbourg, March 1999 *Norbert Schappacher*

Preface to the German Edition

Number Theory, among the mathematical disciplines, occupies an idealized position, similar to the one that mathematics holds among the sciences. Under no obligation to serve needs that do not originate within itself, it is essentially autonomous in setting its goals, and thus manages to protect its undisturbed harmony. The possibility of formulating its basic problems simply, the peculiar clarity of its statements, the arcane touch in its laws, be they discovered or undiscovered, merely divined; last but not least, the charm of its particularly satisfactory ways of reasoning – all these features have at all times attracted to number theory a community of dedicated followers.

But different number theorists may dedicate themselves differently to their science. Some will push the theoretical development only as far as is necessary for the concrete result they desire. Others will strive for a more universal, conceptual clarity, never tiring of searching for the deep-lying reasons behind the apparent variety of arithmetic phenomena. Both attitudes are justified, and they grow particularly effective through the mutual inspirational influence they exert on one another. Several beautiful textbooks illustrate the success of the first attitude, which is oriented towards specific problems. Among them, let us pick out in particular *Number Theory* by *S.I.* BOREVICZ and *I.R.* ŠAFAREVIČ [14]: a book which is extremely rich in content, yet easy to read, and which we especially recommend to the reader.

The present book was conceived with a different objective in mind. It does provide the student with an essentially self-contained introduction to the theory of algebraic number fields, presupposing only basic algebra (it starts with the equation $2 = 1 + 1$). But unlike the textbooks alluded to above, it progressively emphasizes theoretical aspects that rely on modern concepts. Still, in doing so, a special effort is made to limit the amount of abstraction used, in order that the reader should not lose sight of the concrete goals of number theory proper. The desire to present number theory as much as possible from a unified theoretical point of view seems imperative today, as a result of the revolutionary development that number theory has undergone in the last decades in conjunction with 'arithmetic algebraic geometry'. The immense success that this new geometric perspective has brought about – for instance, in the context of the Weil conjectures, the Mordell conjecture, of problems related to the conjectures of Birch and Swinnerton-Dyer – is largely based on the unconditional and universal application of the conceptual approach.

It is true that those impressive results can hardly be touched upon in this book because they require higher dimensional theories, whereas the book deliberately confines itself to the theory of algebraic number fields, i.e., to the 1-dimensional case. But I thought it necessary to present the theory in a way which takes these developments into account, taking them as the distant focus, borrowing emphases and arguments from the higher point of view, thus integrating the theory of algebraic number fields into the higher dimensional theory – or at least avoiding any obstruction to such an integration. This is why I preferred, whenever it was feasible, the functorial point of view and the more far-reaching argument to the clever trick, and made a particular effort to place geometric interpretation to the fore, in the spirit of the theory of algebraic curves.

Let me forego the usual habit of describing the content of each individual chapter in this foreword; simply turning pages will yield the same information in a more entertaining manner. I would however like to emphasize a few basic principles that have guided me while writing the book. The first chapter lays down the foundations of the global theory and the second of the local theory of algebraic number fields. These foundations are finally summed up in the first three sections of chapter III, the aim of which is to present the perfect analogy of the classical notions and results with the theory of algebraic curves and the idea of the Riemann-Roch theorem. The presentation is dominated by "Arakelov's point of view", which has acquired much importance in recent years. It is probably the first time that this approach, with all its intricate normalizations, has received an extensive treatment in a textbook. But I finally decided not to employ the term "Arakelov divisor" although it is now widely used. This would have entailed attaching the name of *Arakelov* to many other concepts, introducing too heavy a terminology for this elementary material. My decision seemed all the more justified as *Arakelov* himself introduced his divisors only for arithmetic surfaces. The corresponding idea in the number field case goes back to *Hasse*, and is clearly highlighted for instance in *S. Lang*'s textbook [94].

It was not without hesitation that I decided to include *Class Field Theory* in chapters IV–VI. Since my book [107] on this subject had been published not long before, another treatment of this theory posed obvious questions. But in the end, after long consideration, there was simply no other choice. A sourcebook on algebraic number fields without the crowning conclusion of class field theory with its important consequences for the theory of L-series would have appeared like a torso, suffering from an unacceptable lack of completeness. This also gave me the opportunity to modify and emend my earlier treatment, to enrich that somewhat dry presentation with quite a few examples, to refer ahead with some remarks, and to add beneficial exercises.

A lot of work went into the last chapter on zeta functions and L-series. These functions have gained central importance in recent decades, but textbooks do not pay sufficient attention to them. I did not, however, include *TATE*'s approach to Hecke L-series, which is based on harmonic analysis, although it would have suited the more conceptual orientation of the book perfectly well. In fact, the clarity of *TATE*'s own presentation could hardly be improved upon, and it has also been sufficiently repeated in other places. Instead I have preferred to turn back to *HECKE*'s approach, which is not easy to understand in the original version, but for all its various advantages cried out for a modern treatment. This having been done, there was the obvious opportunity of giving a thorough presentation of *ARTIN*'s L-series with their functional equation – which surprisingly has not been undertaken in any existing textbook.

It was a difficult decision to exclude *Iwasawa Theory*, a relatively recent theory totally germane to algebraic number fields, the subject of this book. Since it mirrors important geometric properties of algebraic curves, it would have been a particularly beautiful vindication of our oft-repeated thesis that number theory is geometry. I do believe, however, that in this case the geometric aspect becomes truly convincing only if one uses *étale cohomology* – which can neither be assumed nor reasonably developed here. Perhaps the dissatisfaction with this exclusion will be strong enough to bring about a sequel to the present volume, devoted to the cohomology of algebraic number fields.

From the very start the book was not just intended as a modern sourcebook on algebraic number theory, but also as a convenient textbook for a course. This intention was increasingly jeopardized by the unexpected growth of the material which had to be covered in view of the intrinsic necessities of the theory. Yet I think that the book has not lost that character. In fact, it has passed a first test in this respect. With a bit of careful planning, the basic content of the first three chapters can easily be presented in one academic year (if possible including infinite Galois theory). The following term will then provide scarce, yet sufficient room for the class field theory of chapters IV–VI.

Sections 11–14 of chapter I may mostly be dropped from an introductory course. Although the results of section 12 on *orders* are irrelevant for the sequel, I consider its insertion in the book particularly important. For one thing, orders constitute the rings of multipliers, which play an eminent role in many diophantine problems. But most importantly, they represent the analogues of *singular* algebraic curves. As cohomology theory becomes increasingly important for algebraic number fields, and since this is even more true of *algebraic K-theory*, which cannot be constructed without singular schemes, the time has come to give orders an adequate treatment.

In chapter II, the special treatment of henselian fields in section 6 may be restricted to complete valued fields, and thus joined with section 4. If pressed for time, section 10 on higher ramification may be omitted completely.

The first three sections of chapter III should be presented in the lectures since they highlight a new approach to classical results of algebraic number theory. The subsequent theory concerning the theorem of Grothendieck-Riemann-Roch is a nice subject for a student seminar rather than for an introductory course.

Finally, in presenting class field theory, it saves considerable time if the students are already familiar with profinite groups and infinite Galois theory. Sections 4–7 of chapter V, on formal groups, Lubin-Tate theory and the theory of higher ramification may be omitted. Cutting out even more, chapter V, 3, on the Hilbert symbol, and VI, 7 and 8, still leaves a fully-fledged theory, which is however unsatisfactory because it remains in the abstract realm, and is never linked to classical problems.

A word on the exercises at the end of the sections. Some of them are not so much exercises, but additional remarks which did not fit well into the main text. The reader is encouraged to prove his versatility in looking up the literature. I should also point out that I have not actually done all the exercises myself, so that there might be occasional mistakes in the way they are posed. If such a case arises, it is for the reader to find the correct formulation. May the reader's reaction to such a possible slip of the author be mitigated by Goethe's distich:

> "Irrtum verläßt uns nie, doch ziehet ein höher Bedürfnis
> Immer den strebenden Geist leise zur Wahrheit hinan." *

During the writing of this book I have been helped in many ways. I thank the Springer Verlag for considering my wishes with generosity. My students *I. Kausz*, *B. Köck*, *P. Kölcze*, *Th. Moser*, *M. Spiess* have critically examined larger or smaller parts, which led to numerous improvements and made it possible to avoid mistakes and ambiguities. To my friends *W.-D. Geyer*, *G. Tamme*, and *K. Wingberg* I owe much valuable advice from which the book has profited, and it was *C. Deninger* and *U. Jannsen* who suggested that I give a new treatment of Hecke's theory of theta series and L-series. I owe a great debt of gratitude to Mrs. *Eva-Maria Strobel*. She drew the pictures and helped me with the proofreading and the formatting of the text, never tiring of going into the minutest detail. Let me heartily thank all those who assisted me, and also those who are not named here. Tremendous thanks are due to Mrs. *Martina Hertl* who did the typesetting of the manuscript in TEX. That the book can appear is

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* Error is ever with us. Yet some angelic need
Gently coaxes our striving mind upwards, towards truth.
(Translation suggested by *Barry Mazur*.)

essentially due to her competence, to the unfailing and kind willingness with which she worked through the long handwritten manuscript, and through the many modifications, additions, and corrections, always prepared to give her best.

Regensburg, February 1992 *Jürgen Neukirch*

Table of Contents

Chapter I
Algebraic Integers

§ 1. The Gaussian Integers

The equations

$$2 = 1+1,\ 5 = 1+4,\ 13 = 4+9,\ 17 = 1+16,\ 29 = 4+25,\ 37 = 1+36$$

show the first prime numbers that can be represented as a sum of two squares. Except for 2, they are all $\equiv 1 \bmod 4$, and it is true in general that any odd prime number of the form $p = a^2 + b^2$ satisfies $p \equiv 1 \bmod 4$, because perfect squares are $\equiv 0$ or $\equiv 1 \bmod 4$. This is obvious. What is not obvious is the remarkable fact that the converse also holds:

(1.1) Theorem. *For all prime numbers $p \neq 2$, one has:*

$$p = a^2 + b^2 \quad (a, b \in \mathbb{Z}) \quad \Longleftrightarrow \quad p \equiv 1 \bmod 4 .$$

The natural explanation of this arithmetic law concerning the ring $\mathbb{Z}$ of rational integers is found in the larger domain of the **gaussian integers**

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}, \quad i = \sqrt{-1} .$$

In this ring, the equation $p = x^2 + y^2$ turns into the product decomposition

$$p = (x + iy)(x - iy),$$

so that the problem is now when and how a prime number $p \in \mathbb{Z}$ factors in $\mathbb{Z}[i]$. The answer to this question is based on the following result about unique factorization in $\mathbb{Z}[i]$.

(1.2) Proposition. *The ring $\mathbb{Z}[i]$ is euclidean, therefore in particular factorial.*

Proof: We show that $\mathbb{Z}[i]$ is euclidean with respect to the function $\mathbb{Z}[i] \to \mathbb{N} \cup \{0\}$, $\alpha \mapsto |\alpha|^2$. So, for $\alpha, \beta \in \mathbb{Z}[i]$, $\beta \neq 0$, one has to verify the existence of gaussian integers γ, ρ such that

$$\alpha = \gamma\beta + \rho \quad \text{and} \quad |\rho|^2 < |\beta|^2 .$$

It clearly suffices to find $\gamma \in \mathbb{Z}[i]$ such that $\left|\frac{\alpha}{\beta} - \gamma\right| < 1$. Now, the