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Superconvergence Analysis and a Posteriori Error Estimation in Finite Element Methods

Ningning Yan

（有限元超收敛分析及后验误差估计）

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Preface to the Series in Information and Computational Science

Since the 1970s, Science Press has published more than thirty volumes in its series Monographs in Computational Methods. This series was established and led by the late academician, Feng Kang, the founding director of the Computing Center of the Chinese Academy of Sciences. The monograph series has provided timely information of the frontier directions and latest research results in computational mathematics. It has had great impact on young scientists and the entire research community, and has played a very important role in the development of computational mathematics in China.

To cope with these new scientific developments, the Ministry of Education of the People's Republic of China in 1998 combined several subjects, such as computational mathematics, numerical algorithms, information science, and operations research and optimal control, into a new discipline called Information and Computational Science. As a result, Science Press also reorganized the editorial board of the monograph series and changed its name to Series in Information and Computational Science. The first editorial board meeting was held in Beijing in September 2004, and it discussed the new objectives, and the directions and contents of the new monograph series.

The aim of the new series is to present the state of the art in Information and Computational Science to senior undergraduate and graduate students, as well as to scientists working in these fields. Hence, the series will provide concrete and systematic expositions of the advances in information and computational science, encompassing also related interdisciplinary developments.

I would like to thank the previous editorial board members and assistants, and all the mathematicians who have contributed significantly to the monograph series on Computational Methods. As a result of their contributions the monograph series achieved an outstanding reputation in the community. I sincerely wish that we will extend this support to the new Series in Information and Computational Science, so that the new series can equally enhance the scientific development in information and computational science in this century.

Shi Zhongci

2005.7

Preface

It is well known that the finite element method plays an important role in scientific and engineering computing. The high performance finite element methods based on superconvergence analysis and a posteriori error estimates aim at improving the accuracy and the efficiency of the finite element methods. As for the importance and usefulness of these numerical methods for solving partial differential equations, there have been extensive studies on superconvergence analysis and a posteriori error estimates, and the literature on this aspect is huge.

Our purpose in this book is to give an essentially self-contained presentation of the mathematical theory underlying the global superconvergence analysis and the recovery type a posteriori error estimates. Since, as mentioned in the beginning, the literature and research on superconvergence and a posteriori error estimation are huge, it is impossible to include all relevant material in this book. This book tries to summarize most of the research results on global superconvergence analysis completed by the author and her colleagues, especially by Professor Q. Lin's group. Instead of using the Green function theory as the theoretical basis as in most other earlier books on superconvergence, our global superconvergence analysis is based on the so-called integral identity technique. For a posteriori error estimates, this book focuses on the recovery type a posteriori error estimate, which is more close to the superconvergence analysis in its theoretical analysis. The emphases and selection of the topics reflects our (author and colleagues) involvement in the field over the past 15 years.

The following is the outline of the contents of this book:

In Chapter 1, we focus on discussing a simple model problem (Poisson equation) and a simple finite element space (bilinear conforming finite elements on rectangular meshes). For the simple model problem and the simple finite element space, we explain our basic framework of global superconvergence analysis. These basic idea and technique will be used to deal with much more complicated problems and finite element spaces in other chapters. The analysis in this chapter is very simple and easy to understand. The reader can quickly acquire a good understanding of how to analyze the global superconvergence by using the integral identity technique.

In the following two chapters, we extended the basic theory presented in Chapter 1 to the more complicated problems and finite element spaces. In Chapter 2, we provide the integral identities for various finite element spaces, including conforming rectangular finite elements, nonconforming finite elements, mixed finite elements, $\cdots$. Using the integral identities provided in Chapter 2, we present the superconvergence analysis in Chapter 3.

The superclose properties are proved for elliptic equations, parabolic equations, hyperbolic equations, integral equations, integral-differential equations and some nonlinear problems. It is shown that using the integral identity technique, we can obtain the improved error estimate (superclose or improved optimal error estimate) for various partial differential equations (not limited on elliptic equations).

In Chapter 4, we furthermore discuss more high accuracy finite element algorithms (extrapolation, defect correction, local superconvergence, ultraconvergence, and so on) based on the integral identities provided in Chapter 2. The numerical examples demonstrating our theoretical results in Chapter 3 and 4 are provided. The reader will see that the integral identity technique is powerful not only for superconvergence analysis, but also for extrapolation, defect correction, and so on. Moreover, combining the integral identity technique and Green function theory, remarkable results on ultraconvergence and local superconvergence can be obtained.

Chapter 5 is devoted to introducing a posteriori error estimates. Here, we concentrate on the recovery type a posteriori error estimate. It is because the theoretical analysis of the recovery type a posteriori error estimate is more closely related to the superconvergence analysis which is the principal subject of this book.

This book is intended for postgraduate and graduate students, university teachers, scientists and engineers, who study or are engaged in computational mathematics, computational mechanics, applied mathematics, scientific and engineering computation or other related special fields. A desirable mathematical background for reading this book includes the basic knowledge of partial differential equations, functional analysis, numerical PDE, including Sobolev spaces and finite element theory. But because this book does not use the very deep mathematical theory, it also can be understood and used by students and engineers who are only familiar with calculus.

The most basic material in this book is taken from another, earlier Chinese book entitled *Structure and Analysis of Efficient Finite Element Methods*, written by Professor Q. Lin and myself in 1996. I would like to express my special gratitude to Professor Q. Lin for his great help and support of my research, especially my work for this book. I also would like to address my thanks to my teachers or colleagues Chuanmiao Chen, Liu Du, Aixiang Huang, Hongci Huang, Yunqing Huang, Jichun Li, Kaitai Li, Ruo Li, Zicai Li, Huipo Liu, Mingjun Liu, Wenbin Liu, Ping Luo, Jianhua Pan, Dongyang Shi, Xuecheng Tai, Junping Wang, Reifeng Xie, Jinchao Xu, Shuhua Zhang, Zhimin Zhang, Aihui Zhou, Junming Zhou, Qiding Zhu, $\cdots$. They provided a lot of material, suggestions and support for this book. Moreover, I would like to thank Professor H. Brunner and many other friends, who helped me to improve the English of this book, and I also would like to address my thanks to Ms. L. Li and Dr. H. Liu for typing some material and creating the pictures for this book. Furthermore, I would like to thank Prof. Tang Tao, who encouraged me to write this book and gave me a lot of helpful suggestions.

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I welcome comments and corrections to the book and can be reached at ynn@amss.ac.cn.

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Contents

Chapter 1

Basic framework

In this chapter, the general principle of the superconvergence analysis will be provided. In order to explain the framework of the superconvergence analysis clearly, we concentrate our discussion in this chapter on a simple model problem (Poisson equation) and a simple finite element space (bilinear elements on the rectangular meshes). The reader can quickly acquire a good understanding of the basic idea and the technique of the global superconvergence analysis. The interested reader can find more extension to the more complicated finite element spaces (Chapter 2) and the more complicated problems (Chapter 3).

1.1 Preliminaries

We assume that the reader is familiar with the basic theory of Sobolev spaces and the finite element method. In this section we introduce some definitions, notations and a few well-known properties and conclusions. Details can be found in many books, e.g., [1] and [36].

In this book, the partial differential equations are defined on the bounded domain $\Omega \subset \mathcal{R}^n$, $n = 1, 2, 3$, with the boundary $\partial\Omega$. We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces (see, e.g., [1]) on Ω with the norm

$$\|w\|_{m,p,\Omega} = \left(\sum_{k=0}^{m} |w|_{k,p,\Omega}^{p} \right)^{\frac{1}{p}}, \quad \forall w \in W^{m,p}(\Omega),\ 1 \leqslant p < \infty,$$

and the seminorm

$$|w|_{k,p,\Omega} = \left(\int_{\Omega} \sum_{|\alpha|=k} \left(\frac{\partial^{|\alpha|} w}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \right)^{p} \right)^{\frac{1}{p}}.$$

Here, $w = w(x_1, \cdots, x_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the derivatives in above formula are the weak derivatives defined in Sobolev space (see, e.g., [1]). For $p = \infty$, define the norm

$$\|w\|_{m,\infty,\Omega} = \sup_{0 \leqslant k \leqslant m} |w|_{k,\infty,\Omega}, \quad \forall w \in W^{m,\infty}(\Omega),$$

with the seminorm

$$|w|_{k,\infty,\Omega} = \max_{|\alpha|=k} \operatorname*{ess\,sup}_{x\in\Omega} \left\{ \left| \frac{\partial^{|\alpha|} w}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \right| \right\}.$$

Moreover, we set

$$W_0^{m,p}(\Omega) = \left\{ w \in W^{m,p}(\Omega) : \gamma \left(\frac{\partial^{|\alpha|} w}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \right) |_{\partial\Omega} = 0,\ |\alpha| \leqslant m-1 \right\},$$

where γ is the trace operator. We denote $W^{m,2}(\Omega)$ (or $W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ (or $H_0^m(\Omega)$) with the norm $\|\cdot\|_{m,\Omega}$ and the seminorm $|\cdot|_{m,\Omega}$. In particularly, when the domain Ω does not need to be emphasized, we use $\|\cdot\|_m$ ($|\cdot|_m$) simply to denote $\|\cdot\|_{m,\Omega}$ ($|\cdot|_{m,\Omega}$). We also need a fractional Sobolev space with the norm

$$\|w\|_{\frac{1}{2},\Gamma} = \inf_{v\in H^1(\Omega),\ \gamma v = w} \{\|v\|_{1,\Omega}\}.$$

In this book, we use c or C to denote a general positive constant independent of h, which can represent different values in different places.

Sobolev embedding theorems are important in our theoretical analysis. The details can be found in many books, e.g., [1]. In this book, the useful conclusion is that if $\partial\Omega$ satisfies the interior cone condition, then

$$W^{k,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-kp}}(\Omega) & \text{if} \quad kp < n \\ C^l(\Omega) & \text{if} \quad 0 \leqslant l < k - \frac{n}{p} \end{cases},$$

$$W^{k,p}(\Omega) \hookrightarrow W^{l,s}(\Omega), \quad s = \frac{np}{n-(k-l)p}, \quad 0 < (k-l)p < n.$$

We often use the following properties: for $n = 2$,

$$\|w\|_{k,p,\Omega} \leqslant C\|w\|_{k+1,\Omega}, \quad p \in [1,\infty),$$

$$\|w\|_{k,\infty,\Omega} \leqslant C\|w\|_{k+1,p,\Omega}, \quad p > 2.$$

In this book, we will use the finite element space (see, e.g., [36]) as follows. Let Ω^h be a polygonal approximation to Ω with the boundary $\partial\Omega^h$. Let $\mathcal{T}^h$ be a partitioning of Ω^h into disjoint element τ, so that $\bar{\Omega}^h = \bigcup_{\tau\in\mathcal{T}^h} \bar{\tau}$. Let h_τ be the radius of the circumscribed circle of the element τ, $h = \max_{\tau\in\mathcal{T}^h}\{h_\tau\}$. We further require that $P_i \in \partial\Omega^h \Rightarrow P_i \in \partial\Omega$, where $\{P_i\}(i = 1,\cdots,J)$ is the vertex set associated with the partitioning $\mathcal{T}^h$. For ease of exposition we will assume that $\Omega^h = \Omega$. Associated with $\mathcal{T}^h$ is a finite-dimensional subspace V^h defined by

$$V^h = \{v : v|_\tau \in P_k(\tau),\ \forall \tau \in \mathcal{T}^h\},$$

where P_k is the polynomial space of order k. Let

$$V_0^h = \{v \in V^h : v|_{\partial\Omega} = 0\}.$$

If the element τ is regular such that

$$1 \leqslant \frac{h_\tau}{\rho_\tau} \leqslant C,$$

where ρ_τ is the radius of the inscribed circle of the element τ, then we have the following inverse estimate:

$$\|v\|_{k,p,\tau} \leqslant C h_\tau^{l-k+n(\frac{1}{p}-\frac{1}{q})} \|v\|_{l,q,\tau}, \quad l \leqslant k, \ q \leqslant p.$$

If u_I^p is the standard interpolant of u in the finite element space V^h with order p, we have the well-known interpolation error estimate:

$$\|u - u_I^p\|_{m,q,\Omega} \leqslant C h^{k+1-m} |u|_{k+1,q,\Omega}, \quad 0 \leqslant m \leqslant k+1, \ 1 \leqslant k \leqslant p.$$

In addition, we will use the Poincaré inequality:

$$\|w\|_{1,p,\Omega} \leqslant C|w|_{1,p,\Omega}, \quad \forall w \in W_0^{1,p}(\Omega),$$

and the trace theorem:

$$\|\gamma w\|_{k-1,\partial\Omega} \leqslant C\|w\|_{k-\frac{1}{2},\partial\Omega} \leqslant C\|w\|_{k,\Omega}, \quad k \geqslant 1.$$

In the last inequality, the constant C is dependent on the domain Ω. Especially, when the domain is the element τ, and the element τ is regular, then the inequality becomes

$$\|\gamma w\|_{k-1,\partial\tau} \leqslant C\|w\|_{k-\frac{1}{2},\partial\Omega} \leqslant C h_\tau^{-\frac{1}{2}} \|w\|_{k-1,\tau} + C h_\tau^{\frac{1}{2}} |w|_{k,\tau}.$$

1.2 Model problem

Let us start our introduction about superconvergence analysis from a simple model problem – the Poisson equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}, \tag{1.2.1}$$

where $\Omega \subset \mathcal{R}^2$ is a two-dimensional bounded domain with the boundary $\partial\Omega$, $f \in L^2(\Omega)$ is the given function, and $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

It is well known (see, e.g., [36]) that when the solution of (1.2.1) is a classical solution, the problem (1.2.1) is equivalent to the following variational problem: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega), \tag{1.2.2}$$

where $\nabla u = \left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right)$.

In order to approximate the variational problem (1.2.2) by the finite element method, we construct the standard conforming finite element space $V_0^h \subset H_0^1(\Omega)$ on the regular mesh $\mathcal{T}^h$ (see [36] for details). Then, the finite element scheme of (1.2.2) is to look for $u_h \in V_0^h$ such that

$$\int_\Omega \nabla u_h \nabla v_h = \int_\Omega f v_h, \quad \forall v_h \in V_0^h \subset H_0^1(\Omega). \tag{1.2.3}$$

It is well known that the standard error estimate for (1.2.2) and (1.2.3) is

$$\|u - u_h\|_{1,\Omega} \leqslant C\|u - u_I\|_{1,\Omega} \leqslant Ch^k \|u\|_{k+1,\Omega}, \tag{1.2.4}$$

if $u \in H^{k+1}(\Omega)$, where h is the diameter of the largest element in $\mathcal{T}^h$, k is the order of the polynomials of the finite element space, and C is a constant independent of u and Ω.

The proof of the error estimate (1.2.4) can be sketched as follows. Let $v_h = u_h - u_I$, where $u_I \in V^h$ is the interpolant of u. Then, $v_h \in V^h \subset H_0^1(\Omega)$. It follows from (1.2.2) and (1.2.3) that

$$|v_h|_{1,\Omega}^2 = \int_\Omega \nabla(u_h - u_I)\nabla v_h = \int_\Omega \nabla(u - u_I)\nabla v_h.$$

Therefore, the error between u_h and u_I can be bounded by the inequality

$$|u_h - u_I|_{1,\Omega} \leqslant \sup_{v_h \in V^h} \frac{\displaystyle\int_\Omega \nabla(u - u_I)\nabla v_h}{|v_h|_{1,\Omega}}. \tag{1.2.5}$$

The right-hand side of (1.2.5) is the weak error form (a kind of error presented by the weak integral form). If we simply use the Schwarz inequality on the right-hand side of (1.2.5), we obtain

$$|u_h - u_I|_{1,\Omega} \leqslant \sup_{v_h \in V^h} \frac{|u - u_I|_{1,\Omega}|v_h|_{1,\Omega}}{|v_h|_{1,\Omega}} = |u - u_I|_{1,\Omega}. \tag{1.2.6}$$

Using the Poincaré's inequality, we find

$$\|u_h - u_I\|_{1,\Omega} \leqslant C|u_h - u_I|_{1,\Omega} \leqslant C|u - u_I|_{1,\Omega} \leqslant C\|u - u_I\|_{1,\Omega}. \tag{1.2.7}$$

The inequality (1.2.7) implies that

$$\|u - u_h\|_{1,\Omega} \leqslant \|u - u_I\|_{1,\Omega} + \|u_I - u_h\|_{1,\Omega} \leqslant C\|u - u_I\|_{1,\Omega}. \tag{1.2.8}$$

Then (1.2.4) follows from (1.2.8) and the standard interpolation error estimate:

$$\|u - u_I\|_{1,\Omega} \leqslant C\|u - u_I\|_{1,\Omega} \leqslant Ch^k \|u\|_{k+1,\Omega}.$$

(1.2.4) means that the order of the error $\|u - u_h\|_{1,\Omega}$ is optimal. It also has been shown in (1.2.6) that the order of the error $\|u_h - u_I\|_{1,\Omega}$ is optimal. But if we analyze the weak integral form (right-side of (1.2.5)) carefully, instead of simply using the Schwarz inequality to get (1.2.6), a higher convergence order can be proved for the error $\|u_h - u_I\|_{1,\Omega}$ under some stronger conditions. In the following section, we will analyze the weak integral form $\int_\Omega \nabla(u - u_I)\nabla v_h$ carefully using the integral identities technique, so that the superclose property (the higher convergence order for the error $\|u_h - u_I\|_{1,\Omega}$) and the global superconvergence can be proved.

1.3 Integral identity

In this section, we will derive the integral identity which will be used for the global superconvergence analysis (see more details in, e.g., [69] and [88]).

For simplicity, we will only consider the simple finite element space: the bilinear finite element space on the rectangular meshes. Let Ω be a rectangular domain, and $\mathcal{T}^h$ the rectangular mesh. The element $\tau \in \mathcal{T}^h$ is represented by

$$\tau = (x_\tau - h_\tau, x_\tau + h_\tau) \times (y_\tau - k_\tau, y_\tau + k_\tau),$$

where (x_τ, y_τ) is the midpoint of the element τ, $2h_\tau$ and $2k_\tau$ are the sizes of τ in $x-$direction and $y-$direction, respectively. Let l_l, l_r be the left and right edges of the element τ, and l_u, l_b the upper and bottom edges of the element τ (see Fig. 1.3.1).

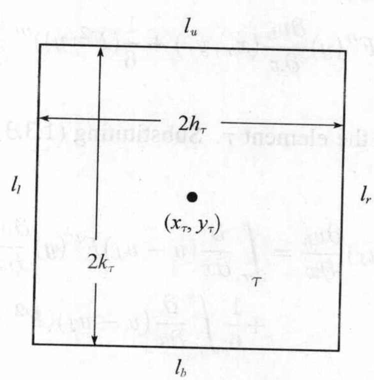

Fig. 1.3.1 The element τ.

The bilinear finite element space V^h is defined on the mesh $\mathcal{T}^h$ such that

$$V^h = \{v_h \in C(\Omega) : v_h|_\tau \in Q_1\},$$

where Q_1 is the bilinear function space, and

$$V_0^h = \{v_h \in V^h : v_h|_{\partial\Omega} = 0\}.$$

Introduce the auxiliary function

$$F(y)=\frac{1}{2}\Big((y-y_\tau)^2-k_\tau^2\Big). \tag{1.3.1}$$

The following lemma can be proved by means of simple integration by parts.

Lemma 1.3.1 Suppose that $u\in H^3(\Omega)$. Then on the element τ, we have that for all $v_h\in V_0^h$

$$\int_\tau \frac{\partial}{\partial x}(u-u_I)\frac{\partial v_h}{\partial x}=\int_\tau \frac{\partial^3 u}{\partial x\partial y^2}\Big(F(y)\frac{\partial v_h}{\partial x}-\frac{2}{3}F(y)(y-y_\tau)\frac{\partial^2 v_h}{\partial x\partial y}\Big), \tag{1.3.2}$$

where u_I is the piecewise bilinear Lagrange interpolant of u, and $F(y)$ is defined by (1.3.1).

Proof Note that

$$F'(y)=y-y_\tau,\qquad F''(y)=1,\qquad (F^2(y))'''=6(y-y_\tau),$$

and v_h is a piecewise bilinear function on the element τ. Then it is easy to see that

$$\begin{aligned}\frac{\partial v_h}{\partial x}&=\frac{\partial v_h}{\partial x}(x_\tau,y_\tau)+(y-y_\tau)\frac{\partial^2 v_h}{\partial x\partial y}\\&=F''(y)\frac{\partial v_h}{\partial x}(x_\tau,y_\tau)+\frac{1}{6}(F^2(y))'''\frac{\partial^2 v_h}{\partial x\partial y},\end{aligned} \tag{1.3.3}$$

where $\frac{\partial^2 v_h}{\partial x\partial y}$ is constant on the element τ. Substituting (1.3.3) into the left-hand side of (1.3.2), we have that

$$\begin{aligned}\int_\tau \frac{\partial}{\partial x}(u-u_I)\frac{\partial v_h}{\partial x}=&\int_\tau \frac{\partial}{\partial x}(u-u_I)F''(y)\frac{\partial v_h}{\partial x}(x_\tau,y_\tau)\\&+\frac{1}{6}\int_\tau \frac{\partial}{\partial x}(u-u_I)(F^2(y))'''\frac{\partial^2 v_h}{\partial x\partial y}.\end{aligned} \tag{1.3.4}$$

Let $w=u-u_I$, let l_u and l_b be the upper edge and the bottom edge of the element τ, respectively (see Fig. 1.3.1). Integrating by parts, it can be proved that

$$\begin{aligned}\int_\tau \frac{\partial w}{\partial x}F''(y)=&\Big(\int_{l_u}-\int_{l_b}\Big)\frac{\partial w}{\partial x}F'(y)dx-\int_\tau \frac{\partial^2 w}{\partial x\partial y}F'(y)\\=&F'(y_\tau+k_\tau)\Big(w(x_\tau+h_\tau,y_\tau+k_\tau)-w(x_\tau-h_\tau,y_\tau+k_\tau)\Big)\\&-F'(y_\tau-k_\tau)\Big(w(x_\tau+h_\tau,y_\tau-k_\tau)-w(x_\tau-h_\tau,y_\tau-k_\tau)\Big)\end{aligned}$$