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A. Majda

Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables

高维守恒律与可压缩流体

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内容简介

这是一部研究高维守恒律与可压缩流体方面的重要著作，书中介绍了高维守恒律的特性，深入探讨了可压缩流体理论，其中所涉及的许多问题是当今相关领域研究的热门课题。

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Andrew Majda

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Chapter 1

Introduction

Conservation laws arise from the modeling of physical processes through the following three steps:

1) The appropriate *physical balance laws* are derived for m -physical quantities, u_1, \dots, u_m with $u = {}^t(u_1, \dots, u_m)$ and $u(x, t)$ defined for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ ($N = 1, 2$, or 3), $t \geq 0$ and with the values $u(x, t)$ lying in an open subset, G , of \mathbb{R}^m , the *state space*. The state space G arises because physical quantities such as the density or total energy should always be positive; thus the values of u are often constrained to an open set G .

2) The flux functions appearing in these balance laws are idealized through prescribed nonlinear functions, $F_j(u)$, mapping G into \mathbb{R}^m , $j = 1, \dots, N$ while source terms are defined by $S(u, x, t)$ with S a given smooth function of these arguments with values in \mathbb{R}^m . In particular, the *detailed microscopic* effects of diffusion and dissipation are ignored.

3) A generalized version of the principle of virtual work is applied (see Antman [1]).

The formal result of applying the three steps (1)-(3) is that the m physical quantities u define a weak solution of an $m \times m$ system of conservation laws,

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}^N} (\psi_t \cdot u + \sum_{j=1}^N \psi_{x_j} \cdot F_j(u) + \psi \cdot S(u, x, t)) dx dt \quad (1.1)$$

for all $\psi \in C_0^1(\mathbb{R}^N \times \mathbb{R}^+)$, $\psi(x, t) \in \mathbb{R}^m$.

A beautiful discussion of the logical subtlety involved in the process (1)-(3) in a single space variable can be found in the article by Antman ([1]). If one is willing to assume at the outset that u is

bounded and is a function of bounded variation, i.e., the first distribution derivatives of u are locally finite Borel measures, then the steps (1)-(3) can be rigorously justified immediately through an application of the generalized Green's formula for BV functions ([9], [31]). Almost any introduction to fluid dynamics or continuum mechanics ([4], [17], and [29] are recommended as well as the companion lectures of Professor Serrin) gives a formal description of the process in (1)-(3).

Systems of conservation laws describe many physical phenomena. In particular, important examples occur in gas dynamics, shallow water theory, combustion theory, nonlinear elasticity, magneto-fluid dynamics, and petroleum reservoir engineering ([7] contains an excellent discussion of many physical examples; see [2] for magnetofluid dynamics).

In these lectures, we will discuss in detail several properties of the solutions of the four examples of conservation laws provided by (1) the Euler equations of gas dynamics, (2) the isentropic Euler gas equations (useful in shallow water theory), (3) the inviscid (and viscous) combustion equations, and (4) the nonlinear wave equation (both as a model for 1-D gas dynamics in Lagrangian coordinates and also as the system of conservation laws for 1-D hyperelasticity). In addition, we will occasionally emphasize the use of highly instructive rational but largely formal arguments from applied perturbation theory which indicate the fashion in which simpler systems such as scalar conservation laws or the Burgers' equation quantitatively model special solutions of these more complex systems.

In these lectures, we will only study C^1 and piecewise C^1 solutions of $m \times m$ systems of conservation laws with an emphasis on several space dimensions. In particular for C^1 solutions of (1.1), it follows easily that

$$\frac{\partial u}{\partial t} + \sum_{j=1}^N \frac{\partial}{\partial x_j} F_j(u) = S(u, x, t) \quad (1.2)$$

pointwise for $(x, t) \in \mathbb{R}^N \times \mathbb{R}^+$. The other simple consequence of (1.1) which we will use applies to piecewise smooth solutions of (1.1). For piecewise smooth solutions, the conditions in (1.1) are equivalent to requiring the following two properties: 1) In regions of smoothness for u , the equations in (1.2) are satisfied; 2) if u has a jump discontinuity across a space-time hypersurface, S , with outward space-time normal (n_t, n) , $n = (n_1, \dots, n_N)$, the following nonlinear boundary conditions are satisfied,

$$(n_t[u] + \sum_{j=1}^N n_j[F_j(u)])|_S = 0 \quad (1.3)$$

with the brackets, $[]$, denoting the jump in a quantity. The conditions in (1.3) are the generalized Rankine-Hugoniot jump conditions. For classical gas dynamics, the surface S where (1.3) is satisfied are either *shock fronts* or *vortex sheets*. Furthermore, the surfaces, S , should not only satisfy (1.3) but *should also include the subtle macroscopic effects of diffusion through entropy conditions*. (See Chapter 4 and the companion lectures by Professor Bardos.) Since the surfaces S are not known in advance, the conditions in (1.3) define a highly nonlinear free boundary value problem for an $m \times m$ system of equations (see Chapter 4 of these lectures).

We will emphasize some of the special properties of the physical equations in multi-D which distinguish these systems from the general case. The remaining three chapters each have a separate introduction so we will not describe their contents here. Several open problems are discussed throughout the text - we believe that many of these problems are tractable with a few new ideas. These lectures contain one glaring omission - numerical methods are not discussed. Not only is this area of great practical importance but designing and analyzing such methods can be remarkably subtle. Furthermore, the theoretical insight into multi-D conservation laws gained from high quality numerical calculations can be substantial. The bibliography at the end of this chapter contains a short and biased section devoted to the design and analysis of numerical methods for conservation laws - for readers who like to browse, the volumes of the Journal of Computational Physics and Mathematics of Computation over the last five years are also recommended. Finally, the author hopes that these lectures stimulate others to work on the numerous open problems in this area.

1.1. Some Physical Examples of Systems of Conservation Laws

1. Gas Dynamics

The compressible Euler equations in three space dimensions are the 5×5 system of conservation laws,

$$\begin{aligned} (a) \quad & \frac{\partial \rho}{\partial t} + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\rho v_j) = 0 \\ (b) \quad & \frac{\partial}{\partial t} (\rho v_i) + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3 \\ (c) \quad & \frac{\partial}{\partial t} (\rho E) + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\rho v_j E + p v_j) = 0 \end{aligned} \quad (1.4)$$

where (a), (b), and (c) represent the respective conservation of mass, momentum, and total energy with δ_{ij} the Kronecker delta. Here ρ is the density, $\vec{v} = (v_1, v_2, v_3)$ is the fluid velocity, p is the scalar pressure, and E is the specific total energy. The quantity E is the sum of contributions from the kinetic energy, $\frac{1}{2} |\vec{v}|^2$, and the internal energy, e , i.e., $E = \frac{1}{2} |\vec{v}|^2 + e$. For a single gas, e becomes a well-defined function of ρ and p through the theory of thermodynamics, $e = e(\rho, p)$. Other interesting physical quantities, the temperature, $T(\rho, p)$ and the entropy, $S(\rho, p)$, are defined through the following consequence of the second law of thermodynamics

$$T dS = de - \frac{p}{\rho^2} d\rho. \quad (1.5)$$

In particular, we will often discuss *ideal gases* where the quantities, e , T , S , have the explicit formulae,

$$\begin{aligned} e(\rho, p) &= \frac{p}{\rho(\gamma-1)} = \frac{T}{\gamma-1} \\ T(\rho, p) &= \frac{p}{\rho} \\ e^S &= p\rho^{-\gamma}, \quad \gamma > 1, \text{ constant.} \end{aligned} \quad (1.6)$$

Strictly speaking, to write the equations in (1.4) in the form of (1.2), the conservative variables $\vec{t}(\rho, \vec{m}, E)$ need to be introduced as \vec{u} where $\vec{m} = \rho \vec{v}$ is the momentum, $E = \rho E$ is the total energy, and the flux functions need to be rewritten in terms of (ρ, \vec{m}, E) but we will not do this here. In fact, in discussing solutions of (1.1) in regions of smoothness for \vec{u} , one often uses the velocity and a convenient choice of two additional variables among the five quantities S , T , p , ρ , e as independent variables. For example, the Euler equations for an ideal gas can be written in terms of the variables p , v , S in regions of smoothness in the equivalent form

$$\begin{aligned} \frac{Dp}{Dt} + \gamma p \operatorname{div} \vec{v} &= 0 \\ \rho \frac{Dv}{Dt} + \nabla p &= 0 \\ \frac{DS}{Dt} &= 0 \end{aligned} \quad (1.7)$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial}{\partial x_j}$, the convective derivative along fluid

particle trajectories. Here $\rho(p, S)$ is defined by the last formula in

(1.6) and since ρ must be positive it is evident that the state space $G \subseteq R^5$ is defined in this example by

$$G = \{^t(p, v, S) \mid p > 0\}.$$

2. Isentropic Gas Dynamics and the Shallow Water Equation

In the isentropic gas equations, one retains the equations (a) and (b) described above in (1.4) and closes this system of four equations for the four quantities ρ , v by postulating a functional relationship between p and ρ , i.e., $p = p(\rho)$ - the ideal gas approximation in this case is $p(\rho) = A\rho^\gamma$, $A > 0$, $\gamma > 1$. In terms of the nonconservative variables, (p, v) , the ideal gas isentropic equations have the form

$$\begin{aligned} \frac{Dp}{Dt} + \gamma p \operatorname{div} v &= 0 \\ \rho \frac{Dv}{Dt} + \nabla p &= 0. \end{aligned} \quad (1.8)$$

In particular, the system in (1.8) follows from the one in (1.7) by setting $S = S_0$ and deducing the relationship

$$\rho = p^{1/\gamma} e^{-S_0/\gamma} \quad (1.9)$$

directly from (1.6). With $N = 1$ or 2 , the case $\gamma \equiv 2$ arises in the shallow water approximation (see [7]).

3. The Inviscid Combustion Equations

Here we will not write down the general equations (see Strehlow [30]) but instead concentrate on an instructive idealized special case. We assume that there are only two species present, the unburnt gas and the burnt gas, and that the unburnt gas is converted to burnt gas through a one-step irreversible exothermic chemical reaction with an Arrhenius kinetics mechanism. With Z denoting the mass fraction of unburnt gas so that $1-Z$ is the mass fraction of burnt gas, we have the continuum chemistry equation,

$$\begin{aligned} (d) \quad \frac{\partial}{\partial t} (\rho Z) + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\rho v_j Z) &= -\rho W(Z, T) \\ \text{with } W(Z, T) &\equiv K e^{-A_0/T} Z \end{aligned} \quad (1.10)$$

adjoined to the equations of compressible fluid flow, (a), (b), and (c) defined in (1.4). As regards the equations in (1.4), a crucial modifi-

cation of the internal energy, e , is the only change in these equations. The internal energy of the mixture, $e(p, \rho, Z)$, is defined within a constant by

$$e(\rho, p, Z) = Ze_u(\rho, p) + (1-Z)e_b(\rho, p)$$

where e_u , e_b are the internal energies of the unburnt and burnt gas. For simplicity we assume that both the burnt and unburnt gases are ideal with the same γ -gas law so that

$$e_u(\rho, p) = \frac{(T-T_0)}{\gamma-1} + q_0, \quad e_b = \frac{(T-T_0)}{\gamma-1}$$

where q_0 is the normalized energy of formation at the reference temperature T_0 for the unburnt gas - $q_0 > 0$ for an exothermic reaction. Then,

$$\begin{aligned} e(\rho, p, Z) &= \frac{T}{\gamma-1} + q_0 Z \\ T(\rho, p) &= \frac{p}{\rho}. \end{aligned} \tag{1.11}$$

(We have ignored the inessential constant shift in e .) The formulae in equation (1.11) together with the equations (a), (b), (c), and (d) from (1.4) and (1.10) define the inviscid combustion equations. The source term S is nonzero here.

4. The Nonlinear Wave Equation

The nonlinear wave equation is the second order equation,

$$u_{tt} - (K(u_x))_x = 0.$$

If one introduces the vector $u = {}^t(u_1, u_2)$ with $u_1 = u_x$ and $u_2 = u_t$, the first order 2×2 system of conservation laws

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} -u_2 \\ -K(u_1) \end{pmatrix}_x = 0 \tag{1.12}$$

is an equivalent system. The natural state space G defined by the requirement of hyperbolicity for (1.12) is given by

$$G = \{{}^t(u_1, u_2) \mid K'(u_1) > 0\}. \tag{1.13}$$

4A. The Nonlinear Wave Equation in Fluid Flow

In fluid dynamics, (1.12) arises as the equations of one-dimensional isentropic gas motion in Lagrangian coordinates - here x is a mass

coordinate (see [7]), in this context, $u_1 = \tau$, with $\tau = \frac{1}{\rho}$, the specific volume, and $u_2 = v$ the fluid velocity. For an isentropic ideal gas (see (1.9)), $K(u_1)$ has the explicit form

$$K(u_1) = -Au_1^{-\gamma}, \quad A > 0, \quad \gamma > 1 \quad (1.14)$$

and in this case $G = \{^t(u_1, u_2) \mid u_1 > 0\}$.

4B. The Nonlinear Wave Equation in Elasticity Theory

In 1-D elasticity, the nonlinear wave equation arises with $u(x, t) + x$ the deformed location at time t of the material initially located at x , $u_1 = u_x$ the strain, and $K(u_1)$ the stress-strain function (see p. 235 of [7]). In particular, for a hard spring or a "non-Hookian" material ([7]), in a neighborhood of zero, typically,

$K(u_1)$ is a smooth odd function

such as $K(u_1) = u_1/\sqrt{1+u_1^2}$ or $u_1 + au_1^3$. The facts that $K(u_1)$ is convex in case 4A while $K(u_1)$ is typically an odd function in case 4B imply that the nonlinear wave motion in these two cases is qualitatively quite different - see various results in the remainder of Chapter 1 and Chapter 3 below (and also [18]).

1.2. The Importance of Dissipative Mechanisms

We begin by writing the combustion equations in terms of $^t(p, v, T)$ but retaining all the diffusion mechanisms including viscosity, heat conduction, and species diffusion (see [32]). The result is the system of equations of *combustion theory*

$$\begin{aligned} (a) \quad & \frac{Dp}{Dt} + \gamma p \operatorname{div} v = (\gamma-1) \operatorname{div}(\kappa \nabla T) + q_0 \rho (\gamma-1) W(Z, T) \\ & + (\gamma-1) \tilde{v} \sum_{i,j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 - \frac{2}{3} \delta_{ij} (\operatorname{div} v)^2 \\ (b) \quad & \rho \frac{Dv_i}{Dt} + \frac{\partial p}{\partial x_i} = \tilde{v} \sum_j \frac{\partial}{\partial x_j} \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3} \delta_{ij} \operatorname{div} v \right), \quad i = 1, 2, 3 \\ (c) \quad & \rho \frac{DT}{Dt} + (\gamma-1)p \operatorname{div} v = (\gamma-1) \operatorname{div}(\kappa \nabla T) + (\gamma-1)q_0 \rho W(Z, T) \quad (1.15) \\ & + (\gamma-1) \tilde{v} \sum_{i,j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 - \frac{2}{3} \delta_{ij} (\operatorname{div} v)^2 \\ (d) \quad & \rho \frac{DZ}{Dt} = \operatorname{div}(\rho d \nabla Z) - \rho W(Z, T) \\ (e) \quad & \rho = \frac{p}{T}. \end{aligned}$$

(Our somewhat nonstandard choice of the dependent variable (p,v,T) rather than (ρ,v,e) , etc. is motivated by the application to be discussed in Section 2.5 where $\tilde{\nu}$ is the viscosity coefficient, κ is the coefficient of heat conduction, and d is the coefficient of species diffusion.) We have written down all of these diffusion coefficients to illustrate the complexity of the physical equations when all transport processes are included. When the equations in (1.15) are nondimensionalized with macroscopic length and time scales the diffusion coefficients, $\tilde{\nu}$, κ , and d are typically of the order 10^{-3} or 10^{-4} . *One of the objectives of the theory of conservation laws is to incorporate the effects of the small scale diffusion processes on the large scale quantities without resolving the small scale effects in detail - one imposes additional entropy conditions on the weak solutions of (1.1) in order to achieve this. Such an approach is very successful in ideal gas dynamics but can be rather subtle in more complex physical systems like the combustion equations. Flame fronts are significant waves which do not move at the characteristic speeds associated with the inviscid combustion equations defined in (1.10), (1.11) and (1.4). These waves are generally slow moving fronts with velocities of 2 or 3 meters/sec and the wave speeds are governed by a subtle balance between the reaction term $W(Z,T)$, and the diffusion coefficients, κ , d . In fact even for detonation fronts, the fast moving supersonic wave fronts in combustion theory which are analogous to shock fronts, the inviscid shock-layer approximation (known as the Chapman-Jouget theory in this context ([7])) which works so well for ideal gas dynamics can break down completely in describing the actual solutions of the equations (see [11], [20]). In fact, for fluid dynamics with very general equations of state, the standard conclusions of shock-layer analysis which allow one to ignore the detailed effects of diffusion can be completely wrong when heat conduction dominates viscosity ([26]). The above remarks indicate the subtlety involved in ignoring dissipative mechanisms in the theory of conservation laws. In fact, in more complex physical systems such as the equations in (1.15), a more detailed assessment of their effect on the macroscopic length scales is always needed. The simpler inviscid combustion equations described in Section 3 above might be an excellent approximation in a given regime (usually a detonation regime) in practice but one always needs to assess the effects of these diffusion mechanisms in that regime through careful analysis of simpler problems.*

1.3. The Common Structure of the Physical Systems of Conservation Laws and Friedrichs' Theory of Symmetric Systems

We first consider the case where the source terms $S(u, x, t)$, vanish. In this case, any constant u_0 , belonging to the state space, G , is a trivial solution of (1.2). By linearizing about this smooth solution u_0 by considering solutions $u(x, t) = u_0 + v$, we obtain the linearized equations,

$$\frac{\partial v}{\partial t} + \sum_{j=1}^N A_j(u_0) \frac{\partial v}{\partial x_j} = 0, \quad t > 0, \quad x \in \mathbb{R}^N \quad (1.16)$$

$$v(x, 0) \equiv v_0(x)$$

where $A_j(u) = \partial F_j / \partial u$, $j = 1, \dots, N$ are the corresponding $m \times m$ Jacobian matrices. A minimum requirement for a general system of conservation laws from (1.2) is that the linearized Cauchy problem from (1.16) defines a well-posed problem. Many years ago, Friedrichs made the important observation that under reasonable conditions, almost all equations of classical physics of the form (1.2) admit the following structure: For all $u \in G$, there is a positive definite symmetric matrix $\tilde{A}_0(u)$ smoothly varying with u so that

$$\begin{aligned} (a) \quad cI &\leq \tilde{A}_0(u) \leq c^{-1}I, \quad \tilde{A}_0 = \tilde{A}_0^*, \\ &\text{with a constant } c \text{ uniform for } u \in G_1 \text{ and any } G_1 \\ &\text{with } \overline{G_1} \subset G \\ (b) \quad \tilde{A}_0(u)A_j(u) &= \tilde{A}_j(u) \text{ with } \tilde{A}_j(u) = \tilde{A}_j^*(u), \quad j = 1, \dots, N. \end{aligned} \quad (1.17)$$

For example, the equations for an ideal gas in (1.7) are symmetrized by the 5×5 matrix

$$\tilde{A}_0(p, S) = \begin{pmatrix} (\gamma p)^{-1} & & 0 \\ & \rho(p, S)I_3 & \\ 0 & & 1 \end{pmatrix}. \quad (1.18)$$

Of course, symmetrizers are not unique - the reader can check that if one writes the ideal fluid equations in nonconservative form using as variables (ρ, v, T) resulting in the system,

$$\begin{aligned} \frac{D\rho}{Dt} + \rho \operatorname{div} v &= 0 \\ \rho \frac{Dv}{Dt} + \rho \nabla T + T \nabla \rho &= 0 \\ \frac{DT}{Dt} + (\gamma - 1) T \operatorname{div} v &= 0. \end{aligned}$$