

Analysis and Computation of
ELECTRIC AND MAGNETIC
FIELD PROBLEMS

SECOND EDITION

by

K. J. BINNS

and

P. J. LAWRENSON

53.67
B 614(2)

Analysis and Computation of ELECTRIC AND MAGNETIC FIELD PROBLEMS

SECOND EDITION

by

K. J. BINNS

Reader in Electrical Machines, University of Southampton

and

P. J. LAWRENSON

Professor of Electrical and Electronic Engineering, University of Leeds

2K146/04



PERGAMON PRESS

OXFORD · NEW YORK · TORONTO
SYDNEY · BRAUNSCHEWIG

Pergamon Press Ltd., Headington Hill Hall, Oxford
Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford,
New York 10523
Pergamon of Canada Ltd., 207 Queen's Quay West, Toronto 1
Pergamon Press (Aust.) Pty. Ltd., 19a Boundary Street,
Rushcutters Bay, N.S.W. 2011, Australia
Vieweg & Sohn GmbH, Burgplatz 1, Braunschweig

Copyright © 1973 K. J. BINNS and P. J. LAWRENSON

*All Rights Reserved. No part of this publication may be
reproduced, stored in a retrieval system, or transmitted, in any
form or by any means, electronic, mechanical, photocopying,
recording or otherwise, without the prior permission of
Pergamon Press Ltd.*

First edition 1962
Second edition 1973

Library of Congress Cataloging in Publication Data

Binns, Kenneth John.

Analysis and computation of electric and magnetic
field problems.

Bibliography: p.

1. Electric fields. 2. Magnetic fields.
3. Mathematical analysis. I. Lawrenson, P. J., joint
author. II. Title. III. Title: Electric and
magnetic field problems.

QC661.B435 1973
ISBN 0-08-016638-5

530.1'2

73-4545

Printed in Hungary

CONTENTS

	PAGE
PREFACE	xi
Part I: Introduction	
CHAPTER 1 INTRODUCTION	3
CHAPTER 2 BASIC FIELD THEORY	7
2.1 Electric Fields	7
2.1.1 The Electrostatic Field Vectors	7
2.1.2 Electric Potential	8
2.1.3 Potential Function and Flux Function	10
2.1.4 Capacitance	12
2.1.5 Laplace's and Poisson's Equations	12
2.1.6 Principle of Superposition	15
2.1.7 Electric Fields of Currents	15
2.2 Magnetic Fields	16
2.2.1 The Field of Magnetic Poles	16
2.2.2 The Magnetic Field of Line Currents	17
2.2.3 The Magnetic Field of Distributed Currents	18
2.2.4 Inductance	22
2.3 Boundary Conditions	23
2.4 Conjugate Functions	25
2.4.1 Laplace's Equation	25
2.4.2 Cauchy-Riemann Equations	27
2.4.3 Flux and Potential Functions as Conjugate Functions	28
2.4.4 Simple Examples of the Use of Conjugate Functions	29
2.5 Equivalent Pole and Charge Distributions	30
2.6 Forces	32
2.6.1 Line Sources	32
2.6.2 Distributed Sources	32
2.6.3 Total Force Acting on a Boundary	32
2.6.4 Force Distribution over a Boundary	33
References	33

61174

Part II: Direct methods

CHAPTER 3 IMAGES	37
3.1 Introduction	37
3.2 Plane Boundaries	38
3.2.1 Single Plane Boundary	38
3.2.2 Parallel Plane Boundaries	41
3.2.3 Intersecting Plane Boundaries	42
3.2.4 Inductance of Parallel Bus-bars near an Iron Surface	45
3.3 Circular Boundaries	46
3.3.1 Charge or Current near a Circular Boundary	46
3.3.2 Doublets: Circular Cylinder in a Uniform Field	48
3.4 General Considerations	54
References	57
 CHAPTER 4 THE SOLUTION OF LAPLACE'S EQUATION BY SEPARATION OF THE VARIABLES	 59
4.1 Introduction	59
4.2 Circular Boundaries	60
4.2.1 The Solution of Laplace's Equation in Circular-cylinder Coordinates	60
4.2.2 Iron Cylinder Influenced by a Current	63
4.2.3 The Screening Effect of a Permeable Cylinder	65
4.2.4 The Force between Rotor and Stator Conductors in a Cylindrical Machine	68
4.2.5 Specified Distributions of Potential or Potential Gradient on the Perimeter of a Circular Boundary	71
4.3 Rectangular Boundaries	72
4.3.1 Solution of Laplace's Equation in Cartesian Coordinates	72
4.3.2 The Semi-infinite Strip and the Rectangle	74
4.3.3 Pole Profile in the Inductor Alternator for a Sinusoidal Flux Distribution	77
4.4 Conclusions	80
References	81
 CHAPTER 5 THE SOLUTION OF POISSON'S EQUATION: MAGNETIC FIELDS OF DISTRIBUTED CURRENT	 83
5.1 Introduction	83
5.2 Non-magnetic Conductors in Air	85
5.2.1 The Method: Vector Potential of a Line Current	85
5.2.2 The Field of a Rectangular Bus-bar	85
5.2.3 The Force between Parallel Rectangular Bus-bars	88

5.3 The Field inside Infinitely Permeable Conductors in Air	91
5.3.1 General Considerations	91
5.3.2 The Field inside a Highly Permeable Rectangular Conductor	92
5.4 Simple Boundaries: Use of the Image Method	94
5.5 The Treatment of Boundaries using Single Fourier Series: Rogowski's Method	95
5.5.1 Rectangular Conductors in an Infinite, Parallel Air Gap	95
5.5.2 Finite Boundaries: Rectangular Conductor in a Slot	99
5.5.3 Scope of the Method	102
5.6 The Treatment of Boundaries using Double Fourier Series: Roth's Method	103
5.6.1 The Method	103
5.6.2 The Forces on, and the Inductance of, a Transformer Winding	107
5.6.3 Conductor in Slot: Calculation of Inductance	109
5.6.4 Scope of the Method	110
References	112

Part III: Transformation methods

CHAPTER 6 INTRODUCTION TO CONFORMAL TRANSFORMATION	117
6.1 Conformal Transformation and Conjugate Functions	117
6.1.1 Conformal Transformation	117
6.1.2 The Solution of Laplace's Equation	120
6.1.3 The Logarithmic Function	122
6.2 Classes of Solvable Problems	123
6.3 General Considerations	124
6.3.1 Choice of Origin	125
6.3.2 Multiple Transformations	125
6.3.3 Field Maps	126
6.3.4 Scale Relationship between Planes	126
6.3.5 Conservation of Flux and Potential	126
6.3.6 Field Strength	127
6.4 The Determination of Transformation Equations	128
References	128
CHAPTER 7 CURVED BOUNDARIES	129
7.1 The Bilinear Transformation	129
7.1.1 Mapping Properties	130
7.1.2 The Cross-ratio	134
7.1.3 The Magnetic Field of Currents inside an Infinitely Permeable Tube	135
7.1.4 The Capacitance of and the Voltage Gradient between Two Cylindrical Conductors	136
7.2 The Simple Joukowski Transformation	140
7.2.1 The Transformation	140
7.2.2 Flow round a Circular Hole	141
7.2.3 Permeable Cylinder Influenced by a Line Current	142

7.3 Curves Expressible Parametrically: General Series Transformations	145
7.3.1 The Method	145
7.3.2 The Field outside a Charged, Conducting Boundary of Elliptical Shape	146
7.3.3 General Series Transformations	147
7.3.4 Field Solutions	148
References	149
 CHAPTER 8 POLYGONAL BOUNDARIES	 151
8.1 Introduction	151
8.2 Transformation of the Upper Half Plane into the Interior of a Polygon	152
8.2.1 The Transformation	152
8.2.2 Polygons with Two Vertices	154
8.2.3 Parallel Plate Condenser: Rogowski Electrode	156
8.2.4 The Choice of Corresponding Points	160
8.2.5 Scale Relationship between Planes	163
8.2.6 The Field of a Current in a Slot	165
8.2.7 Negative Vertex Angles	169
8.2.8 The Forces between the Armature and Magnet of a Contactor	171
8.2.9 A Simple Electrostatic Lens	175
8.3 Transformation of the Upper Half Plane into the Region Exterior to a Polygon	177
8.3.1 The Transformation	177
8.3.2 The Field of a Charged, Conducting Plate	179
8.4 Transformations from a Circular to a Polygonal Boundary	182
8.4.1 The Transformation Equations	182
8.4.2 The Field of a Line Current and a Permeable Plate of Finite Cross-section	184
8.5 Classification of Integrals	186
References	187
 CHAPTER 9 THE USE OF ELLIPTIC FUNCTIONS	 189
9.1 Introduction	189
9.2 Elliptic Integrals and Functions	190
9.2.1 The Elliptic Integral of the First Kind	190
9.2.2 The Principal Jacobian Elliptic Functions	191
9.2.3 The Elliptic Integral of the Second Kind	191
9.2.4 Two Finite Charged Plates	192
9.2.5 Elliptic Integrals of the Third Kind	195
9.3 The Field outside a Charged Rectangular Conductor	196
9.3.1 The Transformation from an Infinite, Straight Line	197
9.3.2 The Transformation from a Circular Boundary	205
9.4 The Field in a Slot of Finite Depth	208
9.5 Conclusions	213
References	214

CHAPTER 10 GENERAL CONSIDERATIONS	217
10.1 Introduction	217
10.2 Field Sources	217
10.2.1 Infinite Boundaries	218
10.2.2 Finite Boundaries	222
10.2.3 Distributed Sources	223
10.3 Curved Boundaries	224
10.3.1 Rounded Corners	224
10.3.2 Curvilinear Polygons	226
10.4 Angles Not Multiples of $\pi/2$	227
10.4.1 Two-vertex Problems	227
10.5 Numerical Methods	228
10.5.1 Numerical Integration of the Function $f(t)$	229
10.5.2 Solution of the Implicit Equations	231
10.5.3 The Centring Force due to Displaced Ventilating Ducts in Rotating Machines	232
10.6 Non-equipotential Boundaries	235
10.6.1 Boundary Value Problems of the First Kind	235
10.6.2 Boundary Value Problems of the Second and Mixed Kinds	237
References	237

Part IV: Numerical methods

CHAPTER 11 FINITE-DIFFERENCE METHODS	241
11.1 Introduction	241
11.2 Finite-difference Representation	242
11.2.1 Regular Distributions of Field Points	242
11.2.2 Basic Equations for the Square and Rectangular Meshes	244
11.2.3 Field Problem as a Set of Simultaneous Equations	246
11.3 Hand Computation: Relaxation	248
11.3.1 Introduction	248
11.3.2 The Basic Method	249
11.3.3 Accelerating Processes	252
11.3.4 Practical Aspects	255
11.3.5 The Use of Point Values of Potential: Capacitance of a Three-core Rectangular Cable	256
11.4 Machine Computation: Iteration	258
11.4.1 Introduction	258
11.4.2 Basic Considerations and Methods	258
11.4.3 The Successive Over-relaxation Method	260
11.4.4 Current Flow in an I-section Conductor	265
11.4.5 Other Rapidly Convergent Methods	266
11.4.6 A Special Technique	267

11.5 Gradient Boundary Conditions	268
11.5.1 Introduction	268
11.5.2 Boundaries Coincident with Nodes	268
11.5.3 Boundaries Non-coincident with Nodes	273
11.5.4 Lines of Symmetry	276
11.5.5 Two Examples	277
11.6 Errors	279
11.6.1 Introduction	279
11.6.2 Mesh Error	279
11.6.3 Computational Errors	283
11.7 Conclusions	284
References	285

CHAPTER 12 THE MONTE CARLO METHOD	289
12.1 Introduction	289
12.2 The Method	289
12.3 Example	291
12.4 Some General Points	292
References	292

Appendixes

APPENDIX I The Sums of Certain Fourier Series	293
APPENDIX II Series Expansions of Elliptic Functions	294
APPENDIX III Table of Transformations	297-299
APPENDIX IV Bibliographies	315
INDEX	319

PREFACE

IN THE first edition we attempted to provide, in a single volume, a comprehensive treatment of both analytical and numerical methods for the derivation of two-dimensional static and quasi-static electric and magnetic fields. The main objectives were to try to present the essence of each method of solution and to indicate and compare the scopes of the different methods having particular regard to the influence of digital computers. In this second edition the aim is largely the same, but the treatment has been revised to include developments which have occurred over the last ten years both in methods of solution and in new applications.

As with the first edition, the book is intended primarily for engineers, physicists, and mathematicians who are faced with problems which can only be solved by an analysis of electromagnetic fields. It is also suitable for degree students towards the end of their courses. An aim at all stages has been to emphasize the physical significance of the mathematics and, to this end, examples of practical interest have been selected wherever possible.

The main text is divided into four parts so arranged that, provided the material contained in the first of these is familiar, study can commence in any of the other three parts. Part I contains a brief introductory chapter and a chapter devoted to the fundamental theory of electric and magnetic fields. The latter has been considerably modified since the first edition so as to give, in as concise a form as possible, the background theory essential to an understanding of the methods of analysis used later in the book. A clear explanation is attempted of the derivation of quantities of physical interest such as force, inductance, and capacitance from the field solution.

Part II deals with the image and variables separation methods of solution. In addition to the topics commonly treated under these headings, the present treatment covers a wide range of field sources; and, in the chapter on images, the basic solutions are developed rigorously from considerations of surface charges and solutions are expressed in complex variable form.

Part III, the longest of the four, is devoted to transformation methods, and the authors believe that it offers the most comprehensive treatment of the subject which is available. Some of the more important topics not normally dealt with include the following: line and doublet sources, which are rarely treated in connection with electromagnetic fields; the transformation of regions exterior to finite boundaries; and the powerful numerical methods which have been developed to enlarge the scope of conformal transformation.

Part IV deals with finite difference methods which can be used to solve any problem relevant to this book. All classes of boundary shape and condition are discussed and Chapter 2 has been enlarged to take account of recent computational developments. It should provide a useful introduction in a particularly important and rapidly developing area.

For their helpful comments we are most grateful to Dr. E. M. Freeman of Brighton Polytechnic and Professor P. Hammond of the University of Southampton.

K.J.B. P.J.L.

PART I

INTRODUCTION

CHAPTER 1

INTRODUCTION

Types of field discussed. All static electric and magnetic fields in a uniform medium are described by Poisson's equation or its particular form, Laplace's equation. Poisson's equation applies within regions of distributed current or charge, and Laplace's equation applies in all other regions of the field. In Chapter 2 the properties of fields described by these equations are reviewed, and the whole of the remainder of the book is devoted to different methods for the solution of the field equations.

In addition to the above static fields, which they describe exactly, Laplace's and Poisson's equations also describe, to a high degree of accuracy, several types of time-varying field. The commonest of these occurs when the frequency and boundaries are such that the effect of eddy currents is negligible. However, Laplacian solutions can also be used when the eddy currents are so strong that negligible flux penetrates a boundary surface. Electromagnetic radiation phenomena are described by the wave equation, but for certain problems, such as the determination of the characteristic impedance of transmission lines, Laplacian solutions are applicable.

All physical fields are, of course, three-dimensional, but for most cases of practical interest exact analytical solutions are not available, and numerical solutions often involve a prohibitive amount of computation. However, approximate solutions of quite sufficient accuracy can be obtained by using a two-dimensional treatment, i.e. by neglecting the variation of the field in one direction. As a result, analysis becomes possible in very many cases, and in the others the labour of numerical solution is greatly reduced. Two examples of two-dimensional treatment occur in the calculation of the magnetic fields in rotating electrical machines. Firstly, the distribution of the main field within the air gap can be found with negligible error by analysing the field at a cross-section perpendicular to the axis (the variation along the length of the machine being neglected). Secondly, the field outside the machine ends can be found, though rather less accurately than in the previous example, by analysing the field in an axial plane (neglecting the peripheral variations).

Types of solution. Most of this book is concerned with solutions of Laplace's equation, though the more general form, Poisson's equation, is discussed in Chapters 5 and 11. There are two reasons for giving more attention to Laplace's equation: firstly, the majority of fields of practical importance are of this simpler type, and, secondly, since Poissonian fields are the more difficult to solve, advantage is frequently taken of the relatively small importance of the Poissonian region to replace it by an equivalent filament, so effectively making the whole field Laplacian. For example, in calculating the inductance of a transmission line, the field is solved for a current concentrated in a central filament of the line.

All solutions fall into one of two classes, analytical or numerical. In the first class a solution is in the form of an algebraic equation in which values of the parameters defining the field can be substituted. A solution in the second class takes the form of a set of numerical

values of the function describing the field for one particular set of values of the parameters. All analytical methods have been in common use for at least sixty years, but it is only within the last thirty years or so that numerical methods have come into prominence. The recent development of numerical methods has been greatly stimulated by the advent of fast digital computing machines which have made possible routine solutions, to a high degree of accuracy, of many types of problems which would otherwise be extremely or even prohibitively laborious.

Where either analytical or numerical methods can be employed for the solution of a particular field, the choice of the most suitable method can sometimes be difficult to make. Analytical methods have the advantage that a general solution can be derived, from which it is possible to gain an overall picture of the effect of the various parameters. In contrast, with numerical methods it is necessary to calculate separately for each set of values of the parameters; a consequent disadvantage is that an overall picture can often be achieved only at the expense of a great amount of computation. However, for some problems for which analytical methods are possible, the determination of an analytical solution can be so involved and the computation so lengthy that numerical methods are simpler and quicker.

Analogous fields. In many aspects of engineering and physics there are physical phenomena which are directly analogous to electric and magnetic field phenomena. Amongst these are the flow of heat in conducting media and the flow of an inviscid liquid. For example, the temperature distribution between two boundaries having a constant temperature difference between them, or the distribution of the stream function of an ideal fluid passing between these boundaries, is identical in form with the voltage distribution between the same boundaries having a constant electric potential difference. Thus a solution to one problem

TABLE 1.1. ANALOGOUS QUANTITIES IN SCALAR POTENTIAL FIELDS

Quantity	Electrostatic	Electric current	Magneto-static	Heat flow	Fluid flow	Gravitational
Potential	Potential V	Potential V	Potential Ω	Temperature	Velocity potential	Newtonian potential
Potential gradient	Electric field strength E	Electric field strength E	Magnetic field strength H	Temperature gradient	Velocity	Gravitation force
Constant of medium	Permittivity ϵ	Conductivity σ	Permeability μ	Thermal conductivity	Density	Reciprocal of gravitation constant
Flux density	Electric flux density D	Current density J	Magnetic flux density B	Heat flow density	Flow rate	
Source strength	Charge density ρ_c	Current density J	Pole density ρ_m	Heat source density	Density of efflux	Mass density
Field conductance	Capacitance C	Conductance G	Permeance A	Thermal conductance		

of a particular physical type is directly applicable to other problems of different types, and methods developed in this book for electric and magnetic fields apply equally to the other fields mentioned above. Table 1.1 shows the equivalence of quantities in the different types of scalar potential field. In addition to the ones tabulated, consideration is given in the book to magnetic fields within regions of distributed current, and it is of interest to note that this type of field is analogous, for example, to that of fluid flow with vorticity.

CHAPTER 2

BASIC FIELD THEORY

THIS chapter provides a very brief review of the basic concepts of stationary electric and magnetic fields in just sufficient detail to cover the background theory required for the methods of analysis described in the book. Initially, the development is based on the point sources of field, but thereafter attention is given primarily to the line sources, the charge, the pole, and the current which are basic to the two-dimensional fields considered in this book.

2.1. Electric fields

2.1.1. The electrostatic field vectors

The concept of electric charge is of fundamental importance in the study of electric fields. A charge of magnitude q coulombs is considered to emit a total electric flux of q units; hence, an electric flux q emanates from any closed surface containing a charge q .

The *electric flux density* at a point is the vector \mathbf{D} , and its direction is that of the flux. Considering a spherical surface of radius r , with its centre at the position of a point charge, it is evident from considerations of symmetry that the direction of the flux is radially outward and that the density of flux crossing the surface is equal to $q/4\pi r^2$, i.e. the magnitude of the flux density is given by

$$D = \frac{q}{4\pi r^2}. \quad (2.1)$$

The force exerted on unit charge placed at a point, a distance r from a charge q , is proportional to q/r^2 , and so to the value of the vector \mathbf{D} at that point due to the charge q . Thus if a vector \mathbf{E} , known as the *electric field strength*, is defined to describe the force acting on the unit charge, then \mathbf{E} is proportional to \mathbf{D} for a given medium and may be expressed as

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E}, \quad (2.2)$$

where ϵ_0 is the primary electric constant and ϵ is the relative permittivity of the surrounding medium. So combining eqns. (2.1) and (2.2) gives

$$E = \frac{q}{4\pi \epsilon_0 \epsilon r^2}. \quad (2.3)$$

In free space this becomes

$$E = \frac{q}{4\pi \epsilon_0 r^2},$$

which, because of the nature of the variation of E with r , is called the inverse square law.

Consider now a charge distributed over a volume. As the volume tends to zero, the limit, at a point, of the outward flux per unit volume is called the *divergence* of the vector \mathbf{D} , and is a scalar. Thus the divergence of \mathbf{D} at any point within the volume is equal to the charge density ρ_c , i.e.

$$\text{div } \mathbf{D} = \rho_c. \quad (2.4)$$

The field of a line charge. When charge is uniformly distributed along an infinite straight line, the direction of the flux leaving the charge is everywhere perpendicular to the line, and the flux emitted per unit length of the line is equal to the linear charge density q . At a radius r about the charge, the flux density \mathbf{D} is given in magnitude by

$$D = \frac{q}{2\pi r}, \quad (2.5)$$

and so

$$E = \frac{\rho}{2\pi\epsilon_0\epsilon r}. \quad (2.6)$$

Thus the field strength varies inversely as the distance from the line charge.

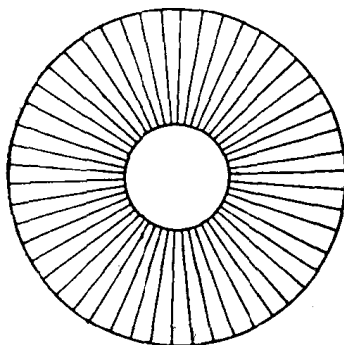


FIG. 2.1

This field is two-dimensional, and in all such fields a quantity of flux may be represented by a *number of flux lines*. At any point the direction of such a line is that of the flux density, and the concentration of the lines is a direct indication of the flux density there. A simple example of the distribution of flux lines is provided by the field of two charged conducting concentric cylinders (Fig. 2.1). From symmetry it is seen that the flux passes radially between the two cylinders and, since the quantity of flux passing each surface is the same, the flux densities on the surfaces of the cylinders are inversely proportional to their circumferences and therefore to their radii.

2.1.2. Electric potential

The scalar quantity, called the *electric potential* V , is a point function defined as the work done in moving unit charge from infinity to the point. Now the work done dV in moving unit charge a small distance dl is given by

$$dV = -\mathbf{E} \cdot d\mathbf{l}, \quad (2.7)$$

since \mathbf{E} is the force on unit charge. The negative sign means that the potential decreases with