Analysis and Computation of ELECTRIC AND MAGNETIC FIELD PROBLEMS

SECOND EDITION

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PREFACE

IN THE first edition we attempted to provide, in a single volume, a comprehensive treatment of both analytical and numerical methods for the derivation of two-dimensional static and quasi-static electric and magnetic fields. The main objectives were to try to present the essence of each method of solution and to indicate and compare the scopes of the different methods having particular regard to the influence of digital computers. In this second edition the aim is largely the same, but the treatment has been revised to include developments which have occurred over the last ten years both in methods of solution and in new applications.

As with the first edition, the book is intended primarily for engineers, physicists, and mathematicians who are faced with problems which can only be solved by an analysis of electromagnetic fields. It is also suitable for degree students towards the end of their courses. An aim at all stages has been to emphasize the physical significance of the mathematics and, to this end, examples of practical interest have been selected wherever possible.

The main text is divided into four parts so arranged that, provided the material contained in the first of these is familiar, study can commence in any of the other three parts. Part I contains a brief introductory chapter and a chapter devoted to the fundamental theory of electric and magnetic fields. The latter has been considerably modified since the first edition so as to give, in as concise a form as possible, the background theory essential to an understanding of the methods of analysis used later in the book. A clear explanation is attempted of the derivation of quantities of physical interest such as force, inductance, and capacitance from the field solution.

Part II deals with the image and variables separation methods of solution. In addition to the topics commonly treated under these headings, the present treatment covers a wide range of field sources; and, in the chapter on images, the basic solutions are developed rigorously from considerations of surface charges and solutions are expressed in complex variable form.

Part III, the longest of the four, is devoted to transformation methods, and the authors believe that it offers the most comprehensive treatment of the subject which is available. Some of the more important topics not normally dealt with include the following: line and doublet sources, which are rarely treated in connection with electromagnetic fields; the transformation of regions exterior to finite boundaries; and the powerful numerical methods which have been developed to enlarge the scope of conformal transformation.

Part IV deals with finite difference methods which can be used to solve any problem relevant to this book. All classes of boundary shape and condition are discussed and Chapter 2 has been enlarged to take account of recent computational developments. It should provide a useful introduction in a particularly important and rapidly developing area.

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K.J.B. P.J.L.

PART I

INTRODUCTION

CHAPTER 1

INTRODUCTION

Types of field discussed. All static electric and magnetic fields in a uniform medium are described by Poisson's equation or its particular form, Laplace's equation. Poisson's equation applies within regions of distributed current or charge, and Laplace's equation applies in all other regions of the field. In Chapter 2 the properties of fields described by these equations are reviewed, and the whole of the remainder of the book is devoted to different methods for the solution of the field equations.

In addition to the above static fields, which they describe exactly, Laplace's and Poisson's equations also describe, to a high degree of accuracy, several types of time-varying field. The commonest of these occurs when the frequency and boundaries are such that the effect of eddy currents is negligible. However, Laplacian solutions can also be used when the eddy currents are so strong that negligible flux penetrates a boundary surface. Electromagnetic radiation phenomena are described by the wave equation, but for certain problems, such as the determination of the characteristic impedance of transmission lines, Laplacian solutions are applicable.

All physical fields are, of course, three-dimensional, but for most cases of practical interest exact analytical solutions are not available, and numerical solutions often involve a prohibitive amount of computation. However, approximate solutions of quite sufficient accuracy can be obtained by using a two-dimensional treatment, i.e. by neglecting the variation of the field in one direction. As a result, analysis becomes possible in very many cases, and in the others the labour of numerical solution is greatly reduced. Two examples of two-dimensional treatment occur in the calculation of the magnetic fields in rotating electrical machines. Firstly, the distribution of the main field within the air gap can be found with negligible error by analysing the field at a cross-section perpendicular to the axis (the variation along the length of the machine being neglected). Secondly, the field outside the machine ends can be found, though rather less accurately than in the previous example, by analysing the field in an axial plane (neglecting the peripheral variations).

Types of solution. Most of this book is concerned with solutions of Laplace's equation, though the more general form, Poisson's equation, is discussed in Chapters 5 and 11. There are two reasons for giving more attention to Laplace's equation: firstly, the majority of fields of practical importance are of this simpler type, and, secondly, since Poissonian fields are the more difficult to solve, advantage is frequently taken of the relatively small importance of the Poissonian region to replace it by an equivalent filament, so effectively making the whole field Laplacian. For example, in calculating the inductance of a transmission line, the field is solved for a current concentrated in a central filament of the line.

All solutions fall into one of two classes, analytical or numerical. In the first class a solution is in the form of an algebraic equation in which values of the parameters defining the field can be substituted. A solution in the second class takes the form of a set of numerical values of the function describing the field for one particular set of values of the parameters. All analytical methods have been in common use for at least sixty years, but it is only within the last thirty years or so that numerical methods have come into prominence. The recent development of numerical methods has been greatly stimulated by the advent of fast digital computing machines which have made possible routine solutions, to a high degree of accuracy, of many types of problems which would otherwise be extremely or even prohibitively laborious.

Where either analytical or numerical methods can be employed for the solution of a particular field, the choice of the most suitable method can sometimes be difficult to make. Analytical methods have the advantage that a general solution can be derived, from which it is possible to gain an overall picture of the effect of the various parameters. In contrast, with numerical methods it is necessary to calculate separately for each set of values of the parameters; a consequent disadvantage is that an overall picture can often be achieved only at the expense of a great amount of computation. However, for some problems for which analytical methods are possible, the determination of an analytical solution can be so involved and the computation so lengthy that numerical methods are simpler and quicker.

Analogous fields.-In many aspects of engineering and physics there are physical phenomena which are directly analogous to electric and magnetic field phenomena. Amongst these are the flow of heat in conducting media and the flow of an inviscid liquid. For example, the temperature distribution between two boundaries having a constant temperature difference between them, or the distribution of the stream function of an ideal fluid passing between these boundaries, is identical in form with the voltage distribution between the same boundaries having a constant electric potential difference. Thus a solution to one problem

TABLE 1.1. ANALOGOUS QUANTITIES IN SCALAR POTENTIAL FIELDS

Quantity	Electrostatic	Electric current	Magneto-static	Heat flow	Fluid flow	Gravita-tional
Potential	Potential V	Potential V	Potential Ω	Temper-ature	Velocity potential	Newtonian potential
Potential gradient	Electric field strength E	Electric field strength E	Magnetic field strength H	Temper-ature gradient	Velocity	Gravitation force
Constant of medium	Permittivity ε	Conductivity σ	Permeability μ	Thermal conduc-tivity	Density	Reciprocal of gravi-tation constant
Flux density	Electric flux density D	Current density J	Magnetic flux density B	Heat flow density	Flow rate	
Source strength	Charge density ϱ_e	Current density J	Pole density ϱ_m	Heat source density	Density of efflux	Mass density
Field conduct-ance	Capacitance C	Conductance G	Permeance Λ	Thermal conduct-ance		

of a particular physical type is directly applicable to other problems of different types, and methods developed in this book for electric and magnetic fields apply equally to the other fields mentioned above. Table 1.1 shows the equivalence of quantities in the different types of scalar potential field. In addition to the ones tabulated, consideration is given in the book to magnetic fields within regions of distributed current, and it is of interest to note that this type of field is analogous, for example, to that of fluid flow with vorticity.

CHAPTER 2

BASIC FIELD THEORY

This chapter provides a very brief review of the basic concepts of stationary electric and magnetic fields in just sufficient detail to cover the background theory required for the methods of analysis described in the book. Initially, the development is based on the point sources of field, but thereafter attention is given primarily to the line sources, the charge, the pole, and the current which are basic to the two-dimensional fields considered in this book.

2.1. Electric fields

2.1.1. *The electrostatic field vectors*

The concept of electric charge is of fundamental importance in the study of electric fields. A charge of magnitude q coulombs is considered to emit a total electric flux of q units; hence, an electric flux q emanates from any closed surface containing a charge q.

The *electric flux density* at a point is the vector $\mathbf{D}$, and its direction is that of the flux. Considering a spherical surface of radius r, with its centre at the position of a point charge, it is evident from considerations of symmetry that the direction of the flux is radially outward and that the density of flux crossing the surface is equal to $q/4\pi r^2$, i.e. the magnitude of the flux density is given by

$$D = \frac{q}{4\pi r^2}. \tag{2.1}$$

The force exerted on unit charge placed at a point, a distance r from a charge q, is proportional to q/r^2, and so to the value of the vector $\mathbf{D}$ at that point due to the charge q. Thus if a vector $\mathbf{E}$, known as the *electric field strength*, is defined to describe the force acting on the unit charge, then $\mathbf{E}$ is proportional to $\mathbf{D}$ for a given medium and may be expressed as

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}, \tag{2.2}$$

where ε_0 is the primary electric constant and ε is the relative permittivity of the surrounding medium. So combining eqns. (2.1) and (2.2) gives

$$E = \frac{q}{4\pi \varepsilon_0 \varepsilon r^2}. \qquad 2.3)$$

In free space this becomes

$$E = \frac{q}{4\pi \varepsilon_0 r^2},$$

which, because of the nature of the variation of E with r, is called the inverse square law.

2*

Consider now a charge distributed over a volume. As the volume tends to zero, the limit, at a point, of the outward flux per unit volume is called the *divergence* of the vector **D**, and is a scalar. Thus the divergence of **D** at any point within the volume is equal to the charge density ϱ_c, i.e.

$$\operatorname{div} \mathbf{D} = \varrho_c. \tag{2.4}$$

The field of a line charge. When charge is uniformly distributed along an infinite straight line, the direction of the flux leaving the charge is everywhere perpendicular to the line, and the flux emitted per unit length of the line is equal to the linear charge density q. At a radius r about the charge, the flux density **D** is given in magnitude by

$$D = \frac{q}{2\pi r}, \tag{2.5}$$

and so

$$E = \frac{p}{2\pi\varepsilon_0\varepsilon r}. \tag{2.6}$$

Thus the field strength varies inversely as the distance from the line charge.

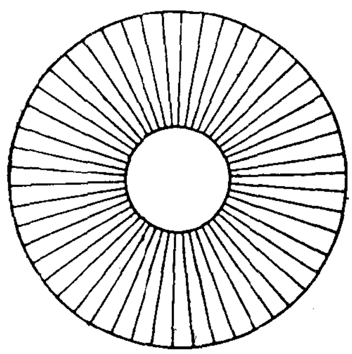

FIG. 2.1

This field is two-dimensional, and in all such fields a quantity of flux may be represented by a *number of flux lines*. At any point the direction of such a line is that of the flux density, and the concentration of the lines is a direct indication of the flux density there. A simple example of the distribution of flux lines is provided by the field of two charged conducting concentric cylinders (Fig. 2.1). From symmetry it is seen that the flux passes radially between the two cylinders and, since the quantity of flux passing each surface is the same, the flux densities on the surfaces of the cylinders are inversely proportional to their circumferences and therefore to their radii.

2.1.2. *Electric potential*

The scalar quantity, called the *electric potential V*, is a point function defined as the the work done in moving unit charge from infinity to the point. Now the work done $\mathrm{d}V$ in moving unit charge a small distance $\mathrm{d}l$ is given by

$$\mathrm{d}V = -\mathbf{E}\cdot\mathbf{d}l, \tag{2.7}$$

since **E** is the force on unit charge. The negative sign means that the potential decreases with