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原 版 系 列

Zeta and q-Zeta Functions and Associated Series and Integrals

zeta函数，q-zeta函数，相伴级数与积分

[加] Srivastava, H. M.（斯利瓦斯塔瓦）[韩] Junesang Choi（催真尚）著

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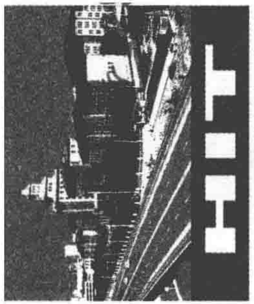

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Preface

This book is essentially a thoroughly revised, enlarged and updated version of the authors' work: ***Series Associated with the Zeta and Related Functions*** (Kluwer Academic Publishers, Dordrecht, Boston and London, 2001). It aims at presenting a state-of-the-art account of the theories and applications of the various methods and techniques which are used in dealing with many different families of series associated with the Riemann Zeta function and its numerous generalizations and basic (or q-) extensions. Systematic accounts of only some of these methods and techniques, which are widely scattered in journal articles and book chapters, were included in the above-mentioned book.

In recent years, there has been an increasing interest in problems involving closed-form evaluations of (and representations of the Riemann Zeta function at positive integer arguments as) various families of series associated with the Riemann Zeta function $\zeta(s)$, the Hurwitz Zeta function $\zeta(s, a)$, and their such extensions and generalizations as (for example) Lerch's transcendent (or the Hurwitz-Lerch Zeta function) $\Phi(z, s, a)$. Some of these developments have apparently stemmed from an over two-century-old theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782), from recent rediscoveries of a fairly rapidly convergent series representation for $\zeta(3)$, which is actually contained in a 1772 paper by Leonhard Euler (1707–1783), and from another known series representation for $\zeta(3)$, which was used by Roger Apéry (1916–1994) in 1978 in his celebrated proof of the irrationality of $\zeta(3)$.

This revised, enlarged and updated version of our 2001 book is motivated essentially by the fact that the theories and applications of the various methods and techniques used in dealing with many different families of series associated with the Riemann Zeta function, its aforementioned relatives and its many different basic (or q-) extensions are to be found so far only in widely scattered journal articles published during the last decade or so. Thus, our systematic (and unified) presentation of these results on the evaluation and representation of the various families of Zeta and q-Zeta functions is expected to fill a conspicuous gap in the existing books dealing exclusively with these Zeta and q-Zeta functions.

The main objective of this revised, enlarged and updated version is to provide a systematic collection of various families of series associated with the Riemann and Hurwitz Zeta functions, as well as with many other higher transcendental functions, which are closely related to these functions (including especially the q-Zeta and related functions). It, therefore, aims at presenting a state-of-the-art account of the theory and applications of many different methods (which are available in the rather scattered literature on this subject, especially since the publication of our aforementioned 2001 book) for the derivation of the types of results considered here.

In our attempt to make this book as self-contained as possible within the obvious constraints, we include in Chapter 1 (Introduction and Preliminaries) a reasonably detailed account of such useful functions as the Gamma and Beta functions, the Polygamma and related functions, multiple Gamma functions, the Gauss hypergeometric function and its familiar generalization, the Stirling numbers of the first and second kind, the Bernoulli, Euler and Genocchi polynomials and numbers, the Apostol-Bernoulli, the Apostol-Euler and the Apostol-Genocchi polynomials and numbers, as well as some interesting inequalities for the Gamma function and the double Gamma function. In Chapter 2 (The Zeta and Related Functions), we present the definitions and various potentially useful properties (and characteristics) of the Riemann, Hurwitz and Hurwitz-Lerch Zeta functions and their generalizations, the Polylogarithm and related functions and the multiple Zeta functions, together with their analytic continuations.

In Chapter 3 (Series Involving Zeta Functions), we begin by providing a brief historical introduction to the main subject of this book. We then describe and illustrate some of the most effective methods of evaluating series associated with the Zeta and related functions. Further developments on the evaluations and (rapidly convergent) series representations of $\zeta(s)$ when $s \in \mathbb{N} \setminus \{1\}$ are presented in Chapter 4 (Evaluations and Series Representations), which also deals with various computational results on this subject.

Chapter 5 (Determinants of the Laplacians) considers the problem involving computations of the determinants of the Laplacians for the n-dimensional sphere $\mathbf{S}^n$ ($n \in \mathbb{N}$). It is here in this chapter that we show how fruitfully some of the series evaluations (which are presented in the earlier chapters) can be applied in the solution of the aforementioned problem.

In a *brand new* Chapter 6 (q-Extensions of Some Special Functions and Polynomials), we first introduce the concepts of the basic (or q-) numbers, the basic (or q-) series and the basic (or q-) polynomials. We then proceed to apply these concepts and present a reasonably detailed theory of the various basic (or q-) extensions of the Gamma and Beta functions, the derivatives, antiderivatives and integrals, the binomial theorem, the multiple Gamma functions, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and so on.

The last chapter (Chapter 7) contains a wide variety of miscellaneous results dealing with (for example) the analysis of several useful mathematical constants, a variety of Log-Sine integrals involving series associated with the Zeta function and Polylogarithms, applications of the Gamma and Polygamma functions involving convolutions of the Rayleigh functions, evaluations of the Bernoulli and Euler polynomials at rational arguments, and the closed-form summation of several classes of trigonometric series.

Each chapter in this book begins with a brief outline summarizing the material presented in the chapter and is then divided into a number of sections. Equations in every section are numbered separately. While referring to an equation in another section of the book, we use numbers like 3.2(18) to represent Equation (18) in Section 3.2 (that is, the second section of Chapter 3).

At the close of each chapter, we have provided a set of carefully-selected problems, which are based essentially upon the material presented in the chapter. Many of these problems are taken from recent research publications, and (in all such instances) we have chosen to include the precise references for further investigation (if necessary). Another valuable feature of this book is the extensive and up-to-date bibliography on the subject dealt with in the book.

Just as its predecessor (that is, the 2001 edition), this book is written primarily as a reference work for various seemingly diverse groups of research workers and other users of series associated with the Zeta and related functions. In particular, teachers, researchers and postgraduate students in the fields of mathematical and applied sciences will find this book especially useful, not only for its detailed and systematic presentations of the theory and applications of the various methods and techniques used in dealing with many different classes of series associated with the Zeta and related functions, or for its stimulating historical accounts of a large number of problems considered here, but also for its well-classified tables of series (and integrals) and its well-motivated presentation of many sets of closely related problems with their precise bibliographical references (if any).

Acknowledgements

Many persons have contributed rather significantly to this thoroughly revised, enlarged and updated version, just as to its predecessor (that is, the 2001 edition), both directly *and* indirectly. Contribution of subject matter is duly acknowledged throughout the text *and* in the bibliography. Indeed, we are greatly indebted to the various authors whose works we have freely consulted and who occasionally provided invaluable references and advice serving for the enrichment of the matter presented in this book. The first-named author wishes to express his deep sense of gratitude to his wife *and* colleague, Professor Rekha Srivastava, for her cooperation and support throughout the preparation of this thoroughly revised, enlarged and updated version of the 2001 book.

The collaboration of the authors on the 2001 book project was conceptualized as long ago as August 1995, and the preparation of a preliminary outline was initiated in December 1997, during the first-named author's visits to Dongguk University at Gyeongju. The first drafts of some of the chapters in this book were written during several subsequent visits of the first-named author to Dongguk University at Gyeongju. The final drafts of most of the chapters in the 2001 book were prepared during the second-named author's visit to the University of Victoria from August 1999 to August 2000, while he was on Study Leave from Dongguk University at Gyeongju. The preparation of this thoroughly revised, enlarged and updated version was carried out, in most part, during the period from January 2008 to January 2009, during the second-named author's visit to the University of Victoria, while he was on Study Leave from Dongguk University at Gyeongju for the second time. Our sincere thanks are due to the appropriate authorities of each of these universities, to the Korea Research Foundation (Support for Faculty Research Abroad under its Research Fund Program) and to the Natural Sciences and Engineering Research Council of Canada, for providing financial support and other facilities for the completion of each of the projects leading eventually to the 2001 edition and this thoroughly revised, enlarged and updated version. We especially acknowledge and appreciate the financial support that was received under the *Basic Science Research Program* through the National Research Foundation of the Republic of Korea.

We take this opportunity to express our thanks to the editorial (and technical) staff of the Elsevier Science Publishers B.V. (especially the Publisher, Ms. Lisa Tickner, for Serials and Elsevier Insights) for their continued interest in this book and for their proficient (and impeccable) handling of its publication. Springer's permission to publish this thoroughly revised, enlarged and updated edition of the 2001 book is also greatly appreciated.

Finally, we should like to record our indebtedness to the members of our respective families for their understanding, cooperation and support throughout this project.

The second-named author and his family would, especially, like to express their appreciation for the first-named author and his family's hospitality and every prudent consideration during their stay in Victoria for over one year, first from August 1999 to August 2000 and then again from January 2008 to January 2009, while the second-named author was on Study Leave from Dongguk University at Gyeongju.

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1 Introduction and Preliminaries

In this introductory chapter, we present the definitions and notations (and some of the important properties and characteristics) of the various special functions, polynomials and numbers, which are potentially useful in the remainder of the book. The special functions considered here include (for example) the Gamma, Beta and related functions, the Polygamma functions, the multiple Gamma functions, the Gaussian hypergeometric function and the generalized hypergeometric function. We also consider the Stirling numbers of the first and second kind, the Bernoulli, Euler and Genocchi polynomials and numbers and the various families of the generalized Bernoulli, Euler and Genocchi polynomials and numbers. Relevant connections of some of these functions with other special functions and polynomials, which are not listed above, are also presented here.

1.1 Gamma and Beta Functions

The Gamma Function

The origin of the *Gamma function* can be traced back to two letters from Leonhard Euler (1707–1783) to Christian Goldbach (1690–1764), just as a simple desire to extend factorials to values between the integers. The first letter (dated October 13, 1729) dealt with the interpolation problem, whereas the second letter (dated January 8, 1730) dealt with integration and tied the two together.

The Gamma function $\Gamma(z)$ developed by Euler is usually defined by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\Re(z) > 0). \tag{1}$$

We also present here several equivalent forms of the Gamma function $\Gamma(z)$, one by Weierstrass:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\} \qquad \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right), \tag{2}$$

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where γ denotes the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right) \cong 0.57721\,56649\,01532\,86060\,6512\ldots, \tag{3}$$

and the other by Gauss:

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty}\left\{\frac{(n-1)!\,n^z}{z(z+1)\cdots(z+n-1)}\right\}\\ &= \lim_{n\to\infty}\left\{\frac{n!\,(n+1)^z}{z(z+1)\cdots(z+n)}\right\}\\ &= \lim_{n\to\infty}\left\{\frac{n!\,n^z}{z(z+1)\cdots(z+n)}\right\}\\ &\qquad (z\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{aligned} \tag{4}$$

since

$$\lim_{n\to\infty}\frac{n}{z+n} = 1 = \lim_{n\to\infty}\frac{n^z}{(n+1)^z}.$$

In terms of the Pochhammer symbol $(\lambda)_n$ defined (for $\lambda\in\mathbb{C}$) by

$$(\lambda)_n := \begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}:=\{1,\,2,\,3,\,\ldots\}),\end{cases} \tag{5}$$

the definition (4) can easily be written in an equivalent form:

$$\Gamma(z) = \lim_{n\to\infty}\frac{(n-1)!\,n^z}{(z)_n} \quad (z\in\mathbb{C}\setminus\mathbb{Z}_0^-). \tag{6}$$

By taking the reciprocal of (2) and applying the definition (3), we have

$$\begin{aligned}\frac{1}{\Gamma(z)} &= z\left[\lim_{n\to\infty}\exp\left\{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)z\right\}\right]\left[\lim_{n\to\infty}\prod_{k=1}^{n}\left\{\left(1+\frac{z}{k}\right)e^{-z/k}\right\}\right]\\ &= z\lim_{n\to\infty}\left[\exp\left\{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)z\right\}\cdot\prod_{k=1}^{n}\left\{\left(1+\frac{z}{k}\right)e^{-z/k}\right\}\right]\\ &= z\lim_{n\to\infty}\left\{n^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right\}\\ &= z\lim_{n\to\infty}\left[\left\{\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^{-z}\right\}\left\{\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\right\}\right]\\ &= z\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)\left(1+\frac{1}{k}\right)^{-z}\right\},\end{aligned}$$