RECENT ADVANCES IN

SCIENCE

PHYSICS AND APPLIED MATHEMATICS

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Physics and Applied Mathematics

Edited by

Morris H. Shamos
Associate Professor of Physics, New York University

George M. Murphy
Professor of Chemistry, New York University

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FOREWORD

At the start of World War I, rumor has it, the old War Department declined with regret the proffer of aid from the American Chemical Society on the grounds that the Department already had a chemist thinking about their problems. No significant scientific advance in armament had taken place by the opening rounds of World War II. However, in the relatively short span since 1940 a scientific and technological revolution has swept this planet with a pace hitherto unknown.

New discoveries have unleashed forces, opened vistas, and laid bare accomplishments of the magnitude that makes addicts of science fiction enthusiasts. New industries literally have sprung forth from the fruits of scientific research, creating unforeseen demands for trained, skilled, and educated personnel. From a nuclear pile under the stands at Stagg Field, Chicago, has grown a multibillion-dollar industry just in its swaddling clothes. From the fertile imagination of mathematicians has emerged the fountainhead of the scientific revolution excited by the high-speed digital computing machine. The electronics industry of 1940 has little resemblance to the lusty giant of today. And many of the tools with which the biologists and medical scientists attack the problems of cancer and other dread diseases were not yet born at the outbreak of World War II.

Not only did these vast new frontiers of science open, but new ones are threatening to push on to the stage at any time. The storehouse of scientific information increases with positive acceleration. Each new basic scientific discovery is multiplied in significance because of its meaning and power for other scientific activity and because of its impact upon our economy and society through application and new industry.

A problem facing scientists for more than a generation has now reached major proportions and threatens to stem the tide of the necessary flow of knowledge: How is it possible to keep abreast of developments in one's own field, as well as being aware, if only through mental osmosis, of developments in peripheral realms? The problem is further complicated. Each expert speaks a language probably meaningful to the few who are likewise expert in that field. But could not the engineer, chemist, or solid state physicist profit significantly from a knowledge of the work done by the mathematician or the nuclear physicist? Is there a common methodology that would bear fruit in all of the physical sciences?

In terms of this thinking, it is eminently clear that an educational and training need exists. To meet this need, the Division of General Education of New York University, with the co-operation of scientific leaders from industry, Government laboratories, and the University, developed a series of co-ordinated lectures by the outstanding men in scientific fields of current interest to industry, research laboratories, and the universities.

The First Symposium on Recent Advances in Science was held at Washington Square during the spring of 1954. It attracted 225 men and women from metropolitan industries and faculties. This book, an outgrowth of that Symposium on applied mathematics and physics, is, we believe, a contribution to the fundamental understanding of the important work explicated by the several lecturers. It is obvious that both the Symposium and this book have required a large degree of active co-operation and participation from many different sources. The enthusiastic response to the need from the Planning Committee was both inspiring and gratifying. It is only fitting that recognition be given to them for their efforts.

To the members of the committee from industry and the research laboratories, Lloyd V. Berkner, Richard Emberson, Elmer W. Engstrom, James Fisk, Mervin J. Kelly, A. B. Kinzel, Eger Murphree, C. G. Suits, I, on behalf of New York University, wish to acknowledge my debt for their ideas, support, and general helpfulness. Without their devotion to the twin causes of education and science, this Symposium would not have been possible. These scientific leaders were always available for advice and took time from their extraordinarily busy lives to consult with university representatives throughout the planning year.

Our university representatives likewise did yeoman work to make the Symposium an important event for the scientific community.

Dean Paul A. McGhee of the Division of General Education, Chancellor Henry T. Heald, and Dean Thorndike Saville of the College of Engineering encouraged and co-operated with the committee to garner the best scientific talent to come to the podium. Professor Serge Korff acted as chairman of the meetings; Professors George Murphy and Morris Shamos did much of the preliminary work before the opening of the Symposium and have performed a superb task against overwhelming odds in getting the Symposium into this permanent form. In addition to these, our grateful acknowledgment for their support goes to the other university members of the Planning Committee: Professors Yardley Beers, Myron A. Coler, Richard Courant, James Mulligan, John Vance, Carel van der Merwe.

SIDNEY G. ROTH

New York University
May 1956

PREFACE

Scientific historians of the future may attach special significance to the fact that the first half of the twentieth century saw the establishment of a number of highly technical industries. Many of the basic discoveries of the first few decades have already been put to commercial practice, and the fundamental ideas now being developed will form the nuclei for new industries of the future. The very rapid growth of the physical sciences quite naturally has led to specialization, with the result that practicing scientists and engineers too often are but dimly aware of the latest developments outside their immediate fields.

This book is a product of the First Symposium on Recent Advances in Science, held at New York University during the spring of 1954. The purpose of the Symposium, which was confined to physics and applied mathematics, was to convey the basic ideas in some of the newest and most active fields of study. The level of presentation is probably best described as intermediate, inasmuch as the lectures presupposed some scientific training, although not necessarily in the particular subject areas. Nevertheless, it was apparent that much benefit was derived from the lectures even by those actively engaged in these fields.

Prominent among the topics will be found several phases of atomic and nuclear physics. This is probably to be expected in view of the enormous advances in these fields and the impact they have made upon modern engineering. The rapid rise of interest during the recent years in the physical properties of solids is reflected in those chapters that treat some of the aspects of solid state physics. Similarly, the techniques of operations research, developed primarily for military application, have only recently been turned to industrial problems, for which they appear to offer great promise. The opening chapter deals with what is probably the most important tool of the engineer, applied mathematics, while the concluding chapter reviews the recent advances in the light of their implications for future trends in industrial development. Although the related topics of information theory and computing machines were presented at the Symposium, it unfortunately has not been possible to include this material in the present volume.

We are grateful to the contributing authors for generously giving their time to the preparation of their manuscripts, and we wish to thank the *American Scientist* for permission to reprint Dr. Shockley's article.

For various reasons it has not been easy to prepare this printed volume, as is evident from the publication date. However, we still believe the title to be exact in that the contents reflect the most recent advances in these fields. The chapters were prepared by some of the most distinguished scientists in this country, and their subjects are those in which they are acknowledged experts. Some repetition and lack of continuity are perhaps inevitable in a volume of this sort, yet we feel that the primary purpose of the book has been realized. We trust it may prove as rewarding to the reader as the original lectures were to the listeners.

G. M. Murphy
M. H. Shamos

May 1956

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Methods of Applied Mathematics

RICHARD COURANT

Introduction

"Applied" mathematics as distinct from "pure" mathematics is a relatively recent phenomenon of scientific specialization. The distinction does not refer to fields of knowledge but rather to human attitudes and motivations. Pure mathematics is directed towards logical crystallization, abstraction, generalization; applied mathematics means close interconnection of mathematical methods with physical reality, and it may mean subordination of logical completeness to the need for results obtained by a mixed approach which may, if necessary, utilize analytical methods as well as physical intuition, numerical computation, and empirical reasoning.

Until the middle of the nineteenth century it was usual for great mathematicians to represent both the pure and applied trends in mathematics. The most striking example was Gauss. As a matter of course, this great creator of modern algebra, number theory, function theory, differential geometry, and non-Euclidean geometry took an active part in the development of geodesy, astronomy, and electrostatics. He built the first electrical telegraph (with Weber) and was fully conscious of the importance of this invention. He laid the foundation of the pension fund for the widows and orphans of his faculty colleagues. Although he valued highest his construction of the regular 17-gon and proclaimed number theory the queen of sciences, he spent most of his professional life enthusiastically on what we would call today applied mathematics. The mathematician Clerk Maxwell, interpreting in the form of partial differential equations the intuitive notions of Faraday and the quantitative formulation of Biot-Savart, created the basis for modern electromagnetic theory and practice. Henri Poincaré, another of the great mathematicians of the past century, made decisive contributions to the understanding of the propagation of radio waves across the surface of the earth. Perhaps one may consider Bernhard Riemann as the last of the mathematical universalists who made a deep mark in pure as well as applied mathematics. His paper on dynamics of compressible gases opened up an entirely novel field. The professional mathematicians remembered only the elegant, purely mathematical appendix to this paper, and it was left to engineers and physicists to develop the field of gas dynamics and aerodynamics, which some of them did with amazing mathematical insight and ingenuity. Likewise in other directions great mathematical contributions have been made by physicists, in particular in connection with quantum theory.

In this country the trend among professional mathematicians toward isolation from other sciences was interrupted when, during World War II, many pure mathematicians volunteered to give needed mathematical help. Sometimes service of high scientific and technological caliber was rendered by mathematicians who before had belonged to the purist camp. Since the war, government agencies, foremost among them the Office of Naval Research, have successfully tried to maintain a better balance between the applied and pure aspects of scientific activities in this country, and it seems certain that these efforts will have a lasting effect.

There are two principles involved in all problems and methods of applied mathematics, the principle of idealization and the principle of approximation. Idealization is fundamental even for the formulation of the basic concepts and laws of nature. For example, the density of a gas or fluid at a point P is determined by an idealized limiting process; we take the total mass of gas in a small sphere of radius ϵ about P, divide this mass by the volume $(4\pi/3)\,\epsilon^3$, and then let ϵ tend to zero. This limiting process is actually an unrealistic idealization, since in a very small volume gas molecules are sparsely and irregularly distributed. Still, without this and innumerable other idealized concepts, physics and mechanics would be utterly impossible. Newton's laws and celestial mechanics deal with ideal mass points, elasticity, and acoustics with ideal continua, although these continuous media consist actually of a finite but large number of discrete individual particles.

Methods

The general scheme of methods of applied mathematics can be represented by the following diagram:

$$\begin{array}{ccc} & I \rightarrow & I_n \\ \nearrow & & \downarrow \\ R & S \leftarrow & S_n \end{array}$$

R symbolizes a problem posed by reality, I the mathematical idealization, S its solution, I_n an approximate problem tending to I for $n \rightarrow \infty$, S_n its solution. The task of the mathematician is to formulate a proper I, to find a proper approximation I_n, and to identify approximately the desired S with S_n.

For many individual problems the step $R \rightarrow I$ is the most decisive; it often requires ingenuity, experience, and intuitive understanding of the realities of physics, engineering, and other fields. Of course, this step leaves great leeway for the applied mathematician. The next step, $I \rightarrow I_n$, is decisive for mathematical success or failure; this step also leaves open many possibilities for constructive imagination. The comparison of S_n with the theoretical solution S is often a difficult theoretical problem which, however, must be tackled to make sure that the whole process is meaningful.

Examples

We now turn to the discussion of a number of concrete examples which will illustrate the general methodical scheme just described.

Domes

The transition $R \rightarrow I$ from reality to an acceptable idealized mathematical model may be first illustrated by a quite special question that arose in the winter of 1942–1943 in connection with underwater warfare. Underwater sound ranging, as is well known, depends on sending out a sound beam in water from a properly designed projector. If the projecting plate is submerged in water and attached to a fast-moving ship, the water streaming around the plate causes serious disturbances. To eliminate these disturbances the projector is enclosed in a so-called "dome," as shown in Fig. 1, which is a convex shell of metal or other material filled with water. Such domes, as first constructed by the British, interfere only slightly with the formation of a concentrated sound beam. However, during that winter a great number of small submarine chasers were built and equipped with sound gear similar to but smaller than the gear used before. While the manufacture of domes to fit this smaller gear was under way, it was discovered that these smaller domes led to an intolerable diffusion of the sound beam. A quick remedy was imperative, and mathematical analysis of the problem was needed to sup-

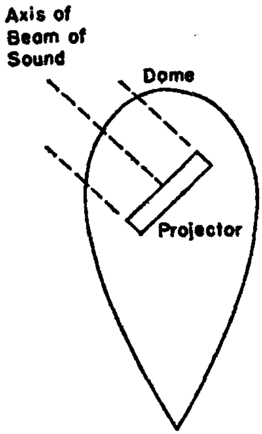

Fig. 1. Dome

port and to speed up experimental work. At first the mathematical problem seemed formidable because it involved the integration of the equation

$$\nabla^2 P + k^2 P = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1}$$

in which the factor $k = \omega/c$ has different values within the shell of the dome and outside, ω denoting the frequency and c the sound velocity. However, a suitably idealized mathematical model was found by the following process. The actual dome of small but finite thickness is replaced by an ideal infinitely thin surface. The influence of the dome is then replaced simply by conditions for jump discontinuities of the disturbance q of the beam across this surface. These conditions are:

$$\begin{aligned} [q] &= \frac{\rho_1}{\rho_0 - 1} \frac{\partial p}{\partial n}, \\ \left[\frac{\partial q}{\partial n}\right] &= \frac{\rho_0}{\rho_1}(k_0^2 - k_1^2)\, p - \left(1 - \frac{\rho_0}{\rho_1}\right)\left(\frac{\partial^2 p}{\partial n^2} + 2H \frac{\partial p}{\partial n}\right), \end{aligned} \tag{2}$$

in which the symbol $[f]$ means the jump of the quantity f across the surface, q is the disturbance of the acoustic pressure p caused by the dome, and the normal derivatives $\partial/\partial n$ are to be evaluated on the surface S. The quantity H is the mean curvature of S, i.e., the average of the curvatures of any two normal plane sections at right angles to each other. In addition to conditions (2) to be satisfied by q on S, q should be a solution of the differential equation

$$\nabla^2 q + k_0^2 q = 0, \tag{3}$$

which is regular everywhere except on S and which has the same behavior as P at ∞. This problem possesses the unique solution

$$q = -\frac{1}{4\pi} \iint_S \left[\frac{\partial q}{\partial n}\right] \frac{e^{ik_0 r'}}{r'}\, dS + \frac{1}{4\pi} \iint_S [q] \frac{\partial}{\partial n}\left(\frac{e^{ik_0 r'}}{r'}\right) dS, \tag{4}$$

in which the integration is to be extended over the dome surface S. The quantities in square brackets are, of course, those given by conditions (2), and r' is the distance from a fixed point (x, y, z) at which $q(x, y, z)$ is to be determined to the point of integration on S. This formula yields the disturbance as the effect due to a layer of point sources and a layer of dipoles distributed on S with intensities which are known as soon as the original pressure p is known, since the quantities in brackets are fixed in value by conditions (2). The relative directional disturbance $\left|\frac{p_1}{p_0}\right|$-Re $h\left(\frac{q_1}{p_1} - \frac{q_0}{p_0}\right)$ would, finally, be obtained from (4).

The solution (4) is valid for a shell of constant thickness, but it could be extended without essential error to cases in which the dome shell is made up of a not too large number of pieces, each of which is of constant thickness. All that would be necessary would be to insert a numerical factor d in the integrands on the right-hand side of expression (4) which would be piecewise constant on S. This formula makes it quite easy to analyze the contribution to the distortion of various factors, such as the curvature of the dome and the density and sound velocity within it. Therefore, this little example of proper idealization, even without detailed numerical computation, proved helpful to the designing engineer.

Shocks

The second, much broader, example of mathematical idealization refers to the dynamics of compressible fluids, a field with many applications of ever-increasing importance. Here the complex reality of a gas or fluid consisting of an enormously large number of discrete particles requires a great deal of mathematical idealization. One has to introduce such averages as density, pressure, temperature, entropy, and flow velocity; one makes further idealizing assumptions by neglecting viscosity and heat conduction. Even so, one arrives at systems of differential equations which are nonlinear and which thus present essentially new mathematical situations leading to phenomena of the greatest practical importance.

Bernhard Riemann, Rankine, Hugoniot, and Rayleigh discovered in the middle of the nineteenth century that the nonlinearity of compressible flow problems necessitated deviations from the traditional belief of Newton and Laplace that solutions to physical problems are determined by differential equations and initial conditions. Indeed, discontinuities may occur in a compressible flow even though the initial data are wholly continuous, so that no continuation of given initial data into a regular solution may be possible. "Shocks," i.e., discontinuities in density, pressure, and entropy which travel through the fluid at high speed, occur in many cases. The problem of determining flows with shock discontinuities which are not known a priori is extremely difficult, though of utmost importance for modern aerodynamics and explosion theory.

The partial differential equations governing compressible flow in the simplest cases are:

1. One-dimensional flow (of a polytropic gas)

$$\rho_t + (\rho u)_x = 0 \qquad \text{(conservation of mass)},$$

$$(\rho u)_t + (\rho u^2)_x + p_x = 0 \qquad \text{(conservation of momentum)},$$

$$\left[\rho\left(\frac{u^2}{2} + e\right)\right]_t + \left[\rho u\left(\frac{u^2}{2} + i\right)\right]_x = 0 \quad \text{(conservation of energy)},$$

with $p = A\rho^\gamma$ (equation of state), $i = \gamma/(\gamma - 1)$, and $(\gamma - 1)e = p/\rho = c^2/(\gamma - 1)$, where c = sound speed, ρ = density, p = pressure, u = particle speed, e = internal energy, and i = enthalpy.

2. Steady irrotational two-dimensional flow

$$\mu^2(u^2 + v^2) + (1 - \mu^2)c^2 = c_*^2 = \text{const.},$$

$$(c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y = 0,$$

$$u_y = v_x,$$

with $\mu^2 = (\gamma - 1)/(\gamma + 1)$, where u and v are the components of the fluid velocity; or

$$(c^2 - u^2)\Phi_{xx} - 2uv\Phi_{xy} + (c^2 - v^2)\Phi_{yy} = 0,$$

with $u = \Phi_x$, $v = \Phi_y$, where Φ is the velocity potential. Flow is supersonic if $u^2 + v^2 > c^2$ everywhere, subsonic if $u^2 + v^2 < c^2$ everywhere, transonic if both inequalities may hold (at different points).

These differential equations can be valid only in regions of continuity. At shock transitions certain "shock conditions" restricting the nature of permissible discontinuities must be satisfied. The principle followed by Riemann, Hugoniot, and others in deriving these conditions, which have the form of finite equations, is that they should express at the shock the same "conservation laws" (in particular, conservation of mass, of momentum, and of energy) as the differential equations of flow in regions of continuity.

It would at first seem that mathematical shock discontinuities do not represent physical reality, since, strictly speaking, fluid flows are continuous. However, the above equations represent an idealized flow without viscosity or heat conduction. Thus it is natural to try to set up the full set of equations governing the flow, taking these effects into account, solve these equations, and then let the viscosity and heat conductivity approach zero. One may expect that for small values of viscosity and heat conductivity the solution, though continuous, will vary extremely rapidly in a narrow strip of the x,t-plane, and that as these parameters approach zero this strip will contract to a curve, while the solution will approach different values on the two sides of this curve. Indeed, in the simple cases in which this has been done the curve so obtained corresponds in position to the shock predicted by the "idealized" set of equations, and the limiting values of the solution on the two sides of the shock satisfy the shock conditions mentioned in the preceding paragraph.

Shocks, or "shock waves" as observed in many phenomena, have striking properties not present in acoustical, electromagnetic, or optical wave propagation. Their speed is "supersonic." The angle at which they are reflected from a rigid wall is different from their angle of incidence, as may be seen in Fig. 2. They may lead to very high pressures.

Problems involving shocks pose many questions of theoretical interest. These questions deal with the existence and uniqueness of the solution of the initial value problem for a partial differential equation if this solution is permitted to have discontinuities but is required to satisfy shock conditions at these discontinuities. Only very rudimentary results have as yet been obtained in this area, since even the simplest problems involving shocks seem to defy the power of pure analysis.

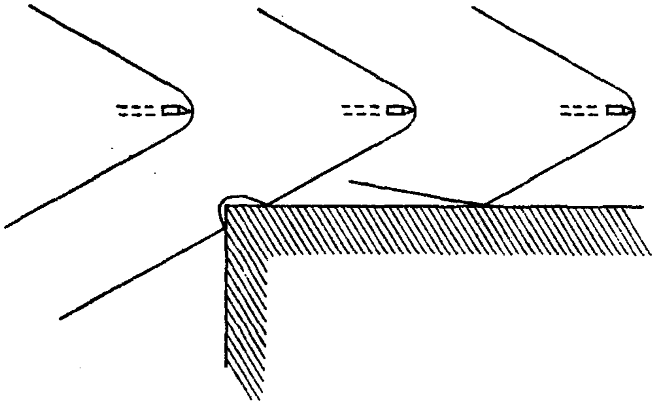

Fig. 2. Shock Reflection

Indeed, the flight of high-speed airplanes and missiles, as well as phenomena occurring in jets and nozzles, in combustion chambers of engines, and in all sorts of explosions, leads to fluid flow problems of such variety and difficulty that in only a few typical but highly simplified cases are analytic results available. It is therefore of the utmost significance that the development of high-speed electronic computing machines has recently made it possible to apply to many of these problems numerical methods which bring their solution within the range of present-day mathematical capabilities.

Numerical Methods

We shall now describe the general procedures involved in the application of numerical methods to the solution of differential equation problems.

Historically, the first such methods arose from consideration of boundary value problems and eigenvalue problems for elliptic partial differential equations which correspond to conditions of equilibrium.

These problems are related to the following "variational" principle: States of equilibrium have minimum potential energy. Indeed the equivalence between boundary value problems of partial differential equations and problems of the calculus of variations has been a central point in analysis since the time of Gauss. At first, theoretical interest in existence proofs was dominant; only much later were practical applications envisaged by two physicists, Lord Rayleigh and Walther Ritz, who independently conceived the idea of utilizing this equivalence for numerical calculation of the solutions by substituting for the variational problems simpler approximating extremum problems in which only a finite number of parameters need be determined. In the works of Rayleigh, especially in his classical *Theory of Sound*, this procedure was first used. However, Ritz gave a masterly account of the theory and at the same time applied his method to the calculation of the nodal lines of vibrating plates, a problem of classical physics that previously had not been satisfactorily treated.

Thus methods emerged which could not fail to attract engineers and physicists; after all, the minimum principles of mechanics are more suggestive than the differential equations. Great successes in applications were soon followed by further progress in the understanding of the theoretical background, and such progress in turn has resulted in advantages for the applications.

It turned out that the specific procedure used by Ritz and Rayleigh was practical only in particular cases and that the use of finite difference methods was preferable. Methods of the latter type have since become universally accepted as the most direct and promising tools of numerical analysis.

Finite Differences

In these methods, the continuum of values which can be assumed by the independent variables $x, y, \cdots, t$ is replaced by a finite set of "net points" whose coordinates are integral multiples of certain fixed "mesh widths," one for each variable. Usually equal mesh widths are chosen for the space dimensions, with possibly a different one in time. Differential equations become equations involving a finite number of difference quotients; integrals are replaced by finite sums. Thus the differential equation problem is reduced to a problem with only a finite number of unknowns.