FOUNDATIONS OF ELECTRODYNAMICS

S.R. DE GROOT AND L.G. SUTTORP

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S. R. de Groot and L. G. Suttorp

Institute of Theoretical Physics
University of Amsterdam

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Preface

Electrodynamics may be said to consist of two parts, at different levels: microscopic and macroscopic theory. The first contains the laws that govern the interaction of fields and point particles – often grouped into stable sets such as atoms and molecules – and the second those that describe the interaction of fields and continuous media. The two theories are linked together, since the phenomena at the macroscopic level may be looked upon as being the result of the interplay of many particles. Therefore one should be able to obtain the electromagnetic laws for continuous media from those for point particles. Such a derivation, together with a discussion of the microscopic starting points, forms the subject of this monograph.

The programme will be carried out in the framework of both classical and quantum theory. The classical theory is given in the non-relativistic approximation and then in covariant formulation. In the latter various topics will receive special attention: among these figure the covariant description of composite particles, the obtention of statistical averages in a relativistically invariant way and a discussion of the energy–momentum tensor for continuous media. The quantum-mechanical theory will be formulated in such a fashion that the analogy with classical theory can be exploited as far as possible. This is achieved by representing the physical quantities by ordinary functions rather than by operators. Again the non-relativistic approximation will be studied first. Subsequently magnetic effects are discussed in a 'semi-relativistic' theory, which goes one step beyond the non-relativistic treatment. The completely covariant extension of quantum theory will be confined to the discussion of the motion of single particles with and without spin in slowly varying external fields. The covariant generalization to statistical assemblies of particles moving in each other's fields would require quantization of the electromagnetic field together with its sources: this forms the subject of quantum electrodynamics not dealt with here.

The subject matter of the various chapters is, roughly spoken, of two kinds.

Part is meant especially to serve as textbook material for graduate students who take courses in electromagnetic theory. By reading the first two chapters they will get acquainted with the way in which the macroscopic laws of electrodynamics are obtained from a microscopic basis, albeit in the framework of classical, non-relativistic theory. In the relativistic part the third chapter may be useful as an exposé of the covariant equations for fields and particles with the inclusion of the effects of radiation damping, while the final results of the fourth and fifth chapters give an idea of the way in which the non-relativistic laws may be generalized. Similarly the results of chapters VI and VII show the consequences of the use of quantum mechanics. The special formulation of quantum mechanics in terms of Weyl transforms and Wigner functions can be studied independently from the appendix of chapter VI.

More advanced students will be interested in the covariant formulation of the equations of motion for composite particles in chapter IV, relativistic statistics as discussed in chapter V, the covariant quantum-mechanical equations of motion for particles with spin 0 and ½ in chapter VIII, and in the semi-relativistic treatment of magnetic effects given in chapters IX and X.

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S.R. de G.

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CHAPTER I

Particles: their fields and motion

1 Introduction

The aim of this chapter is to obtain a description of the electromagnetic behaviour of composite particles in the framework of classical, non-relativistic theory. Such composite particles, like atoms, molecules or ions are supposed to consist of charged point particles: the electrons and nuclei. The equations which govern their motion and describe their fields will be derived from the corresponding basic equations valid for charged point particles without structure. The latter microscopic equations are the Maxwell–Lorentz field equations and the Newton equation with the Lorentz force inserted. A series expansion in terms of multipoles leads then to the field equations and the momentum, energy and angular momentum equations for the composite particles.

2 The microscopic field equations

The electric and magnetic fields $\boldsymbol{e}(\boldsymbol{R}, t)$ and $\boldsymbol{b}(\boldsymbol{R}, t)$ at the point with coordinates $\boldsymbol{R}$ and at time t, generated by a collection of point particles $i = 1, 2, \ldots$ with charges e_i, positions $\boldsymbol{R}_i(t)$ and velocities $\dot{\boldsymbol{R}}_i(t)$, satisfy the Maxwell–Lorentz field equations (in the rationalized Gauss system[1])

$$\begin{aligned} \nabla\cdot\boldsymbol{e} &= \sum_i e_i\,\delta(\boldsymbol{R}_i-\boldsymbol{R}), \\ -\partial_0\,\boldsymbol{e}+\nabla\wedge\boldsymbol{b} &= c^{-1}\sum_i e_i\,\dot{\boldsymbol{R}}_i\,\delta(\boldsymbol{R}_i-\boldsymbol{R}), \\ \nabla\cdot\boldsymbol{b} &= 0, \\ \partial_0\,\boldsymbol{b}+\nabla\wedge\boldsymbol{e} &= 0, \end{aligned} \tag{1}$$

[1] In the Giorgi system different numerical coefficients appear: the factors c^{-1} (both explicitly and in ∂_0) are absent, while in the first two equations $\boldsymbol{e}$ and $\boldsymbol{b}$ are replaced by $\varepsilon_0\boldsymbol{e}$ and $\mu_0^{-1}\boldsymbol{b}$ respectively.

where ∇ and ∂_0 are differentiations with respect to $\boldsymbol{R}$ and ct (with c the speed of light) and the dot and the symbol $\wedge$ scalar and vector products of vectors. The sources contain the three-dimensional delta functions of $\boldsymbol{R}_i - \boldsymbol{R}$.

In non-relativistic theory one is interested in solutions of these equations up to order c^{-1}. To find them it is convenient to introduce potentials. From the third equation it follows that

$$\boldsymbol{b} = \nabla \wedge \boldsymbol{a} \tag{2}$$

with the vector potential $\boldsymbol{a}(\boldsymbol{R}, t)$. Then with the fourth equation one has

$$\boldsymbol{e} = -\nabla\varphi - \partial_0 \boldsymbol{a}, \tag{3}$$

where $\varphi(\boldsymbol{R}, t)$ is the scalar potential. Insertion of these expressions into the first two equations of (1) gives, if one omits terms in c^{-2},

$$\begin{aligned} \Delta\varphi + \partial_0 \nabla\cdot\boldsymbol{a} &= -\sum_i e_i \delta(\boldsymbol{R}_i - \boldsymbol{R}), \\ \Delta\boldsymbol{a} - \nabla(\nabla\cdot\boldsymbol{a} + \partial_0\varphi) &= -c^{-1}\sum_i e_i \dot{\boldsymbol{R}}_i \delta(\boldsymbol{R}_i - \boldsymbol{R}), \end{aligned} \tag{4}$$

where $\Delta = \nabla\cdot\nabla$ is the Laplace operator. The potentials are not fixed in a unique way by the relations (2) and (3). The same electromagnetic fields are described by potentials $\boldsymbol{a}'$ and φ' which are related to the original potentials $\boldsymbol{a}$ and φ by a gauge transformation

$$\begin{aligned} \varphi' &= \varphi - \partial_0 \psi, \\ \boldsymbol{a}' &= \boldsymbol{a} + \nabla\psi \end{aligned} \tag{5}$$

with an arbitrary function ψ. This property is utilized to choose the potentials in such a way that they satisfy

$$\partial_0 \varphi + \nabla\cdot\boldsymbol{a} = 0, \tag{6}$$

the Lorentz condition. The reason for imposing this condition is that then the equations (4) become two uncoupled Poisson equations for φ and $\boldsymbol{a}$:

$$\begin{aligned} \Delta\varphi &= -\sum_i e_i \delta(\boldsymbol{R}_i - \boldsymbol{R}), \\ \Delta\boldsymbol{a} &= -c^{-1}\sum_i e_i \dot{\boldsymbol{R}}_i \delta(\boldsymbol{R}_i - \boldsymbol{R}), \end{aligned} \tag{7}$$

where again a term of order c^{-2} has been dismissed. The solutions follow from the property for the delta function:

$$\Delta \frac{1}{|\boldsymbol{r}|} = -4\pi\delta(\boldsymbol{r}). \tag{8}$$

Using this property, one finds from (7) the non-relativistic potentials in the Lorentz gauge:

$$\varphi = \sum_i \frac{e_i}{4\pi|\boldsymbol{R}_i - \boldsymbol{R}|},$$
$$\boldsymbol{a} = c^{-1} \sum_i \frac{e_i \dot{\boldsymbol{R}}_i}{4\pi|\boldsymbol{R}_i - \boldsymbol{R}|}, \tag{9}$$

so that the non-relativistic fields (2) and (3) are

$$\boldsymbol{e} = \sum_i \boldsymbol{e}_i, \qquad \boldsymbol{e}_i = -\nabla \frac{e_i}{4\pi|\boldsymbol{R}_i - \boldsymbol{R}|},$$
$$\boldsymbol{b} = \sum_i \boldsymbol{b}_i, \qquad \boldsymbol{b}_i = c^{-1}\nabla \wedge \frac{e_i \dot{\boldsymbol{R}}_i}{4\pi|\boldsymbol{R}_i - \boldsymbol{R}|}. \tag{10}$$

These formulae show that the non-relativistic electric field is of order c^0 (a term in c^{-1} does not appear), while the non-relativistic magnetic field is of order c^{-1} (no term in c^0 arises). From the first line of (10) it follows that $\boldsymbol{e}$ is irrotational. This is in agreement with the fourth field equation in (1), since $\partial_0 \boldsymbol{b}$ is of order c^{-2} and hence has to be neglected in non-relativistic theory. So strictly spoken one should write in a non-relativistic theory the truncated equation $\nabla \wedge \boldsymbol{e} = 0$ instead of the fourth field equation.

3 *The equation of motion for a point particle*

The equation of motion for a particle with charge e, mass m, position $\boldsymbol{R}_1(t)$, velocity $\dot{\boldsymbol{R}}_1(t)$ and acceleration $\ddot{\boldsymbol{R}}_1(t)$ in an external electromagnetic field $(\boldsymbol{E}_e, \boldsymbol{B}_e)$ is:

$$m\ddot{\boldsymbol{R}}_1 = e\{\boldsymbol{E}_e(\boldsymbol{R}_1, t) + c^{-1}\dot{\boldsymbol{R}}_1 \wedge \boldsymbol{B}_e(\boldsymbol{R}_1, t)\}, \tag{11}$$

where at the right-hand side the Lorentz force appears. The equation of motion of one particle of a set labelled by the index $i = 1, 2, \ldots, N$ reads

$$m_i \ddot{\boldsymbol{R}}_i = e_i\{\boldsymbol{e}_t(\boldsymbol{R}_i, t) + c^{-1}\dot{\boldsymbol{R}}_i \wedge \boldsymbol{b}_t(\boldsymbol{R}_i, t)\}, \tag{12}$$

where the total electric and magnetic fields are the sums of the external fields and the fields (10) generated by the other particles:

$$\boldsymbol{e}_t(\boldsymbol{R}_i, t) = \sum_{j(\neq i)} \boldsymbol{e}_j(\boldsymbol{R}_i, t) + \boldsymbol{E}_e(\boldsymbol{R}_i, t),$$
$$\boldsymbol{b}_t(\boldsymbol{R}_i, t) = \sum_{j(\neq i)} \boldsymbol{b}_j(\boldsymbol{R}_i, t) + \boldsymbol{B}_e(\boldsymbol{R}_i, t). \tag{13}$$

Since in the equation of motion (12) the magnetic field is accompanied by a factor c^{-1}, one needs there as fields

$$\begin{aligned} e_i(\boldsymbol{R}_i, t) &= -\sum_{j(\neq i)} \nabla_i \frac{e_j}{4\pi|\boldsymbol{R}_i - \boldsymbol{R}_j|} + \boldsymbol{E}_e(\boldsymbol{R}_i, t), \\ \boldsymbol{b}_i(\boldsymbol{R}_i, t) &= \boldsymbol{B}_e(\boldsymbol{R}_i, t), \end{aligned} \tag{14}$$

instead of the complete expressions (13) with (10).

The equations of motion (12) with (14) may be written in Hamiltonian form

$$\frac{\partial H}{\partial \boldsymbol{P}_i} = \dot{\boldsymbol{R}}_i, \qquad \frac{\partial H}{\partial \boldsymbol{R}_i} = -\dot{\boldsymbol{P}}_i \tag{15}$$

with the Hamiltonian

$$H = \sum_i \frac{\boldsymbol{P}_i^2}{2m_i} + \sum_{i,j(i\neq j)} \frac{e_i e_j}{8\pi|\boldsymbol{R}_i - \boldsymbol{R}_j|} + \sum_i e_i \left\{\varphi_e(\boldsymbol{R}_i, t) - c^{-1}\frac{\boldsymbol{P}_i}{m_i}\cdot \boldsymbol{A}_e(\boldsymbol{R}_i, t)\right\}, \tag{16}$$

with φ_e and $\boldsymbol{A}_e$ potentials for the external fields. Indeed insertion of (16) into (15) leads to (12) with (14).

4 *The equations for the fields due to composite particles*

a. *The atomic series expansion*

Charged point particles (electrons and nuclei) are often grouped into stable sets, like atoms, molecules or ions. (For convenience we shall sometimes refer to such composite particles simply as 'atoms'.) The starting point for the derivation of the equations for the fields due to such atoms is the set of microscopic field equations (1). It will be convenient in the present case to replace the numbering i of the point particles by a numbering k of the stable groups and i of their constituent particles. The position vector $\boldsymbol{R}_i$, written as $\boldsymbol{R}_{ki}$ now, can be split into two parts:

$$\boldsymbol{R}_{ki} = \boldsymbol{R}_k + \boldsymbol{r}_{ki}. \tag{17}$$

Here $\boldsymbol{R}_k$ is the position of some privileged point of the stable group k (e.g. the nucleus of an atom or the centre of mass, etc.), while the $\boldsymbol{r}_{ki}$ ($i = 1, 2, \ldots$) are the internal coordinates, which specify the positions of the constituent particles ki with respect to that of the privileged point of the stable group k.

The case will now be studied in which the solutions $\boldsymbol{e}$ and $\boldsymbol{b}$ of the field equations can be considered as converging series expansions in $|\boldsymbol{r}_{ki}|/|\boldsymbol{R}_k - \boldsymbol{R}|$.