Precalculus

Algebra, Trigonometry and Geometry

Roy H. McLeod
Charles H. Stolze

Precalculus:

ALGEBRA, TRIGONOMETRY, AND GEOMETRY

Roy H. McLeod
F. H. LaGuardia Community College

Charles H. Stolze
Manhattan College

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Precalculus: Algebra, Trigonometry, and Geometry

To my loving daughters
HEATHER and SAMANTHA
Roy H. McLeod

To my dear wife
GLORIA
and to my children
ERIC and AMY
Charles H. Stolze

Preface

A precalculus course serves as the gateway to the study of calculus and more advanced mathematics. The materials in the precalculus course will provide a firm summary of advanced high school algebra, trigonometry, and introductory analytical geometry.

In keeping with the need to prepare students for more advanced work in mathematics, the unifying concept of "function" serves as the firm base throughout the book. As the title suggests, the material in this book will adequately suffice for the requirements of a three- or four-semester-hour course without the need for an elaborate selection process. Problem solving is stressed throughout, and a wealth of illustrative examples serves to underscore this emphasis. Relevant application problems are judiciously included. Detailed proofs are provided for clarification and mathematical preciseness. Nevertheless, where rigor would tend to deter full comprehension, intuitive discussions are exploited.

As previously mentioned, the course consolidates concepts that the student encountered in earlier courses. It also builds on these concepts and sharpens the skills necessary to cope with more advanced mathematics. For example, the circular/trigonometric functions as well as the exponential and logarithmic functions are carefully treated with the recognition that these are vitally important topics to the successful understanding of calculus and other advanced mathematics courses.

There is also a recognized need to deal with a large number of problems that relate to the conic sections. A concise treatment of these topics is included so that students can gain a fundamental knowledge through clear exposition buttressed by detailed illustrations and carefully worked-out examples.

In summary, the material in this text is characterized by informality in style with the cognizance that the text must be read to be fully appreciated. There is also the need to synthesize the material in each chapter; in keeping with this notion, a comprehensive set of review problems appears at the end of each chapter. The problems are wide ranging in the level of difficulty—from routine problems to some thought-provoking ones.

For helping in the preparation of this text we acknowledge with gratitude the contribution of Professor Thomas J. Smith of Manhattan College. His detailed criticism has improved many sections of the book. We also owe thanks to our many colleagues at LaGuardia Community College for their help in teaching from this material and improving the notes from which this book has developed. In particular, Professor Giangrasso offered many constructive comments that helped us to clarify the materials.

To the people at Macmillan we express our appreciation for their fine support and encouragement. We are particularly indebted to Mrs. Elaine W. Wetterau, whose meticulous and painstaking efforts contributed immensely to the production of this book.

Finally, to Mrs. Marguerite Kissane we express our sincere thanks for her fine work in typing this manuscript.

R. H. McL.

C. H. S.

Precalculus: Algebra, Trigonometry, and Geometry

Contents

Review of Elementary Algebra

1

1.1 The Real Numbers

In this section we give a brief description of the set R of real numbers. We begin with the set N of *natural* or *counting numbers:*

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

This set is also called the set of *positive integers.* If we extend it by including the number 0 and the negative of each natural number, we have the set Z of integers:

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

An extension of the set Z can be made to obtain the rational numbers. Recall that a *rational number* is any number that can be expressed as the ratio of two integers. In symbols, q is a rational number if and only if $q = a/b$, where a and b are integers and $b \neq 0$. The set of rational numbers will be denoted by the letter Q.

$$Q = \left\{\frac{a}{b} \,\middle|\, a, b, \in Z, \quad b \neq 0\right\}$$

The fact that we can represent each integer as a fraction with denominator 1 makes it clear that every integer is also a rational number. There are, however, rational numbers which are not integers. For example, the number $\frac{2}{3}$ is rational but does not represent an integer.

Rational numbers can be given a geometric interpretation. To accomplish this, we select a line L and an arbitrary point on that line. The point will be identified with the number 0. Next, choose a unit of length and use it to mark off points on L, as indicated in Figure 1.1.

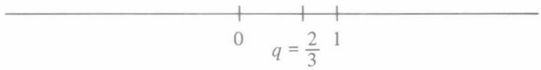
This procedure enables us to associate with each integer in Z a unique point on the line L. It is also possible to associate with each rational number in Q a unique point on L. We shall illustrate with some specific examples.

Example 1
Find the point corresponding to $q = \frac{2}{3}$.

Solution
We take the line segment from 0 to 1 and divide it into three equal parts. The point corresponding to $q = \frac{2}{3}$ is indicated in Figure 1.2.

Figure 1.1

Figure 1.2



Example 2

Find the point corresponding to $q = \frac{10}{7}$.

Solution

$q = \frac{10}{7} = 1\frac{3}{7}$. In this case we use the line segment from 1 to 2 and divide it into seven equal parts. The point corresponding to $q = \frac{10}{7}$ is indicated in Figure 1.3.

From Examples 1 and 2 it should be clear that to each rational number q in Q there corresponds a unique point on the line L.

We now raise the following question: Have we exhausted all the points on L or are there points remaining that do not correspond to any rational number?

It turns out (and it is by no means obvious) that there are points left over. In fact, there are so many points left over that we cannot even *count* them. These "leftover points" correspond to numbers that we call irrational numbers. Thus an *irrational number* is a number that is not rational and hence cannot be written as the ratio of two integers. Examples of such numbers are $\sqrt{2}$, $\sqrt{7}$, and the number π.

The set R of real numbers consists of the set Q of rational numbers together with the set of irrational numbers. Each real number can be interpreted geometrically as a point on the line L, which we henceforth call a *coordinate line*.

Rational and irrational numbers can also be distinguished by their decimal representation. The decimal representations of rational numbers are always infinite *repeating* decimals. Examples are:

$$\frac{22}{7} = 3.142857\ 142857\ 142857\ldots$$
$$\frac{1}{3} = 0.333333\ldots$$
$$\frac{1}{7} = 0.142857\ 142857\ 142857\ldots$$
$$\frac{1}{8} = 0.125 = 0.125000000\ldots$$

In the fourth example the decimal representation for $\frac{1}{8}$ actually terminates, but we may still think of it as an infinite repeating decimal having an infinite repetition of 0 after the number 5. The sequence of integers that repeats itself is referred to as the *repetend*. For example, the repetend for $\frac{1}{7}$ is 142857.

The decimal representation of an irrational number is always an infinite *nonrepeating* decimal. For example, the decimal representations of $\sqrt{2}$ and of π never terminate and have no repetend.

It should be mentioned here that the real numbers are distributed over the coordinate line L in such a way that every line segment, regardless of its length, contains an infinite number of rationals and an infinite number of irrationals. One consequence of this is that an irrational number can always be approximated by a rational number. We know, for example, that the irrational number π is approximately equal to the rational number $\frac{22}{7}$.

Figure 1.3

NOTE. In the preceding paragraphs we have mentioned a number of properties of R that are by no means obvious. Although we plan to use these properties, their proofs are beyond the scope of this book.

Now denote the set of all positive real numbers by R^+. Then, geometrically, a real number x is *positive* if and only if it lies *to the right* of 0 along the coordinate line. If x is a positive real number, we say that x is greater than 0 and we write this symbolically as $x > 0$.

Example 3

$$\begin{aligned} 3 &> 0 \qquad \text{(3 is greater than 0)} \\ \tfrac{1}{2} &> 0 \qquad (\tfrac{1}{2} \text{ is greater than 0}) \\ 2 &> 0 \qquad \text{(2 is greater than 0)} \end{aligned}$$

(see Figure 1.4).

Definition 1. Let a and b be two real numbers. We say that a is *less than* b if and only if the point corresponding to a is to the left of the point corresponding to b along the coordinate line. When a is less than b, we write $a < b$.

Example 4

Clearly,

$$\begin{aligned} 2 &< 3 \\ -1 &< 0 \\ -3 &< -1 \\ -2.5 &< -1.7 \end{aligned}$$

(see Figure 1.5).

Algebraically, $a < b$ if and only if the number $b - a$ is positive. That is, $a < b$ if and only if $b - a > 0$.

Similarly, a is less than or equal to b, denoted $a \leq b$, if and only if the number $b - a$ is greater than or equal to 0. Symbolically, $a \leq$ b if and only if $b - a \geq 0$.

Example 5

$$\begin{aligned} 5 &< 7 \quad \text{since } 7 - 5 = 2 > 0 \\ -1 &< 2 \quad \text{since } 2 - (-1) = 2 + 1 = 3 > 0 \\ -5 &< -2 \quad \text{since } -2 - (-5) = -2 + 5 = 3 > 0 \\ 2 &\leq x \quad \text{if and only if} \quad x - 2 \geq 0 \end{aligned}$$

NOTE. A real number x is *negative* if and only if $x < 0$.

We assume that the reader is familiar with the binary operations of addition and multiplication of real numbers. These numbers are governed by certain basic properties (axioms), which we list here for future reference.

Figure 1.4

Figure 1.5

If a, b, and c are real numbers, then:

1. $a + b$ is a real number.
2. ab is a real number.
3. $a + b = b + a$.
4. $ab = ba$.
5. $(a + b) + c = a + (b + c)$.
6. $(ab)c = a(bc)$.
7. $a(b + c) = ab + ac$.
8. $a + 0 = a$.
9. $1 \cdot a = a$.
10. For each real number a, there is a unique real number $-a$ such that

$$a + (-a) = 0$$

11. For each real number $a \neq 0$ there is a unique real number $1/a$ such that

$$a\left(\frac{1}{a}\right) = 1$$

NOTE. Properties (1) and (2) are called the *closure* properties; (3) and (4) are called the *commutative* properties; (5) and (6) are called the *associative* properties; (7) is called the *distributive* property; (8) and (9) are called *identity* properties; and (10) and (11) are called the *inverse* properties.

Any mathematical system that satisfies all 11 axioms is called a *field*. The system R of real numbers is one example of a field. There are others; for example, the informed reader is probably aware that the system of complex numbers (which we discuss in Chapter 6) is also a field.

Many properties of the real number system can be derived from the 11 field axioms (see Exercises 1.1). To illustrate, we prove the following two theorems.

THEOREM 1. *If a is a real number, then $a \cdot 0 = 0$.*

PROOF

$$0 = 0 + 0$$
$$a \cdot 0 = a(0 + 0)$$
$$a \cdot 0 = a \cdot 0 + a \cdot 0$$

Now add $-(a \cdot 0)$ to both sides, getting

$$0 = a \cdot 0$$

THEOREM 2. *If a and b are real numbers such that*

$$ab = 0$$

then either $a = 0$ or $b = 0$ or both.

Proof

$$ab = 0$$

If both numbers are equal to zero, there is nothing to prove; so let us assume that one number is different from 0. Without loss of generality, we can assume that number is a. So if $a \neq 0$, then $1/a$ exists and we can proceed as follows:

$$ab = 0$$
$$\frac{1}{a}(ab) = \frac{1}{a} \cdot 0$$
$$b = 0$$

Thus, if one number in the product ab is not zero, then the other must be zero.

Exercises 1.1

1. Field axiom 10 states that for each real number a there is a unique real number $-a$ such that $a + (-a) = 0$. The number $-a$ is called the *additive inverse of a*. Is $-a$ always a negative number? Give examples.

2. Show that if a, b, and c are real numbers and $a + c = b + c$, then $a = b$.

3. Show that if a, b, and c are real numbers with $c \neq 0$ and if $ac = bc$, then $a = b$.

4. Prove that for any real number a, $a \cdot 0 = 0$.

In Problems 5 and 6, express each rational number in decimal notation.

5. $\frac{6}{7}$

6. $\frac{7}{4}$

In Problems 7 to 10, express each rational number as the ratio of two integers.

7. 0.34

8. 1.73

9. -0.25

10. 0.72

11. Prove that $\sqrt{2}$ is irrational. (*Hint:* Assume that $\sqrt{2} = p/q$ where p and q are integers and show that this leads to a contradiction.)

12. Show that if $a + b = 0$, then $b = -a$.

13. Show that if $ab = 1$ and $a \neq 0$, then $b = 1/a$.

14. Prove that if $a = b$, then $-a = -b$.

15. Prove that $ab = 0$ if and only if $a = 0$ or $b = 0$.

16. Show that $-(-a) = a$ for any real number a.

17. Prove that if a and b are real numbers, then

$$-(ab) = (-a)b$$

18. Does the set Z of integers form a field? Give reasons.

19. Show that for any real number a, $-a = (-1)a$.

20. Prove that for any two real numbers a and b,

$$(-a)(-b) = ab$$

21. Prove that $(-1)(-1) = 1$.

22. If a and b are real numbers, then we subtract b from a according to the following definition:

$$a - b = a + (-b)$$

For example,

$$6 - 3 = 6 + (-3) = 3$$

and

$$\begin{aligned} 7 - (-2) &= 7 + [-(-2)] \\ &= 7 + 2 \\ &= 9 \end{aligned}$$

Is subtraction a commutative operation? Give examples.

23. Show that if a, b, and c are real numbers, then

$$a(b - c) = ab - ac$$

24. Suppose that a and b are real numbers and $b \neq 0$. We *divide* a by b according to the following definition:

$$\frac{a}{b} = a\frac{1}{b}$$

Show that if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

25. Show that if $a \neq 0$ and $b \neq 0$, then

$$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$$

26. Show that if $b \neq 0$ and $x \neq 0$, then

$$\frac{ax}{bx} = \frac{a}{b}$$

27. Show that if $b, d \neq 0$, then

$$\frac{a}{b} = \frac{c}{d}$$

if and only if $ad = bc$.

28. Show that if $c \neq 0$, then

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

29. Show that if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

30. Show that if $b, c, d \neq 0$, then

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

31. We know that $1 \neq 0$. Use this to discuss why division by zero is undefined.

1.2 Integral Exponents

Suppose that x is a real number and n is a positive integer. The symbol x^n is defined as follows:

$$x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$$

Example 6

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$
$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$
$$4^3 = 4 \cdot 4 \cdot 4 = 64$$
$$5^1 = 5$$

In the expression x^n, x is called the *base* and n is called the *exponent*.

Example 7

Find $(-2)^5$.

Solution

$$(-2)^5 = (-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2) = -32.$$

Example 8
Find $(3^2)(4^3)$.

Solution

$$\begin{aligned}(3^2)(4^3) &= 3\cdot 3\cdot 4\cdot 4\cdot 4\\ &= (9)\cdot(64)\\ &= 576\end{aligned}$$

Example 9
Find $[(-2)^3]^2$.

Solution

$$\begin{aligned}[(-2)^3]^2 &= (-2)^3\cdot(-2)^3\\ &= (-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2)\\ &= (-2)^6\\ &= 64\end{aligned}$$

Example 10
Find $(3^2)(3^4)$.

Solution

$$\begin{aligned}(3^2)(3^4) &= (3)\cdot(3)\cdot(3)\cdot(3)\cdot(3)\cdot(3)\\ &= 3^6\\ &= 729\end{aligned}$$

THEOREM 3. *If n is a positive integer and x and y are real numbers, then*

$$(xy)^n = x^n\cdot y^n$$

PROOF

$$\begin{aligned}(xy)^n &= \underbrace{(xy)\cdot(xy)\cdot(xy)\cdot\ldots\cdot(xy)}_{n\text{ factors}}\\ &= (\underbrace{x\cdot x\cdot x\cdot\ldots\cdot x}_{n\text{ factors}})\cdot(\underbrace{y\cdot y\cdot y\cdot\ldots\cdot y}_{n\text{ factors}})\\ &= x^n\cdot y^n\end{aligned}$$

THEOREM 4. *If n is a positive integer and x and y are real numbers with $y \neq 0$, then*

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$