INTRODUCTION TO DIFFERENTIAL EQUATIONS WITH APPLICATIONS

FRED BRAUER · JOHN A. NOHEL

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University of Wisconsin

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Introduction to Differential Equations with Applications

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Dedicated
to the fond memory of our teacher and friend
Norman Levinson

Preface

This book introduces differential equations to students who have had a sound course in calculus, including functions of several variables. It is intended not only for students in engineering or in the physical or mathematical sciences, but is also suitable for those studying the biological or social sciences. By choice of topics and degree of emphasis the instructor can use this book in either standard or "honors" courses, ranging in length from one quarter to a full academic year.

Differential equations serve as mathematical models for numerous problems in science and engineering. These models originate in Sir Isaac Newton's mathematical description of the laws of motion of particles. By postulating his famous laws of mechanics, Newton was able to model the motion of a particle by an equation involving an unknown function and one or more of its derivatives. Such an equation is called a *differential equation.* The objective of this book is to study methods of solution of differential equations. We use examples of problems in population growth, radioactive decay, compound interest, particle motion, electrical circuits, heat flow, chemical reactions, and vibrations as motivation for and applications of the methods developed. These examples are self-contained, and include the essential underlying physical or biological concepts, making them accessible to readers who are not familiar with these ideas.

The mathematical models for these problems are for the most part very simple, and therefore quite crude. Such simple models lead to differential equations that can sometimes be solved exactly. More refined mathematical models describing physical behavior more precisely usually lead to differential equations for which no explicit solutions in terms of elementary functions can be obtained. Except for infinite series solutions, models of the latter type will be described only briefly, and simplifying assumptions will be made, in order to make them more manageable. Although this course emphasizes mathematical techniques rather than modeling, the simple modeling

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presented throughout the text forms a useful basis for understanding how to construct more sophisticated mathematical models.

The formulation of mathematical models for physical problems is often misunderstood. In our view, a reasonable procedure is to start from a concrete physical situation (e.g., the motion of a pendulum), pass to an "idealized" physical model (e.g., a rod of zero weight, a pivot of zero friction, zero air resistance, etc.), and then using physical laws (e.g., Newton's laws of motion), construct a mathematical description of such an idealized physical model. It is this procedure that is often difficult and belongs in advanced science and engineering courses. Our main objective here is to use mathematical techniques to explore properties of simple models in the attempt to answer specific questions about the behavior of the physical system under consideration. Such answers would make it possible to compare mathematical results with physical results as measured by careful experiments, thereby testing both the validity of the assumed physical laws and the idealizations used in the formulation of the models.

The simplest and crudest mathematical models that we discuss lead to the study of a single differential equation in which the unknown function depends on only one variable. Such an equation is called an *ordinary differential equation.* If the unknown function depends on more than one variable, the resulting equation is called a *partial differential equation.* If the physical system to be described involves more than one unknown function, the resulting mathematical model to be studied is a *system of differential equations* (e.g., coupled masses and springs, electrical circuits, etc.). Although this book deals primarily with single ordinary differential equations, our firm view is that an introduction to the study of partial differential equations and to systems of ordinary differential equations is essential for the adequate preparation of university students.

A proper understanding of methods of solution requires a certain amount of theoretical background. For example, a technique yielding a particular solution of a problem is useless without the knowledge that the problem has exactly one solution. Where theoretical background is necessary it is motivated and the facts are stated as theorems, sometimes given without proof. Readers interested in the proofs may find them either elsewhere in the text or references. The development of differential equations as a flourishing branch of modern mathematics occurred in the nineteenth and twentieth centuries. Recent advances continue to lead to interesting new applications and to a deeper understanding of complex scientific and engineering systems described by sophisticated mathematical models that cannot be solved exactly. Such models must be analyzed in other ways. In some cases the theory, which continues to evolve, provides a qualitative rather than a quantitative description of solutions. However, in most instances the practitioner, particularly in the sciences and engineering, will have to obtain quantitative information by utilizing available techniques, developing approximate numerical methods, and by using a modern high-speed computer. In a first course it is only possible to introduce the student to simpler qualitative methods and provide some simple numerical methods (which, incidentally, form the basis for much commercially available software).

Many students think of mathematics as merely a collection of tricks, techniques, and skills. With the experience gained in courses for which this book is intended, they will learn to consider mathematics as an essential tool for understanding basic phenomena in the physical world. At the same time, it is very important for the prospective mathematician to become acquainted with mathematical problems that arise in applications. Indeed, such problems often lead to questions of independent and continuing mathematical interest. It is our belief that this book will serve these purposes.

Chapters 1, 2, and 3 form the core of any course taught from this book. Students who have already been exposed to this material in their calculus sequence can rapidly cover much of Chapters 1 and 2. In Chapter 3 Sections 3.1, 3.2, 3.8, and 3.9 form a self-contained unit on linear differential equations that does not use linear algebra and matrices. Sections 3.3 through 3.7 discuss linear systems of differential equations by employing elementary linear algebra, of which the principal results are stated in the text without proof.

Chapter 4 (on the use of the Laplace transform in solving linear differential equations), Chapter 5 (on series solutions of linear differential equations), and Chapter 6 (on elementary qualitative study of differential equations) are independent units. Chapters 7 and 8 form a self-contained introduction to linear partial differential equations. Each unit makes use of some of the core material. With the exception of Section 4.5, no knowledge of linear algebra is required. Some necessary background material, primarily from calculus, is collected for reference in five appendices.

Certain sections, indicated in the contents, can be omitted without disturbing the continuity. These sections deal with some elementary methods for numerical approximation of solutions (Sections 1.9, and 2.8) or with theoretical material that the reader may choose to omit, accepting the results without proof (Sections 1.6, 1.8, 7.5, 7.6, 8.4, 8.5, 8.6). In addition, parts of certain sections, indicated in the text by a bullet at the beginning and the end, can likewise be omitted.

For students with no previous exposure to differential equations or linear algebra, a typical one-semester course might include the core material (omitting Sections 3.3 through 3.7) and any two of Chapters 4 (omitting Section 4.5), 5, and 6. For such students, a one-quarter course would only deal with parts of Chapters 4 through 6. The entire book, with the possible exception of some of the more theoretical sections, can be covered in two quarters or two semesters.

On the other hand, students with a more thorough preparation have several options. One possibility for a one-semester course would be to study Chapters 1 through 6 in depth. Another option would be to cover Chapters 7 and 8 in place of Sections 3.3 through 3.7 and Chapter 6; courses following this option could omit Sections 7.5 and 7.6, or Sections 8.4 through 8.6 as desired. Honors courses should include most of Chapter 6 and the theoretical sections that have been designated as optional.

We believe that we have provided students with a readable text, with summary boxes to highlight the important concepts. Students must realize that any mathematics book should be read with paper and pencil at hand in order to fill in missing details.

An essential part of this book consists of the numerous worked examples interspersed throughout the text, and the more than 1000 accompanying exercises, some of which are given at the end of virtually every section. More difficult exercises are marked with an asterisk following the exercise number. The miscellaneous exercises at the ends of chapters are intended to help students review and to supplement the content. Answers to odd-numbered exercises are provided at the end of the book. Students should realize that some answers have more than one equivalent form.

We acknowledge with pleasure and gratitude the direct and indirect help of colleagues, teaching assistants, and students at the University of Wisconsin in Madison over the past 25 years. We are particularly grateful to Jerrold Marsden, Constantine Dafermos, William Hrusa, Michael Renardy, and several reviewers including Dennis R. Dunninger, Michigan State University; Warren S. Loud, University of Minnesota; Carol L. Shilepsky, Wells College; W. E. Conway, University of Arizona; C. L. Dolph, University of Michigan; Thomas R. Kiffe, Texas A & M University; Lars B. Wahlbin, Cornell University; Jay R. Walton, Texas A & M University; Louis Grimm, University of Missouri—Rolla; and Edward J. Scott, University of Illinois, for their painstaking reading of different parts of the manuscript and their many helpful suggestions and corrections in the text and in statements and answers to Exercises. Of course, any remaining errors are our responsibility. Thanks are also due to Mrs. Grace Krewson for typing most of the manuscript, Mrs. Judith Siesen for additional typing and to Mrs. Vera Nohel for valuable assistance with proofreading. Finally, we thank our publisher, Harper & Row, particularly Eleanor Castellano for her patience during the preparation of this book.

Madison, Wisconsin
October, 1985

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Chapter 1

Introduction to First-Order Differential Equations

In this chapter we study methods of solving first-order differential equations, that is, equations involving an unknown function and its first derivative. These methods are applicable to specific types of equations arising in a variety of applications. Sections 1.1 to 1.3 form a unit; in Section 1.1 a mathematical description of the growth of a population leads to a differential equation of a type called *separable equations.* We develop in detail a procedure for solving this class of equations in Section 1.2 and illustrate in Section 1.3 with several applications, including population growth, radioactive decay, and chemical mixtures. In Section 1.4 we learn how to solve the class of *linear first-order equations* and in Section 1.5 we apply the method to problems involving electrical circuits, motion of particles, compound interest, and chemical mixtures.

Not all first-order differential equations fall into the preceding two categories: separable and linear first order. Sections 1.6 to 1.8 are intended to provide geometrical and theoretical comprehension, particularly for dealing with first-order differential equations that cannot be solved explicitly. Sections 1.6 to 1.8 are optional but Section 1.7 is essential for understanding first-order differential equations. The optional Section 1.9 discusses the numerical approximation of solutions of first-order equations by some methods that are simple enough to be carried out with the aid of a programmable hand-held calculator.

1.1 SOME POPULATION GROWTH PROBLEMS

To motivate the study of differential equations of first order, we formulate a mathematical model for the growth of a population of organisms of a single species. The resulting differential equation is then solved. The method of solution will be applied to a large class of first-order differential equations in Section 1.2.

Consider a population of organisms of a single species. Let $y(t)$ be the number of members of the population at time t. Our goal is to determine this function y. We make the following assumptions:

1. At time $t = 0$ the population has A members.
2. $y(t)$ is a differentiable function of t for $t > 0$ [i.e., $y'(t) = d/dt\, y(t)$ exists for $t > 0$].
3. For *small* $h > 0$ the change $y(t + h) - y(t)$ in the size of the population over the time interval from t to $t + h$ is approximately proportional to the length h of the interval and to the population size $y(t)$ at time t with constant of proportionality k (which may be positive, negative, or zero).

We will comment on the significance of these assumptions latter.

We translate these assumptions into a mathematical problem by noting that assumption 3 states:

$$y(t + h) - y(t) \approx khy(t) \tag{1}$$

for any $t \geq 0$ and any number $h > 0$. Dividing Eq. (1) by h, we obtain:

$$\frac{y(t + h) - y(t)}{h} \approx ky(t)$$

Now letting $h \to 0$ and using the definition of the derivative, we obtain the equation:

$$y'(t) = ky(t) \tag{2}$$

Observe that this limiting process makes use of assumption 2. To Eq. (2) we add the statement $y(0) = A$, which is assumption 1.

Summarizing, we have:

Law of Simple Population Growth

$$y'(t) = ky(t) \qquad (t > 0)$$
$$y(0) = A$$

where $y(t)$ is the size of the population at time t and k is a given constant.

The mathematical problem is to find a function $y(t)$ satisfying Eq. (2) for $t > 0$ such that $y(0) = A$. Equation (2) for the unknown function y is called a *differential equation* (of first order) and the condition $y(0) = A$ is called the *initial condition*. We now solve this problem.

If the constant k in Eq. (2) is zero, then (2) becomes $y'(t) = 0$ for $t > 0$. Since $y(t)$ has derivative zero, $y(t)$ is a constant, and the initial condition $y(0) = A$ implies that $y(t) = A$ for $t \geq 0$.

Next, we assume $k \neq 0$. Since $y(t)$ represents the size of the population, we expect that $y(t) \geq 0$. If $y(t) \neq 0$, we can divide (2) by $y(t)$ and obtain:

$$\frac{y'(t)}{y(t)} = k$$

We recall by the chain rule that $d(\log|y(t)|)/dt = y'(t)/y(t)$.* Thus the previous equation can be written as:

$$\frac{d}{dt}\log|y(t)| = k$$

and therefore integration from zero to t yields:

$$\log|y(t)| - \log|y(0)| = kt$$

Using the initial condition $y(0) = A$, assuming that $A > 0$ and using the law $\log a - \log b = \log(a/b)$ of logarithms, we obtain:

$$\log|y(t)| - \log|A| = \log\left|\frac{y(t)}{A}\right| = kt$$

and therefore $$e^{\log|y(t)|/A} = e^{kt}$$

Since $e^{\log a} = a$ for all a, this becomes:

$$\frac{|y(t)|}{A} = e^{kt}$$

$$|y(t)| = Ae^{kt}$$

Assuming $y(t) > 0$, we have $|y(t)| = y(t)$,

and thus $$y(t) = Ae^{kt} \qquad (0 \leq t \leq \infty) \tag{3}$$

Up to this point we have shown that if $y(t) > 0$ is a solution of Eq. (2) such that $y(0) = A$, then $y(t)$ is given by formula (3). We now check that $y(t) = Ae^{kt}$ is a solution for $t \geq 0$. But this is easy because:

$$\frac{dy}{dt}(t) = Ake^{kt} = kAe^{kt} = ky(t)$$

for $t \geq 0$, and $y(0) = Ae^0 = A$. The reader should notice that we confined ourselves to the interval $t \geq 0$ in the formulation of the population problem since t represents time from some beginning; the function y defined by Eq. (3) is actually a solution of Eq. (2) on the interval $-\infty < t < \infty$ for every value of the constant A.

In solving Eq. (2), we have assumed that $y(t)$ is never zero. Indeed, solution (3) is never zero. The method of solution cannot be applied if $A = 0$ because the logarithm of zero is undefined. However, the function $y(t) \equiv 0$ (which has $y'(t) \equiv 0$) satisfies

* In your calculus course you probably used the notation ln for the natural logarithm. Postcalculus books usually write log in place of ln.

Eq. (2) by inspection. Thus if $A = 0$, a solution is $y(t) \equiv 0$. An interpretation of this result is that if the initial size of the population is zero, then the population remains zero (i.e., even mathematically you do not get something for nothing).

We summarize the result as follows:

The Solution of the Equation of Simple Population Growth

$$y' = ky, \qquad y(0) = A \qquad (A \geq 0)$$

is

$$y(t) = Ae^{kt}$$

1.1.1 Discussion of the Assumptions for the Law of Simple Population Growth

Assumption 1 involves picking an initial time which we call $t = 0$ at which the size of the population is counted; note that the result A of this count is a positive integer or zero. Evidently, the value $y(t)$ of the population size at every time $t \geq 0$ must also be a positive integer or zero. However, assumption 2—that $y(t)$ is a continuously varying function for $t > 0$—is a reasonable approximation to the situation for large populations.

Assumption 3 is a particular biological statement. For example, if the population size changes only through births of individuals, and if in a small time interval of length h the fraction of the population giving birth is proportional to h, say, bh, for some constant $b \geq 0$, then the number of members added to the population between time t and time $(t + h)$ is $bhy(t)$. Thus this process satisfies assumption 3 with $k = b \geq 0$.

For a process in which the population size changes only through deaths of individuals (rather than births), and for which in a small time interval of length h the fraction of the population dying is proportional to h, say, dh, for some constant $d \geq 0$, the number of members *added* to the population between time t and time $(t + h)$ is $-dhy(t)$. Thus again assumption 3 is satisfied with $k = -d \leq 0$.

We can combine the birth and death processes. Suppose that in a small time interval of length h the fraction of the population giving birth is bh and the fraction of the population dying is dh, and that the population changes only through births and deaths (migration into or out of the population is not permitted). Then assumption 3 is satisfied, with $k = b - d$; the constant k may be positive, negative, or zero.

From the solution $y(t) = Ae^{kt}$ of the equation of simple population growth, we observe that the population size grows exponentially if $k > 0$ (i.e., if $b > d$), remains constant if $k = 0$ (or $b = d$), and decreases exponentially to zero if $k < 0$ (or $b < d$).

In Eq. (2) we can interpret $y'(t)/y(t)$ as a *growth rate*. It is, in fact, the time rate of increase of population per member. Thus in the model represented by Eq. (2) we are saying that the growth rate is constant. This leads to the biological conclusion that the population continues to increase.

EXAMPLE 1

Suppose that the birthrate of a given population of protozoa is 0.7944 per member per day (i.e., $k = 0.7944$). Let the population on day zero consist of two members. Assuming no deaths or migration, find the population at the end of days 1 through 6.

Solution. Here differential Eq. (2) is $y'(t) = 0.7944y(t)$ $[k = 0.7944]$ and the initial condition is $y(0) = 2$. Solution (3) is $y(t) = 2e^{0.7944t}$. Thus $y(1) = 2e^{0.7944} = 4$, $y(2) = 2e^{2(0.7944)} = 10$, $y(3) = 2e^{3(0.7944)} = 22$, $y(4) = 2e^{4(0.7944)} = 48$, $y(5) = 2e^{5(0.7944)} = 106$, and $y(6) = 2e^{6(0.7944)} = 235$, which gives the population sizes rounded off to the nearest integer. ■ ■

EXAMPLE 2

Suppose that a population satisfies the law of simple population growth. Suppose also that at $t = 0$ the population has 100 members and at the end of 100 days the population has 50 members. Find the population at the end of 150 days.

Solution. Since the population satisfies the law of simple population growth, the population size $y(t)$ satisfies $y(t) = Ae^{kt}$, with $A = 100$ because $y(0) = 100$. In order to calculate k, we substitute $t = 100$, obtaining $y(100) = 100e^{100k}$. Combining this with $y(100) = 50$, we obtain:

$$100e^{100k} = 50$$

Then

$$e^{100k} = \tfrac{1}{2}$$

$$100k = \log \tfrac{1}{2} = -\log 2$$

$$k = -\tfrac{1}{100} \log 2 = -6.93 \times 10^{-3}$$

Now

$$y(150) = 100e^{150k} = 100 \exp[(150)(-6.93) \times 10^{-3})] = 35.36$$

Rounding off the population size to the nearest integer, we have $y(150) = 35$. ■ ■

The ratio $y'(t)/y(t)$ is called the (instantaneous) per capita growth rate of the population. Note that for all populations obeying the law of simple population growth this growth rate is the constant k from Eq. (2).

In many situations the growth rate is not constant. For example, one might assume that for a given population the growth rate depends on the food supply, which decreases as the population size increases. An assumption that is often made for such populations is that the per capita growth rate is $k - cy(t)$, where k and c are positive constants. This is called the law of logistic population growth. (The law of simple population growth corresponds to the case $c = 0$). Under this assumption:

$$y'(t)/y(t) = k - cy(t)$$

Thus if the initial size of the population is A, the population size satisfies the differential equation:

$$y' = ky - cy^2 \tag{4}$$

and the initial condition is: $y(0) = A$.

Law of Logistic Population Growth

$$y' = ky - cy^2 \qquad (t > 0)$$
$$y(0) = A$$

where $y(t)$ is the size of the population at time t and k and c are given constants.

We solve this equation in Section 1.3.

EXERCISES

In each of the following exercises, assume that the population satisfies the law of simple population growth.

1. Suppose that the birthrate of a given population of protozoa is 0.36 per member per day and that there are no deaths. If the population on day zero is 50, find the population 10 days later.

2. Suppose that the birthrate of a given population of protozoa is 0.21 per member per day and that there are no deaths. If the population on day zero is 100, find the population 7 days later.

3. Suppose that a given population has birthrate 0.73 per member per week and death rate 0.28 per member per week. If the population has initial size 170, find the population size after 4 weeks.

4. Suppose that a given population has birthrate 0.56 per member per week and death rate 0.87 per member per week. If the population has initial size 252, find the population size after 6 weeks.

5. Suppose that a population has 173 members at $t = 0$ and 262 members at $t = 10$. Estimate the population size at $t = 5$.

6. Suppose that a population has 87 members at $t = 0$ and 125 members at $t = 4$. Estimate the population size at $t = 6$.

7. Suppose that a population has 24 members at $t = 5$ and 15 members at $t = 15$. What was the population size at $t = 0$?

8. Suppose that a population has 39 members at $t = 8$ and 60 members at $t = 12$. What was the population size at $t = 0$?

9.*Show that if a population contains A members at time $t = t_0$, then for $t \neq t_0$ the population size is $y(t) = Ae^{k(t-t_0)}$.

1.2 EQUATIONS WITH VARIABLES SEPARABLE

The purpose of this section is to define first-order differential equations and learn how to recognize and solve differential equations whose variables are separable.

The two laws of population growth discussed in Section 1.1 are modeled by differential equations of the general form:

$$y' = f(t, y) \tag{1}$$

where f is a given function of two variables (t, y) and y is the unknown function. For

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