ALM 8

Advanced Lectures in Mathematics

Recent Developments in Algebra and Related Areas

代数及相关领域最新进展

Editors: Chongying Dong · Fu-an Li

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Preface

This volume contains papers presented at the International Conference on Algebra and Related Areas, held in Tsinghua University, Beijing, China, during August 18–20, 2007. The conference was dedicated to Professor Zhexian Wan in honor of his 80th birthday. About two hundred researchers, including graduate students and young mathematicians from China, Japan, Singapore, Australia, the Netherlands, Italy, and the United States, participated in this conference. There were fifteen invited lectures by well-known experts on algebraic geometry, combinatorics, coding theory, Lie algebras, representation theory of finite groups and algebraic groups, vertex operator algebras and their applications.

Professor Wan's contributions to mathematics are legendary. His extensive research covers many areas on mathematics, such as classical groups, geometry of matrices, finite fields and finite geometry, Lie algebras, combinatorics, graph theory, lattice theory, coding theory and cryptology, design theory with many fundamental results. In classical groups, Professor Wan investigated the structure and automorphism groups of various subgroups and quotient groups of classical groups over fields and skew fields. In particular, he and his former students Hongshou Ren and Xiaolong Wu proved in 1986 that all automorphisms of the two-dimensional special linear group over an arbitrary skew field are standard, and all isomorphisms between two-dimensional special linear groups over skew fields are standard with only one exception. This completely solved the very difficult problem on automorphisms and isomorphisms of linear groups over skew fields. In geometry of matrices, he systematically investigated the geometry of symmetric matrices, the geometry of alternate matrices, the geometry of hermitian and skew-hermitian matrices, generalizing the Fundamental Theorem of Projective Geometry to the geometry over arbitrary fields and skew fields with involution, and giving some applications to graph theory. The study of finite geometry and its applications in China was initiated by Professor Wan. He studied the action of various classical groups on vector spaces over finite fields. He developed a new theory to classify the orbits and to determine the lengths of orbits and related. He also applied these results to combinatorial design, information security, coding theory and graph theory, and obtained many important results. Besides, he gave a beautiful proof for a graphic method for solving the transportation problem and he solved a problem on linear shift register sequences. There is no doubt that Professor Wan is the leader in the Chinese algebra community, and the influence of his work over the half century will last for many years to come.

We are very grateful to the China and U.S. National Science Foundations, International Mathematical Union, Tsinghua University, Institute of Systems Science of Chinese Academy of Sciences, and many individuals for the organizing and support of this conference. We would like to sincerely thank all the participants, speakers, and authors for all their efforts and timely submissions, thereby making the conference a success. We appreciate the referees for their excellent review work. Thanks also go to the Higher Education Press and the International Press to publish these conference proceedings as one of the series Advanced Lectures in Mathematics.

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Contents

Preface

Eiichi Bannai and Etsuko Bannai: Spherical Designs and Euclidean Designs 1

Yu Chen: Minimal Representation Degree of Affine Kac-Moody Groups 39

Huah Chu, Shou-Jen Hu and Ming-chang Kang: A Rationality Problem of Certain A_4 Action 53

Zongduo Dai, Kunpeng Wang and Dingfeng Ye: Characterization of Multi-Continued Fractions for Multi-Formal Laurent Series 59

Yi Fang and Zongzhu Lin: Eulerian Trails and Hamiltonian Paths in Digraphs with Anti-involutions 81

Rongquan Feng and Hongfeng Wu: Efficient Pairing Computation on Curves 99

Wenbin Guo: Some Ideas and Results in Group Theory 111

Tayuan Huang, Lingling Huang and Miaow-Ing Lin: On a Class of Strongly Regular Designs and Quasi-semisymmetric Designs 129

Huishi Li: Γ-Leading Homogeneous Algebras and Gröbner Bases 155

Jianbo Liu and Kaiming Zhao: Automorphism Groups of Lie Algebras from Quantum Tori 201

Yucai Su: Quasifinite Representations of Some Lie Algebras Related to the Virasoro Algebra 213

George Szeto and Lianyong Xue: On Galois Extensions with an Inner Galois Group 239

Nanhua Xi: Representations of Algebraic Groups: Some Basics and Progresses 247

Jiping Zhang and Zhikai Zhang: Broué's Conjecture for Finite Groups with Abelian Sylow p-Subgroups 263

Sujing Zhou and Dongdai Lin: An Interesting Member ID-based Group Signature 279

Curriculum Vitae and Publications of Zhexian Wan 303

Recent Developments in
Algebra and Related Areas
ALM 8, pp. 1–37

Spherical Designs and Euclidean Designs

Eiichi Bannai† Etsuko Bannai‡

Abstract

The purpose of this paper is basically to give a survey on the theory of spherical t-designs and Euclidean t-designs. We will first recall the definitions of spherical t-designs and Euclidean t-designs. We discuss some examples, and look at the problem of finding and classifying tight spherical designs and tight Euclidean designs. We plan to discuss the connection with the cubature formulas in numerical analysis on one hand, and the connection with groups and sphere packing problems on the other hand.

2000 Mathematics Subject Classification: primary 05E99; secondary 05B99, 51M99, 62K99

Keywords: Euclidean designs, spherical designs, tight designs, association schemes, universally optimal codes, Assmus-Mattson type theorem

1 Introduction

This paper is an extended version of the talk titled "On Euclidean t-Designs" given by the first author at the conference in honor of Professor Zhexian Wan's 80th birthday in Beijing in August 18-20, 2007. The abstract given above reflects the actual talk. In the talk, we discussed extensively on the connections of cubature formulas in numerical analysis with the theory of spherical and Euclidean designs. Since the exposition discussing this part was already written up in [9], we will not treat this part much in this paper, but we concentrate on the new results which were obtained just before the conference and briefly discussed in the talk.

The contents of this paper are as follows:

1. Introduction
2. Notation

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3. Spherical t-Designs
4. Euclidean t-Designs
5. Tight Euclidean 7-Designs on Two Concentric Spheres
6. Tight Euclidean 9-Designs on Two Concentric Spheres
7. Antipodal Spherical t-Designs of Degree s Satisfying $t \geqslant 2s-3$
8. Antipodal Spherical 5-Designs of Degree 4
9. Antipodal Spherical 7-Designs of Degree 5
10. Speculations and Concluding Remarks

The new results are mentioned in Sections 5–9 and main results are Theorems 5.1, 7.1 and 8.1. This paper is not a ultimate paper, but a working paper which describes the ongoing research by us. We hope that this direction of the research will reach higher level of understanding of the subjects in the near future, and we hope this paper is useful for that purpose.

2 Notation

First we give the notation for the vector spaces of polynomials we use in this paper. Let S^{n-1} be the unit sphere in the Euclidean space $\mathbb{R}^n$. Let

$$\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, x_2, \ldots, x_n]$$

be the vector space of polynomials in n variables $x_1, x_2, \ldots, x_n$. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ spanned by all the homogeneous polynomials of degree l. Let

$$\mathcal{P}_l(\mathbb{R}^n) = \bigoplus_{i=0}^{l} \mathrm{Hom}_i(\mathbb{R}^n) \quad \text{and} \quad \mathcal{P}_l^*(\mathbb{R}^n) = \bigoplus_{i=0}^{[\frac{l}{2}]} \mathrm{Hom}_{l-2i}(\mathbb{R}^n).$$

Let $\mathrm{Harm}(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of all the harmonic polynomials. Let $\mathrm{Harm}_l(\mathbb{R}^n) = \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$. Let $h_l = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$ for any non-negative integer l. For a subset $Y \subset \mathbb{R}^n$, let $\mathcal{P}(Y)$, $\mathcal{P}_l(Y)$, $\mathcal{P}_l^*(Y)$, $\mathrm{Hom}_l(Y)$, $\mathrm{Harm}(Y)$, $\mathrm{Harm}_l(Y)$ be the subspaces of corresponding polynomials restricted to Y. For example, $\mathcal{P}_l^*(Y) = \{f|_Y \mid f \in \mathcal{P}_l^*(\mathbb{R}^n)\}$.

3 Spherical t-Designs

The concept of spherical t-designs was given by Delsarte, Goethals and Seidel in [28].

Definition 3.1. (spherical t-designs) Let X be a finite set on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Let t be a natural number. Then with the notation mentioned above, we say that X is a *spherical t-design* if the following condition is satisfied:

$$\frac{1}{|S^{n-1}|} \int_{\boldsymbol{x} \in S^{n-1}} f(\boldsymbol{x}) d\sigma(\boldsymbol{x}) = \frac{1}{|X|} \sum_{\boldsymbol{u} \in X} f(\boldsymbol{u})$$

for any polynomial $f(\boldsymbol{x}) \in \mathcal{P}_t(\mathbb{R}^n)$, where σ denotes the usual Haar measure on the unit sphere.

They gave the following natural lower bounds for the cardinalities of spherical t-designs.

Theorem 3.1. [28] *Let X be a spherical t-design.*

(1) *If $t = 2e$, then the following holds:*

$$|X| \geqslant \binom{n+e-1}{e} + \binom{n+e-2}{e-1} \left(= \dim(\mathcal{P}_e(S^{n-1}))\right).$$

(2) *If $t = 2e+1$, then the following holds:*

$$|X| \geqslant 2\binom{n+e-1}{e} \left(= 2\dim(\mathcal{P}_e^*(S^{n-1}))\right).$$

They defined the following concept of tight spherical t-designs.

Definition 3.2. (tight spherical t-designs) If the equality holds in any of the inequalities of Theorem 3.1, then X is a *tight spherical t-design.*

In [28], they studied the upper bounds for the cardinalities of s-distance sets X in S^{n-1}. Let

$$A(X) = \{\boldsymbol{x} \cdot \boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{y} \in X, \boldsymbol{x} \neq \boldsymbol{y}\},$$

where $\boldsymbol{x} \cdot \boldsymbol{y}$ denote the usual inner product between the vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^n$. If $|A(X)| = s$, then $X \subset S^{n-1}$ is called an s-distance set. They proved the following theorems.

Theorem 3.2. [28] *Let $X \subset S^{n-1}$ be an s-distance set. Then*

$$|X| \leqslant \dim(\mathcal{P}_s(S^{n-1}))$$

holds. Moreover, if X is antipodal, then

$$|X| \leqslant 2\dim(\mathcal{P}_{s-1}^*(S^{n-1}))$$

holds.

Theorem 3.3. [28]

(1) *Let $|X| = \dim(\mathcal{P}_e(S^{n-1}))$. Then X is an e-distance set if and only if X is a spherical tight $2e$-design.*

(2) *Let $|X| = 2\dim(\mathcal{P}_e^*(S^{n-1}))$. Then X is an antipodal $(e+1)$-distance set if and only if X is a spherical tight $(2e+1)$-design.*

The existence of spherical t-designs X in S^{n-1} was proved by Seymour-Zaslavsky for any t, n and $|X|$ if $|X|$ is sufficiently large (see [43]). However, spherical tight t-designs are very special and hardly exist for $n \geqslant 3$ or $t \geqslant 4$. The following are the known results about the classification of spherical tight t-designs at this stage (as for more information, see [5], [15], etc.).

- $n=2$: X is a spherical tight t-design if and only if X is a regular $(t+1)$-gon.
- 1-designs: $\{\boldsymbol{x}, -\boldsymbol{x}\}$ is a spherical tight 1-design for any $\boldsymbol{x} \in S^{n-1}$.
- 2-designs: $X \subset S^{n-1}$ is a spherical tight 2-design if and only if X is a regular simplex.
- 3-designs: $X \subset S^{n-1}$ is a spherical tight 3-design if and only if X is isometric to $\{\pm \boldsymbol{e}_i \mid 1 \leqslant i \leqslant n\}$ (cross polytope), where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n$.
- If $n \geqslant 3$ and X is a spherical tight t-design, then $t = 1, 2, 3, 4, 5, 7, 11$.
- The spherical tight 11-design X is unique up to isometry. That is, X is isometric to the set of 196560 minimal vectors in the Leech lattice in $\mathbb{R}^{24}$.
- The classifications of the spherical tight 4-, 5-, 7-designs are still open problems. The following are the only known examples so far:
  $t=4$: the 27-point set on S^5, the 275-point set on S^{21}.
  $t=5$: the set of 12 vertices of the icosahedron on S^2, the set of 56 weight vectors of the E_7 root system on S^6, the 552-point set on S^{22}.
  $t=7$: the set of 240 vectors of the E_8 root system on S^7, the 4600-point set on S^{22}.

4 Euclidean t-Designs

In this section, we introduce the concept of Euclidean t-designs and give some basic facts given in the papers [41], [29] and [9].

First we give some more notation. Let X be a finite set in $\mathbb{R}^n$ possibly containing 0. Let

$$\{r_1, r_2, \ldots, r_p\} = \{\|\boldsymbol{x}\| \mid \boldsymbol{x} \in X\},$$

where $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$. Let

$$S_i = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = r_i\}$$

and $X_i = X \cap S_i$ for $i = 1, 2, \ldots, p$. Let $w(\boldsymbol{x})$ be a positive real valued weight function on X. Let $w(X_i) = \sum_{\boldsymbol{x} \in X_i} w(\boldsymbol{x})$ $(1 \leqslant i \leqslant p)$. On each S_i we consider the Haar measure σ_i. For $S_i \neq \{0\}$, we assume

$$|S_i| = \int_{S_i} d\sigma_i(\boldsymbol{x}) = r_i^{n-1} \int_{S^{n-1}} d\sigma(\boldsymbol{x}).$$

If $S_i = \{0\}$, then we define

$$\frac{1}{|S_i|} \int_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x}) = f(0).$$

Finally, let $S = \bigcup_{i=1}^{p} S_i$ and we say that X is supported by the union S of p concentric spheres, or S is the support of X.

We define an inner product on the vector space $\mathcal{P}(\mathbb{R}^n)$ by

$$\langle f, g\rangle = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(\boldsymbol{x})g(\boldsymbol{x})d\sigma(\boldsymbol{x}).$$

We give one more notation. For any subsets X_i and X_j of a Euclidean t-design, we define

$$A(X_i, X_j) = \{\boldsymbol{x}\cdot\boldsymbol{y} \mid \boldsymbol{x}\in X_i,\ \boldsymbol{y}\in X_j,\ \boldsymbol{x}\neq\boldsymbol{y}\}.$$

Definition 4.1. (Euclidean t-designs) A finite subset X in $\mathbb{R}^n$ is a *Euclidean t-design* if there exists a positive weight function $w(\boldsymbol{x})$ on X and

$$\sum_{i=1}^{p}\frac{w(X_i)}{|S_i|}\int_{S_i} f(\boldsymbol{x})d\sigma_i(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})f(\boldsymbol{x})$$

holds for any polynomial $f(\boldsymbol{x})\in\mathcal{P}_t(\mathbb{R}^n)$.

The following theorem was proved by Neumaier-Seidel [41] which gives an equivalent condition for Euclidean t-designs (see also [6]).

Theorem 4.1. (Neumaier-Seidel) *Let X be a finite set in $\mathbb{R}^n$. Let w be a positive weight function defined on X. Then the following conditions are equivalent:*

(1) *X is a Euclidean t-design with weight w.*

(2) $\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})f(\boldsymbol{x}) = 0$ *for any polynomial* $f\in\|\boldsymbol{x}\|^{2j}\mathrm{Harm}_l(\mathbb{R}^n)$ *with* $1\leqslant l\leqslant t$ *and* $0\leqslant j\leqslant[\frac{t-l}{2}]$.

Proof. The fact that the integral of any harmonic polynomial of positive degree on a sphere centered at the origin vanishes implies Theorem 4.1. □

By definition, the formula given by a Euclidean t-design is a cubature formula on $S=\bigcup_{i=1}^{p} S_i$ of degree t with $|X|$ points. Since the integrals given in the definition of Euclidean t-designs are centrally symmetric, we can apply Möller's method [39, 40] for the Euclidean t-designs.

Theorem 4.2. *Let $X\subset\mathbb{R}^n$ be a Euclidean $2e$-design supported by a union S of p concentric spheres. Then*

$$|X|\geqslant\dim(\mathcal{P}_e(S))$$

holds.

Theorem 4.3. *Let $X\subset\mathbb{R}^n$ be a Euclidean $(2e+1)$-design supported by a union S of p concentric spheres. Then*

$$|X|\geqslant\begin{cases}2\dim(\mathcal{P}_e^*(S))-1 & \text{for } e \text{ even and } 0\in X,\\ 2\dim(\mathcal{P}_e^*(S)) & \text{otherwise.}\end{cases}$$

Theorem 4.2 was also proved in [29]. As for Euclidean $(2e+1)$-designs, Delsarte-Seidel and also Bannai gave the same lower bound assuming X is antipodal (see [29, 17]).

We also have the following theorem.

Theorem 4.4. [9, Theorem 2.3.5] *Let X be a Euclidean $(2e+1)$-design. Assume that the following* (1) *or* (2) *holds:*

(1) *e is even, $0 \in X$ and $|X| = 2\dim(\mathcal{P}_e^*(S)) - 1$.*

(2) *e is odd and $|X| = 2\dim(\mathcal{P}_e^*(S))$.*

Then X is antipodal and the weight function $w(\boldsymbol{x})$ is centrally symmetric.

We apply Möller's method further and obtain the following theorem.

Theorem 4.5. [9, Theorem 2.3.6] *Let X be a Euclidean $(2e+1)$-design. Assume e is even, $0 \notin X$ and the following conditions are satisfied:*

(1) *$|X| = 2\dim(\mathcal{P}_e^*(S))$.*

(2) *Let Y be a subset in X. Assume $\boldsymbol{x} \neq -\boldsymbol{y}$ for any $\boldsymbol{x}, \boldsymbol{y} \in Y \cap L$ and any line L passing through the origin. Then $|Y| \leqslant \frac{e}{2} + 1$.*

Then X is antipodal and the weight function $w(\boldsymbol{x})$ is centrally symmetric.

We define the tightness of Euclidean t-designs in the following way.

Definition 4.2. (tight t-designs on p concentric spheres) Let X be a Euclidean t-design. Let S be the union of p concentric spheres which supports X. If one of the following conditions holds, then X is a *tight* t-design on p concentric spheres:

(1) $t = 2e$ and $|X| = \dim(\mathcal{P}_e(S))$.

(2) $t = 2e+1$, e is even, $0 \in X$ and $|X| = 2\dim(\mathcal{P}_e^*(S)) - 1$.

(3) $t = 2e+1$, e is even and $0 \notin X$ or e is odd, and $|X| = 2\dim(\mathcal{P}_e^*(S))$.

Definition 4.3. (tight t-designs of $\mathbb{R}^n$) Let X be a tight t-design on p concentric spheres. If one of the following conditions holds, then X is a *tight* t-design of $\mathbb{R}^n$:

(1) $t = 2e$ and $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$.

(2) $t = 2e+1$ and $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$.

Theorem 4.1 implies that if $0 \in X \subset \mathbb{R}^n$, then X is a Euclidean t-design if and only if $X \backslash \{0\}$ is a Euclidean t-design. Therefore, if X is a tight t-design on p concentric spheres satisfying $0 \notin X$ and $p \leqslant [\frac{t}{4}]$, then $X \cup \{0\}$ is a tight t-design on $p+1$ concentric spheres. We also have the following propositions.

Proposition 4.1. [6, Proposition 1.7] *Let X be a tight $2e$-design of $\mathbb{R}^n$. If $0 \in X$, then e is even and $p = \frac{e}{2} + 1$.*

Proposition 4.2. [9, Proposition 2.4.5] *Let X be a tight $(2e+1)$-design of $\mathbb{R}^n$. If $0 \in X$, then e is even and $p = \frac{e}{2} + 1$.*

Proposition 4.3. [9, Proposition 2.4.6] *Let X be a tight $(2e+1)$-design on p concentric spheres. Then the following hold:*

(1) *If e is odd, then X is antipodal, $0 \notin X$, and the weight function w is centrally symmetric.*

(2) *If e is even and $0 \in X$, then X is antipodal and the weight function w is centrally symmetric.*

(3) *If e is even, $0 \notin X$, and $p \leqslant \frac{e}{2} + 1$ (exactly speaking if Theorem 4.5(2) holds), then X is antipodal and the weight function w is centrally symmetric.*

5 Tight Euclidean 7-Designs on Two Concentric Spheres

In this section, we prove the following theorem.

Theorem 5.1. *Any tight 7-design on two concentric spheres is similar to one of the 7-designs whose parameters are given in the following table:*

n	$\lvert X\rvert$	$\lvert X_1\rvert$	$\lvert X_2\rvert$	r_1	r_2	$A(X_1)$	$A(X_2)$	$A(X_1, X_2)$	w_1	w_2
2	12	6 (tight)	6 (tight)	1	r	$-1,\ \pm\frac{1}{2}$	$-r^2,\ \pm\frac{1}{2}r^2$	$0,\ \pm\frac{\sqrt{3}}{2}r$	1	$\frac{1}{r^6}$
4	48	24	24	1	r	$-1,\ 0,\ \pm\frac{1}{2}$	$-r^2,\ 0,\ \pm\frac{1}{2}r^2$	$0,\ \pm\frac{1}{\sqrt{2}}r$	1	$\frac{1}{r^6}$
7	182	56 (tight)	126	1	r	$-1,\ \pm\frac{1}{3}$	$-r^2,\ 0,\ \pm\frac{1}{2}r^2$	$0,\ \pm\frac{1}{\sqrt{3}}r$	1	$\frac{32}{27}\frac{1}{r^6}$

In the table given above, $r \neq 1$, and "tight" means that X_i is tight as a spherical 5-design. The precise structures of the designs will be given at the end of this section.

In Theorem 5.1, if $r = 1$ for $n = 2, 4$, then X is a spherical 7-design, however, it is not tight as a spherical 7-design. For $n = 7$ and $r = 1$, we obtain a non-constant weight spherical 7-design which is not tight as a Euclidean 7-design on a sphere.

In the rest of this section, we give the proof of Theorem 5.1.

Let $X = X_1 \cup X_2$ be a tight 7-design on two concentric spheres. Then Proposition 4.3 implies that X is antipodal and $0 \notin X$. Let X^* be any antipodal half of X, that is, $X = X^* \cup (-X^*)$ and $X^* \cap (-X^*) = \emptyset$. Then by definition, we have

$$|X^*| = \dim(\mathcal{P}_3^*(S)) = \sum_{l=0}^{1} \binom{n+3-2l-1}{3-2l} = \binom{n+2}{3} + \binom{n}{1} = \frac{n(n^2+3n+8)}{6}.$$

Let $X_i^* = X_i \cap X^*$. Then Lemma 1.7 in [17] implies that the weight function $w(\boldsymbol{x}) = w_i$ is constant on each X_i, X_i^* is at most a 3-distance set, and X_i is at most a 4-distance set ($i = 1, 2$). Also, Theorem 2.3 in [17] implies that X_1 and X_2 are both antipodal spherical 5-designs, we have $|X_i^*| \geqslant \binom{n+1}{2} = \frac{n(n+1)}{2}$. Let $N_i = |X_i^*|$. Then

$$\frac{n(n+1)}{2} \leqslant N_1, N_2 \leqslant \frac{n(n^2+3n+8)}{6}.$$

In [17], we introduced the following set of basis for $\mathcal{P}_3^*(S)$. Namely, we defined polynomials $g_{l,j}(\|\boldsymbol{x}\|^2)$ which is a linear combination of $1, \|\boldsymbol{x}\|^2, \ldots, \|\boldsymbol{x}\|^{2j}$ satisfying

$$\sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x})\, \|\boldsymbol{x}\|^{2l} g_{l,j}(\|\boldsymbol{x}\|^2)\, g_{l,j'}(\|\boldsymbol{x}\|^2) = \delta_{j,j'}.$$

Let $\varphi_{l,1}(\boldsymbol{x}), \ldots, \varphi_{l,h_l}(\boldsymbol{x})$ be an orthonormal basis of $\mathrm{Harm}_l(S) \cong \mathrm{Harm}_l(\mathbb{R}^n)$ with respect to the inner product

$$\langle \varphi, \psi \rangle = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \varphi(\boldsymbol{x})\psi(\boldsymbol{x}) d\sigma(\boldsymbol{x}),$$

where $h_l = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$. Let

$$\mathcal{H}_l = \{g_{3-2l,j}\varphi_{3-2l,i} \mid 0 \leqslant l \leqslant 1,\ 1 \leqslant i \leqslant h_{3-2l},\ 0 \leqslant j \leqslant \min\{1, l\}\}.$$

Then $\mathcal{H}_0 \cup \mathcal{H}_1$ is a basis of $\mathcal{P}_3^*(S)$. Let $Q_l(x)$ be the Gegenbauer polynomial of degree l associated to the unit sphere S^{n-1}. We use the normalization so that $Q_l(1) = h_l = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$ holds. We may assume $N_1 \leqslant N_2$, $w(\boldsymbol{x}) = w_1 = 1$ for $\boldsymbol{x} \in X_1$ and $r_1 = 1$. Let $R = r_2^2$. Then the equations (3.1) in [17] implies

$$g_{3,0}(1)^2 Q_3(1) + g_{1,0}(1)^2 Q_1(1) + g_{1,1}(1)^2 Q_1(1) = 1, \tag{5.1}$$

$$R^3 g_{3,0}(R)^2 Q_3(1) + R g_{1,0}(R)^2 Q_1(1) + R g_{1,1}(R)^2 Q_1(1) = 1/w_2, \tag{5.2}$$

and the equation (3.2) in [17] implies

$$g_{3,0}(1)^2 Q_3(\boldsymbol{x}\cdot\boldsymbol{y}) + g_{1,0}(1)^2 Q_1(\boldsymbol{x}\cdot\boldsymbol{y}) + g_{1,1}(1)^2 Q_1(\boldsymbol{x}\cdot\boldsymbol{y}) = 0 \tag{5.3}$$

for $\boldsymbol{x}, \boldsymbol{y} \in X_1^*$ with $\boldsymbol{x} \neq \boldsymbol{y}$,

$$R^3 g_{3,0}(R)^2 Q_3\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{R}\Big) + R g_{1,0}(R)^2 Q_1\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{R}\Big) + R g_{1,1}(R)^2 Q_1\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{R}\Big) = 0 \tag{5.4}$$

for $\boldsymbol{x}, \boldsymbol{y} \in X_2^*$ with $\boldsymbol{x} \neq \boldsymbol{y}$, and

$$\begin{aligned}\sqrt{R}^{\,3} g_{3,0}(1)\, g_{3,0}(R)\, Q_3\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\sqrt{R}}\Big) &+ \sqrt{R}\, g_{1,0}(1)\, g_{1,0}(R)\, Q_1\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\sqrt{R}}\Big)\\ &+ \sqrt{R}\, g_{1,1}(1)\, g_{1,1}(R)\, Q_1\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\sqrt{R}}\Big) = 0\end{aligned} \tag{5.5}$$

for $\boldsymbol{x} \in X_1^*$ and $\boldsymbol{y} \in X_2^*$.

Let $a_i = \sum_{\boldsymbol{x}\in X^*} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2i}$. Then using the formula given in [6], we obtain the following:

$$g_{0,0}(\|\boldsymbol{x}\|^2) = \frac{1}{\sqrt{a_0}} = \sqrt{\frac{6}{6N_1 + w_2(n^3+3n^2+8n-6N_1)}}, \tag{5.6}$$

$$g_{1,0}(\|\boldsymbol{x}\|^2) = \frac{1}{\sqrt{a_1}} = \sqrt{\frac{6}{6N_1 + (n^3+3n^2+8n-6N_1)w_2 R}}, \tag{5.7}$$

$$g_{3,0}(\|\boldsymbol{x}\|^2) = \frac{1}{\sqrt{a_3}} = \sqrt{\frac{6}{6N_1 + (n^3+3n^2+8n-6N_1)w_2 R^3}}, \tag{5.8}$$

$$\begin{aligned}g_{1,1}(\|\boldsymbol{x}\|^2) &= \sqrt{\frac{a_1}{a_1 a_3 - a_2^2}}\left(\|\boldsymbol{x}\|^2 - \frac{a_2}{a_1}\right)\\ &= \frac{6N_1(R-1)}{6N_1 + (n^3+3n^2+8n-6N_1)w_2 R}\\ &\quad\times\sqrt{\frac{6N_1 + (n^3+3n^2+8n-6N_1)w_2 R}{N_1 w_2 R (R-1)^2 (n^3+3n^2+8n-6N_1)}}.\end{aligned} \tag{5.9}$$

By substituting these formulas, (5.1) and (5.2) both imply

$$w_2(N_1-n)(6N_1-n^3-8n-3n^2)R^3-N_1(6N_1-2n-3n^2-n^3)=0.$$

Since $N_1=|X_1^*|\geqslant\frac{n(n+1)}{2}\geqslant n+1$, we obtain

$$w_2=\frac{N_1(6N_1-2n-3n^2-n^3)}{(N_1-n)(6N_1-n^3-8n-3n^2)R^3}. \tag{5.10}$$

Now (5.3), (5.4) and (5.5) imply

$$N_1(n+4)(n+2)\alpha^3 - \left(3(n+2)N_1+(6N_1-3n^2-8n-n^3)w_2R^3\right)\alpha=0, \tag{5.11}$$

$$w_2R(n+4)(n+2)(n^3+8n+3n^2-6N_1)\beta^3 + \left(3(n+2)(6N_1-n^3-8n-3n^2)w_2R^3+36N_1\right)\beta=0, \tag{5.12}$$

$$(n+2)\gamma^3-3R\gamma=0, \tag{5.13}$$

where $\alpha\in A(X_1^*)$, $\beta\in A(X_2^*)$ and $\gamma\in A(X_1^*,X_2^*)$. Then (5.10) with (5.11), (5.12) implies

$$(n+2)(N_1-n)\alpha^3+(2n-3N_1+n^2)\alpha=0, \tag{5.14}$$

$$(n+2)(6N_1-n^3-3n^2-2n)\beta^3 -3(6N_1-n^3-4n-n^2)R^2\beta=0. \tag{5.15}$$

Since $X = X_1\cup X_2$ is a tight 7-design on two concentric spheres, X is antipodal. Moreover, X_1 and X_2 are antipodal spherical 5-designs. Both are at most 4-distance sets. Hence, $A(X_1)\subseteq\{0,\pm\alpha,-1\}$, $A(X_2)\subseteq\{0,\pm\beta,-R\}$ and $A(X_1,X_2)\subset\{0,\pm\gamma,-\sqrt{R}\}$, where α is the positive solution of (5.14), β is the positive solution of (5.15), and γ is the positive solution of (5.13).

Proposition 5.1. *Definitions and notation are as given before. Then the following hold:*

(1) *If X_1 is a spherical tight 5-design and $n\geqslant 4$, then $\frac{1}{\alpha}$ and $\frac{R^2}{\beta^2}$ are integers.*

(2) *If X_1 is not a spherical tight 5-design, then $\frac{1}{\alpha^2}$ and $\frac{R^2}{\beta^2}$ are integers.*

Proof. (1) If X_1 is a tight spherical 5-design and $n\geqslant 4$, then $N_1=\frac{n(n+1)}{2}$ and X_1 is of degree 3, i.e., $A(X_1)=\{-1,\alpha,-\alpha\}$ and $\frac{1}{\alpha}$ is an integer (see [11, 12]). On the other hand,

$$N_2=\frac{|X|}{2}-N_1=\frac{n(n^2+3n+8)}{6}-\frac{n(n+1)}{2}=\frac{n(n^2+5)}{6}>\frac{n(n+1)}{2}$$

holds. Hence, X_2 is not a tight spherical 5-design. Therefore, X_2 is of degree 4 and $A(X_2)=\{-R,\beta,-\beta,0\}$. Then Proposition 8.1(1) implies that X_2^* is a strongly regular graph. If X_2^* is a conference graph, then Proposition 8.1(3) implies $N_2=n^2+n-1$. Hence, we must have $\frac{n(n^2+5)}{6}=n^2+n-1$. So $n=6$. However, there is no spherical tight 5-design in $\mathbb{R}^6$. This contradicts the assumption that X_1 is a tight spherical 5-design. Hence, X_2^* is not a conference graph. Therefore, Proposition 8.1(2) implies that $\frac{R^2}{\beta^2}$ is an integer.

(2) In this case, both X_1 and X_2 are of degree 4. Therefore, X_1^* and X_2^* are strongly regular graphs. If X_1^* is a conference graph, then Proposition 8.1(3) implies that $|X_1^*| = N_1 = n^2+n-1$ and $\alpha^2 = \frac{2(n+1)}{(n+2)n}$. On the other hand, (5.14) implies

$$\alpha^2 = \frac{3N_1 - n^2 - 2n}{(n+2)(N_1-n)} = \frac{2n+3}{(n+1)(n+2)}.$$

This is a contradiction. Hence, X_1^* is not a conference graph. If X_2^* is a conference graph, then $N_2 = n^2+n-1$ and $\frac{\beta^2}{R^2} = \frac{2(n+1)}{(n+2)n}$. Thus,

$$N_1 = \frac{n(n^2+3n+8)}{6} - N_2 = \frac{(n+1)(n^2-4n+6)}{6}.$$

Hence, (5.15) implies $\frac{\beta^2}{R^2} = \frac{3(n^2-4n-6)}{(n^2-2n-6)(n+2)}$. This is impossible. Therefore, X_1^* and X_2^* are not conference graphs. Then Proposition 8.1(2) implies that $\frac{1}{\alpha^2}$ and $\frac{R^2}{\beta^2}$ are integers. □

Proposition 5.2. *Notations are given as before.*

(1) *If* $N_1 = \frac{n(n+1)}{2}$ *and* $n \geqslant 4$, *then* $\sqrt{n+2}$ *and* $\frac{(n+2)(n+1)}{3(n-1)}$ *are integers.*

(2) *If* $N_1 > \frac{n(n+1)}{2}$, *then* $\frac{(n+2)(N_1-n)}{3N_1-n^2-2n}$ *and* $\frac{(6N_1-n^3-3n^2-2n)(n+2)}{3(6N_1-n^3-4n-n^2)}$ *are integers.*

Proof. Proposition 5.1, (5.14) and (5.15) imply the proposition. □

Proposition 5.3. *Notations are given as before.*

(1) *Assume* $n \geqslant 4$. *Then* $\sqrt{n+2}$ *and* $\frac{(n+2)(n+1)}{3(n-1)}$ *are both integers if and only if* $n = 7$.

(2) *Assume* $\frac{n(n+1)}{2} < N_1 \leqslant \frac{1}{12}n(n^2+3n+8)$. *Then*

$$\frac{(n+2)(N_1-n)}{3N_1-n^2-2n} \quad \text{and} \quad \frac{(6N_1-n^3-3n^2-2n)(n+2)}{3(6N_1-n^3-4n-n^2)}$$

are both integers if and only if $n = 4, 8, 10, 16$ *with the values of* N_1 *listed below:*

n	4	8	10	16
N_1	12	64	70	256

Proof. (1) We have

$$\frac{(n+2)(n+1)}{3(n-1)} = \frac{n+4}{3} + \frac{2}{n-1}.$$

If $n \geqslant 8$, then $\frac{2}{n-1} \leqslant \frac{2}{7} < \frac{1}{3}$. Therefore, $\frac{(n+2)(n+1)}{3(n-1)}$ cannot be an integer. Since $n+2$ must be the square of an integer and $n \geqslant 4$, we must have $n = 7$.