Similarity Methods for Differential Equations

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PREFACE

The aim of this book is to provide a systematic and practical account of methods of integration of ordinary and partial differential equations based on invariance under continuous (Lie) groups of transformations. The goal of these methods is the expression of a solution in terms of quadrature in the case of ordinary differential equations of first order and a reduction in order for higher order equations. For partial differential equations at least a reduction in the number of independent variables is sought and in favorable cases a reduction to ordinary differential equations with special solutions or quadrature.

In the last century, approximately one hundred years ago, Sophus Lie tried to construct a general integration theory, in the above sense, for ordinary differential equations. Following Abel's approach for algebraic equations he studied the invariance of ordinary differential equations under transformations. In particular, Lie introduced the study of continuous groups of transformations of ordinary differential equations, based on the infinitesimal properties of the group. In a sense the theory was completely successful. It was shown how for a first-order differential equation the knowledge of a group leads immediately to quadrature, and for a higher order equation (or system) to a reduction in order. In another sense this theory is somewhat disappointing in that for a first-order differential equation essentially no systematic way can be given for finding the groups or showing that they do not exist for a first-order differential equation.

Lie also investigated thoroughly first-order partial differential equations which are essentially equivalent to systems of first-order ordinary differential equations by the theory of characteristics. He also made a preliminary investigation of some second-order equations, for example, the heat equation, but did not develop the integration theory. During the last century these methods have been developed by various mathematicians, engineers, and physicists. A summary of the mathematical approach based on infinitesimal transformations is given in a recent Russian book, Group Properties of Differential Equations, by L. Ovsjannikov.

In the first part of this book the material on ordinary differential equations is reproduced in detail. A typical result is (1.14-2): the criterion that a first-order equation

$$\Omega(x,y,y') = 0$$

admit a given group, defined by infinitesimal generators $\{\xi(x,y),\ \eta(x,y)\}$ is that $\xi\frac{\partial\Omega}{\partial x} + \eta\frac{\partial\Omega}{\partial y} + \eta'\frac{\partial\Omega}{\partial(y')} = 0$ on $\Omega = 0$, for all (x,y) where

$$\eta' \equiv \frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right)y' - \frac{\partial\xi}{\partial y}y'^2 .$$

From a knowledge of the group the (general) solution can be expressed by quadrature.

In the second part of the book this method is extended to partial differential equations and several other connections are made. First the concept of the extended infinitesimal transformation is developed for several variables. It is next shown how invariance under a group can be used to reduce the number of independent variables. The resulting solutions are connected with the usual "similarity solutions" of partial differential equations. Since these solutions are sometimes obtained by physical dimensional analysis §2.5 discusses the connection between transformation theory and that method. A principal example which is treated here is the construction of the general similarity solution of the heat equation. Other examples, including some with non-linearities and some with boundary conditions are sketched but, of course, no complete catalog can be given.

In view of the pioneering work of Sophus Lie in pointing out the importance and use of infinitesimal transformations, the authors would respectfully like to dedicate this book to his memory.

The authors are indebted to F. Milinazzo for carefully reading drafts of the manuscript and suggesting numerous corrections and clarifications. Special thanks are due to Mrs. Vivian Davies and Mrs. Yit-Sin Choo for patiently deciphering the authors' handwriting in typing several drafts of the manuscript. The authors appreciate the technical assistance of the staff at Brown University.

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INTRODUCTION

The work presented here falls naturally into two parts; the aims and partial contents of these parts is now sketched for purposes of orientation.

1. Ordinary Differential Equations

For ordinary differential equations the aim is a theory of integration or reduction to quadratures. For first-order equations this means that special cases can be reduced to essentially the same case, one of quadrature. The canonical case of quadrature occurs when a variable is missing. If the general equation is

$$F\left(x, y, \frac{dy}{dx} \right) = 0 \tag{1}$$

the special case is

$$F\left(x, \frac{dy}{dx} \right) = 0. \tag{2}$$

For one of the (possibly many) branches of (2) we can write

$$\frac{dy}{dx} = G(x), \tag{3}$$

and the general solution is

$$y = \int_{x_0}^{x} G(\zeta)d\zeta + \alpha, \quad \alpha = \text{const.} \tag{4}$$

In the sense represented by (3) and (4) the problem is regarded as solved or reduced to quadrature. In this sense the theory accomplishes all that can reasonably be expected of it.

For higher order equations or systems the aim is the reduction of the problem to lower order plus a suitable number of quadratures, and this can be carried out for a definite class of problems.

2. Partial Differential Equations

For first-order partial differential equations we take the (restricted) point of view that a sufficiently complete integration theory is given by the theory of characteristics. This connects the solution of partial differential equations with the integration of systems of ordinary differential equations and hence with results of part 1. It may, however, be useful to look at some first order equations directly from the point of view of transformations and invariance.

For higher order equations or systems the aim is a reduction in the number of variables. A typical result is the statement that a solution $u(x,y,t)$ of a particular P.D.E. in three independent variables must be representable as

$$u(x,y,t) = \frac{1}{t} F\left(\frac{x}{t} , \frac{y}{t} \right). \tag{5}$$

This procedure can possibly be repeated more than one time. The special case when a partial differential equation contains only two independent variables is particularly important since the problem is reduced to an ordinary differential equation. Further, the methods of part 1 may be applied. In many physical problems of interest the resulting equation which needs to be studied (together with a suitable number of quadratures) is of first order. In this favorable case the structure of all possible solutions in the phase plane provides complete information on the structure of a class of solutions to the original partial differential equation. It also may provide the basis for a method of numerical integration.

Another method, which can be used to obtain the same results in special cases, arises not directly from transformation theory but rather from dimensional analysis. The basic idea is that all physical problems must be expressible in dimensionless variables. This idea is applied to the variables entering a problem for a partial differential equation. For example if (x,y), some independent variables, which are space coordinates with the physical dimensions of "length", enter the problem, then it can be concluded that only the combination (x/y) (or equivalent) can enter the problem. Evidently, this represents a reduction in the number of variables. However, it should be remarked that the failure of dimensional analysis to predict similarity (i.e., a reduction in the number of essential variables) does not necessarily rule out similarity for the problem. The connection between dimensional analysis and similarity is discussed later.

What has been outlined above is the main content of this book. However, various related topics which enter naturally are discussed as the opportunity arises. Among these are asymptotic and local behavior, superposition of similarity solutions for linear cases.

Finally, it should be remarked that the methods used apply equally well to non-linear and linear cases. The ideas used represent one of the few systematic methods of attacking non-linear problems, with an eye to obtaining exact solutions.

1. ORDINARY DIFFERENTIAL EQUATIONS

1.0. Ordinary Differential Equations

The essential ideas of the method occur for first-order equations and these are discussed first. For first-order equations of first degree, which form the main subject matter of the first part of this book, the difference between the case when a variable is missing in the right hand side and the general case should be noted:

$$\frac{dy}{dt} = F(x,y) \qquad \text{general,} \qquad (1.0\text{-}1)$$

$$\frac{dy}{dx} = F(x) \qquad y \text{ missing.} \qquad (1.0\text{-}2)$$

In the general case the complete integration is represented by all the integral curves in the (x,y) plane, one curve passing through each nonsingular point (Fig. 1.0-1) according to the local direction field at each point P; these form ∞^1 (number of) curves.[1] Their construction demands in general ∞^1 integrations.

In the special case (1.0-2), again all the curves are needed for the complete solution. However, the complete solution, representing all the integral curves, is given indirectly by integration

$$y = \int_{x_0}^{x} F(\zeta)d\zeta + \alpha, \qquad \alpha = \text{const.} \qquad (1.0\text{-}3)$$

Thus, essentially only one integration is needed; the problem is one of quadrature. This fact is reflected in the geometric properties of Fig. (1.0-2). The slope of each integral curve is the same at a fixed value of x. The integral curves are thus congruent and a

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[1] Old fashioned notation: ∞^1 = single infinity of curves (characterized by all continuous values of one parameter).

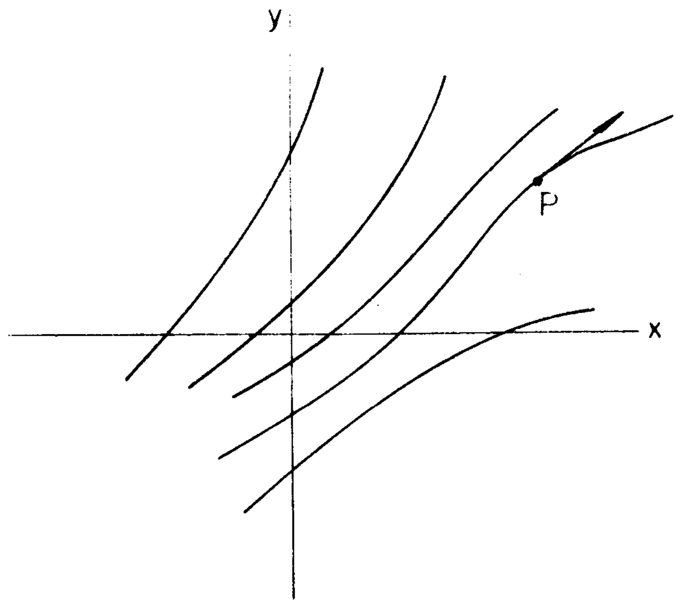

Figure 1.0-1

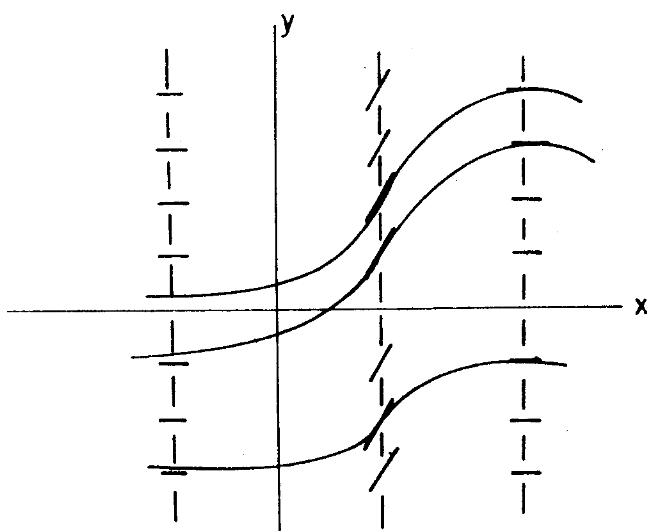

Figure 1.0-2

translation in the y-direction brings one into another. Thus, we can summarize the special properties of this case:

under the transformation $y \to y + \beta$:

- integral curves $\to$ integral curves (1.0-4)
- the differential equation (1.0-2) is invariant

The reduction to quadrature is the aim of the transformation theory for these first-order equations. According to the above remarks we might expect that invariance under transformation is the basic property which allows a reduction to quadrature. That this is so is illustrated by the special example of the next seotion. It is in fact possible to connect all transformations and invariances with that of (1.0-4).

1.1. Example: Global Similarity Transformation, Invariance and Reduction to Quadrature

This section demonstrates, in a special case, how invariance under a transformation can be used to reduce a problem to quadrature. Consider

$$\frac{dy}{dx} = F(x,y) \tag{1.1-1}$$

$F(x,y)$ is at first arbitrary but will soon be restricted by transformation requirements. Assume that the differential equation is invariant under the special transformation

$$\left.\begin{aligned} x^* &= \alpha x \\ y^* &= \beta y \end{aligned}\right\} \quad 0 < \alpha,\ \beta < \infty \tag{1.1-2}$$

(α,β)[1] are the parameters of the transformation. We consider the

[1] Greek letters will be used to denote parameters, as far as possible.

transformation of the original space (x,y) to an image space (x^*,y^*); this can also be thought of as the mapping of the plane into itself. The transformation assigns an image point $P^*(x^*,y^*)$ to each point $P(x,y)$ in the plane and vice-versa. The special transformation (1.1-2) is a stretching or similitudinous transformation. $\alpha = \beta = 1$ is the identity which is included in all transformations. A direction field at P^* is also assigned by the transformation of the differential equation

$$\frac{dy^*}{dx^*} = \frac{\beta}{\alpha} F\left(\frac{x^*}{\alpha} , \frac{y^*}{\beta} \right). \qquad (1.1\text{-}3)$$

To the integral curve of (1.1-1) through P corresponds an integral curve of (1.1-3) through P^*.

Now we can define invariance precisely: the differential equation (1.1-1) is said to be invariant under the transformation (1.1-2) when the differential equation reads the same in the new coordinates. That is, the right hand side of (1.1-3) is equal to $F(x^*,y^*)$.

$$\frac{\beta}{\alpha} F\left(\frac{x^*}{\alpha} , \frac{y^*}{\beta} \right) = F(x^*,y^*) \quad \text{for invariance.} \qquad (1.1\text{-}4)$$

We will assume that F is such that (1.1-1) is invariant.

Before considering the restrictions of $F(x,y)$ let us consider some other consequences of invariance. Consider a definite integral curve

$$y = f(x) \qquad (1.1\text{-}5)$$

in the original space; that is, $f(x)$ is such that $f'(x) = F(x,f(x))$ for some range of x. The fact that the equation is invariant implies that the same curve in the (x^*,y^*) space is an integral curve of (1.1-3)

$$y^* = f(x^*) . \tag{1.1-6}$$

But each integral curve in the "star" space is the image of an integral curve in the original space. Upon transforming (1.1-6) back to the original (x,y) space we have, as integral curves (for various α, β)

$$y = \frac{1}{\beta} f(\alpha x) . \tag{1.1-7}$$

Thus, as a consequence of invariance, we can say that any integral curve in the original space such as (1.1-5) is a member of some family, such as (1.1-7). The identity member of this family has $\alpha = \beta = 1$. With the aid of (1.1-7) we can actually find the integral curve passing through any point of the plane. Thus, essentially only one integral curve needs to be calculated; the problem should be reducible to quadrature.

A procedure for doing this is now indicated. First, we find the form of $F(x,y)$. Two cases need to be considered:

(i) α, β independent parameters (the differential equation is invariant under a two parameter group); rewrite the invariance condition (1.1-4) as

$$\beta F(x,y) = \alpha F(\alpha x, \beta y) . \tag{1.1-8}$$

Then $\partial/\partial\alpha$ implies

$$0 = F(\alpha x, \beta y) + \alpha x \frac{\partial F}{\partial (1)} (\alpha x, \beta y)$$

or (1.1-9)[1]

$$0 = F(x^*, y^*) + x^* \frac{\partial F}{\partial x^*} (x^*, y^*) .$$

[1] The notation $\partial/\partial(1)$ means the partial derivative with respect to the first argument of the function; $\partial/\partial(2)$ denotes the partial derivative with respect to the second argument, etc.

Direct integration yields

$$F(x^*,y^*) = \frac{g(y^*)}{x^*} . \tag{1.1-10}$$

Thus, the basic functional equation (1.1-8) becomes

$$\beta g(y) = g(\beta y). \tag{1.1-11}$$

$\partial/\partial\beta$ of this functional equation yields

$$g(y) = yg'(\beta y) = \frac{1}{\beta} g(\beta y)$$

or (1.1-12)

$$\frac{g'(y^*)}{g(y^*)} = \frac{1}{y^*} .$$

The solution of (1.1-12) is

$$g(y^*) = by^* \qquad b = \text{const.} \tag{1.1-13}$$

and the resulting functional form of F is

$$F(x,y) = b\,\frac{y}{x} . \tag{1.1-14}$$

For this special differential equation

$$\frac{dy}{dx} = b\,\frac{y}{x}$$

a separation of variables provides the reduction to quadrature.

(ii) $\beta = \beta(\alpha)$ (the differential equation is invariant under a one parameter group).

The functional form of the dependence $\beta(\alpha)$ is not arbitrary but must be found in the course of finding the functional form of F. The basic functional equation (1.1-4) is now

$$\beta(\alpha)F(x,y) = \alpha F(\alpha x,\beta(\alpha)y) \tag{1.1-15}$$