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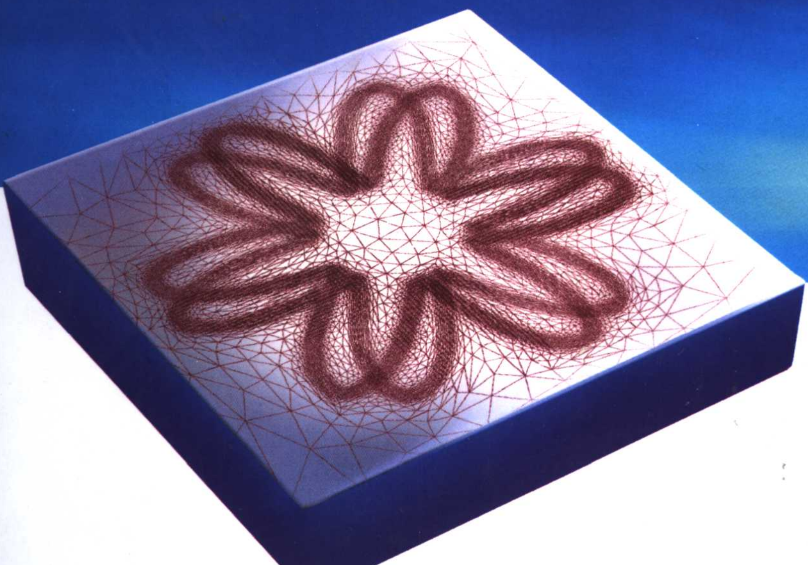
Differential Geometry: Theory and Applications

Philippe G Ciarlet
Ta-Tsien Li
editors



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Preface

The ISFMA-CIMPA School on "Differential Geometry: Theory and Applications" was held on 07 August – 18 August 2006, in the building of the Chinese-French Institute for Applied Mathematics (ISFMA), Fudan University, Shanghai, China. This school was jointly organized by the ISFMA and the CIMPA (International Centre for Pure and Applied Mathematics), Nice, France. About sixty participants from China, Hong Kong, France, Cambodia, India, Iran, Pakistan, Philippines, Romania, Russia, Sri-Lanka, Thailand, Turkey, Uzbekistan and Vietnam attended this highly successful event.

The first objective of this school was to lay down in a self-contained and accessible manner the basic notions of differential geometry, such as the metric tensor, the Riemann curvature tensor, the fundamental forms of a surface, covariant derivatives, and the fundamental theorem of surface theory etc. Although this field is with good reasons often considered as a "classical" one, it has been recently "rejuvenated", thanks to the manifold applications where it plays an essential role.

The second objective of this school was to present some of these applications, such as the theory of linearly and nonlinearly elastic shells, the implementation of numerical methods for shells, and mesh generation in finite element methods.

To fulfill these objectives, four series of lectures, each series comprising ten 50min-lectures, were delivered under the following titles: "Introduction to differential geometry", "Introduction to shell theory", "A differential geometry approach to mesh generation", and "Numerical methods for shells". This volume gathers the materials covered in these lectures. As such, this volume should be very useful to graduate students and researchers in pure and applied mathematics.

The organizers take pleasure in thanking the various organizations for their generous support: The ISFMA, the CIMPA, the French Embassy in Beijing, the Consulate General of France in Shanghai, the National Natural Science Foundation of China, Fudan University, Higher Education Press and World Scientific. Finally, our special thanks are due to Mrs. Zhou Chun-Lian for her patient and effective work in editing this book.

Philippe G. Ciarlet and Ta-Tsien Li
February 2007

Contents

An Introduction to Differential Geometry in $\mathbb{R}^3$

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Introduction

These notes[1] are intended to give a thorough introduction to the basic theorems of differential geometry in $\mathbb{R}^3$, with a special emphasis on those used in applications.

The treatment is essentially self-contained and proofs are complete. The prerequisites essentially consist in a working knowledge of basic notions of analysis and functional analysis, such as differential calculus, integration theory and Sobolev spaces, and some familiarity with ordinary and partial differential equations.

In Part 1, we review the basic notions, such as the metric tensor and covariant derivatives, arising when a three-dimensional open set is equipped with curvilinear coordinates. We then prove that the vanishing of the Riemann curvature tensor is sufficient for the existence of isometric immersions from a simply-connected open subset of $\mathbb{R}^3$ equipped with a Riemannian metric into a three-dimensional Euclidean space. We also prove the corresponding uniqueness theorem, also called rigidity theorem.

In Part 2, we study basic notions about surfaces, such as their two fundamental forms, the Gaussian curvature, and covariant derivatives. We then prove the fundamental theorem of surface theory, which asserts that the Gauss and Codazzi-Mainardi equations constitute sufficient conditions for two matrix fields defined in a simply-connected open subset of $\mathbb{R}^2$ to be the two fundamental forms of a surface in a three-dimensional Euclidean space. We also prove the corresponding rigidity theorem.

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[1] With the kind permission of Springer-Verlag, these notes are extracted and adapted from my book "An Introduction to Differential Geometry with Applications to Elasticity", Springer, Dordrecht, 2005, the writing of which was substantially supported by two grants from the Research Grants Council of Hong Kong Special Administrative Region, China [Project No. 9040869, CityU 100803 and Project No. 9040966, CityU 100604].

1 Three-dimensional differential geometry

Outline

Let Ω be an open subset of $\mathbb{R}^3$, let $\mathbf{E}^3$ denote a three-dimensional Euclidean space, and let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be a smooth injective immersion. We begin by reviewing (Sections 1.1 to 1.3) basic definitions and properties arising when the three-dimensional open subset $\boldsymbol{\Theta}(\Omega)$ of $\mathbf{E}^3$ is equipped with the coordinates of the points of Ω as its *curvilinear coordinates.*

Of fundamental importance is the *metric tensor* of the set $\boldsymbol{\Theta}(\Omega)$, whose covariant and contravariant components $g_{ij} = g_{ji} : \Omega \to \mathbb{R}$ and $g^{ij} = g^{ji} : \Omega \to \mathbb{R}$ are given by (Latin indices or exponents take their values in $\{1,2,3\}$):

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \text{ and } g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \text{ where } \boldsymbol{g}_i = \partial_i \boldsymbol{\Theta} \text{ and } \boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j.$$

The vector fields $\boldsymbol{g}_i : \Omega \to \mathbb{R}^3$ and $\boldsymbol{g}^j : \Omega \to \mathbb{R}^3$ respectively form the *covariant*, and *contravariant, bases* in the set $\boldsymbol{\Theta}(\Omega)$.

It is shown in particular how *volumes*, *areas*, and *lengths*, in the set $\boldsymbol{\Theta}(\Omega)$ are computed in terms of its curvilinear coordinates, by means of the functions g_{ij} and g^{ij} (Theorem 1.3-1).

We next introduce in Section 1.4 the fundamental notion of *covariant derivatives* $v_{i\|j}$ *of a vector field* $v_i \boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ defined by means of its covariant components v_i over the contravariant bases $\boldsymbol{g}^i$. Covariant derivatives constitute a generalization of the usual partial derivatives of vector fields defined by means of their Cartesian components. In particular, covariant derivatives naturally appear when a system of partial differential equations with a vector field as the unknown, e.g., the displacement field in elasticity, is expressed in terms of curvilinear coordinates.

It is a basic fact that the symmetric and positive-definite matrix field (g_{ij}) defined on Ω in this fashion cannot be arbitrary. More specifically (Theorem 1.5-1), its components and some of their partial derivatives must satisfy *necessary conditions* that take the form of the following relations (meant to hold for all $i,j,k,q \in \{1,2,3\}$): Let the functions Γ_{ijq} and Γ_{ij}^p be defined by

$$\Gamma_{ijq} = \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \text{ and } \Gamma_{ij}^p = g^{pq}\Gamma_{ijq}, \text{ where } (g^{pq}) = (g_{ij})^{-1}.$$

Then, necessarily,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0 \text{ in } \Omega.$$

The functions Γ_{ijq} and Γ^p_{ij} are the *Christoffel symbols of the first, and second, kind* and the functions

$$R_{qijk} = \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma^p_{ij}\Gamma_{kqp} - \Gamma^p_{ik}\Gamma_{jqp}$$

are the covariant components of the *Riemann curvature tensor* of the set $\boldsymbol{\Theta}(\Omega)$.

We then focus our attention on the reciprocal questions:

Given an open subset Ω of $\mathbb{R}^3$ and a smooth enough symmetric and positive-definite matrix field (g_{ij}) defined on Ω, when is it the metric tensor field of an open set $\boldsymbol{\Theta}(\Omega) \subset \mathbf{E}^3$, i.e., when does there exist an immersion $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ such that $g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$ in Ω?

If such an immersion exists, to what extent is it unique?

As shown in Theorems 1.6-1 and 1.7-1, the answers turn out to be remarkably simple to state (but not so simple to prove, especially the first one!): *Under the assumption that Ω is simply-connected, the necessary conditions*

$$R_{qijk} = 0 \text{ in } \Omega$$

are also sufficient for the existence of such an immersion $\boldsymbol{\Theta}$.

Besides, *if Ω is connected, this immersion is unique up to isometries of* $\mathbf{E}^3$. This means that, if $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbf{E}^3$ is any other smooth immersion satisfying

$$g_{ij} = \partial_i \widetilde{\boldsymbol{\Theta}} \cdot \partial_j \widetilde{\boldsymbol{\Theta}} \text{ in } \Omega,$$

there then exist a vector $\boldsymbol{c} \in \mathbf{E}^3$ and an orthogonal matrix $\mathbf{Q}$ of order three such that

$$\boldsymbol{\Theta}(x) = \boldsymbol{c} + \mathbf{Q}\widetilde{\boldsymbol{\Theta}}(x) \text{ for all } x \in \Omega.$$

Together, the above existence and uniqueness theorems constitute an important special case of the *fundamental theorem of Riemannian geometry* and as such, constitute the core of Part 1.

We conclude this chapter by indicating in Section 1.8 that *the equivalence class of* $\boldsymbol{\Theta}$, *defined in this fashion* modulo *isometries of* $\mathbf{E}^3$, *depends continuously on the matrix field* (g_{ij}) *with respect to various topologies.*

1.1 Curvilinear coordinates

To begin with, we list some notations and conventions that will be consistently used throughout.

All spaces, matrices, etc., considered here are *real.*

Latin indices and exponents range in the set $\{1, 2, 3\}$, save when otherwise indicated, e.g., when they are used for indexing sequences, and the summation convention with respect to repeated indices or exponents is systematically used in conjunction with this rule. For instance, the relation

$$\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x)$$

means that

$$\boldsymbol{g}_i(x) = \sum_{j=1}^{3} g_{ij}(x)\boldsymbol{g}^j(x) \text{ for } i = 1, 2, 3.$$

Kronecker's symbols are designated by δ_i^j, δ_{ij}, or δ^{ij} according to the context.

Let $\mathbf{E}^3$ denote a *three-dimensional Euclidean space*, let $\boldsymbol{a}\cdot\boldsymbol{b}$ and $\boldsymbol{a}\wedge\boldsymbol{b}$ denote the Euclidean inner product and exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{E}^3$, and let $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ denote the Euclidean norm of $\boldsymbol{a} \in \mathbf{E}^3$. The space $\mathbf{E}^3$ is endowed with an *orthonormal basis* consisting of three vectors $\widehat{\boldsymbol{e}}^i = \widehat{\boldsymbol{e}}_i$. Let $\widehat{x}_i$ denote the *Cartesian coordinates* of a point $\widehat{x} \in \mathbf{E}^3$ and let $\widehat{\partial}_i := \partial/\partial\widehat{x}_i$.

In addition, let there be given a *three-dimensional vector space* in which three vectors $\boldsymbol{e}^i = \boldsymbol{e}_i$ form a basis. *This space will be identified with* $\mathbb{R}^3$. Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

Let there be given an *open* subset $\widehat{\Omega}$ of $\mathbf{E}^3$ and assume that there exist an *open* subset Ω of $\mathbb{R}^3$ and an *injective* mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ such

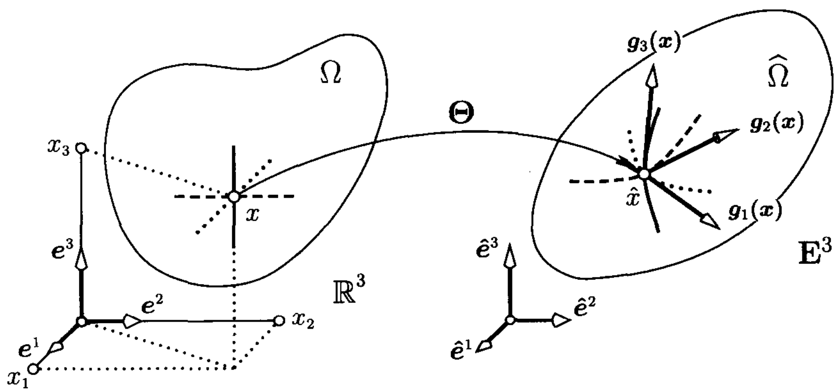

Figure 1.1-1: *Curvilinear coordinates and covariant bases in an open set* $\widehat{\Omega} \subset \mathbf{E}^3$. The three coordinates x_1, x_2, x_3 of $x \in \Omega$ are the curvilinear coordinates of $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$. If the three vectors $\boldsymbol{g}_i(x) = \partial_i\boldsymbol{\Theta}(x)$ are linearly independent, they form the covariant basis at $\widehat{x} = \boldsymbol{\Theta}(x)$ and they are tangent to the coordinate lines passing through $\widehat{x}$.

that $\boldsymbol{\Theta}(\Omega) = \widehat{\Omega}$. Then each point $\widehat{x} \in \widehat{\Omega}$ can be unambiguously written as

$$\widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega,$$

and the three coordinates x_i of x are called the **curvilinear coordinates** of $\widehat{x}$ (Figure 1.1-1). Naturally, *there are infinitely many ways of defining curvilinear coordinates in a given open set* $\widehat{\Omega}$, depending on how the open set Ω and the mapping $\boldsymbol{\Theta}$ are chosen!

Examples of curvilinear coordinates include the well-known *cylindrical* and *spherical coordinates* (Figure 1.1-2).

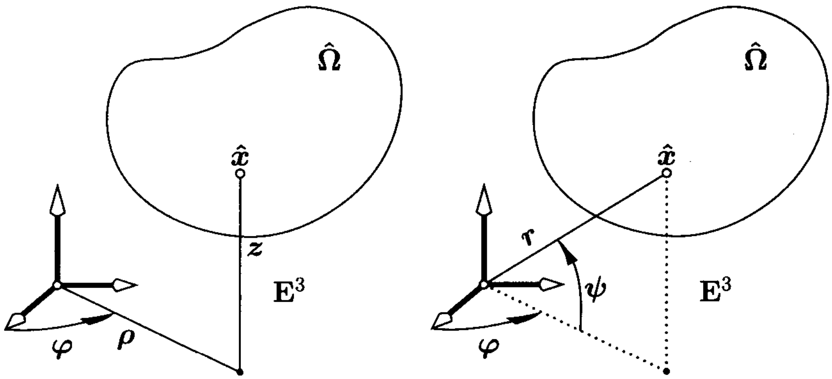

Figure 1.1-2: *Two familiar examples of curvilinear coordinates.* Let the mapping $\boldsymbol{\Theta}$ be defined by

$$\boldsymbol{\Theta} : (\varphi, \rho, z) \in \Omega \to (\rho\cos\varphi, \rho\sin\varphi, z) \in \mathbf{E}^3.$$

Then (φ, ρ, z) are the cylindrical coordinates of $\widehat{x} = \boldsymbol{\Theta}(\varphi, \rho, z)$. Note that $(\varphi + 2k\pi, \rho, z)$ or $(\varphi + \pi + 2k\pi, -\rho, z)$, $k \in \mathbb{Z}$, are also cylindrical coordinates of the same point $\widehat{x}$ and that φ is not defined if $\widehat{x}$ is the origin of $\mathbf{E}^3$.

Let the mapping $\boldsymbol{\Theta}$ be defined by

$$\boldsymbol{\Theta} : (\varphi, \psi, r) \in \Omega \to (r\cos\psi\cos\varphi, r\cos\psi\sin\varphi, r\sin\psi) \in \mathbf{E}^3.$$

Then (φ, ψ, r) are the spherical coordinates of $\widehat{x} = \boldsymbol{\Theta}(\varphi, \psi, r)$. Note that $(\varphi + 2k\pi, \psi + 2\ell\pi, r)$ or $(\varphi + 2k\pi, \psi + \pi + 2\ell\pi, -r)$ are also spherical coordinates of the same point $\widehat{x}$ and that φ and ψ are not defined if $\widehat{x}$ is the origin of $\mathbf{E}^3$.

In a different, but equally important, approach, an open subset Ω of $\mathbb{R}^3$ together with a mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ are instead *a priori* given.

If $\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbf{E}^3)$ and $\boldsymbol{\Theta}$ is injective, the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is *open* by the *invariance of domain theorem* (for a proof, see, e.g., Nirenberg [1974, Corollary 2, p. 17] or Zeidler [1986, Section 16.4]), and curvilinear coordinates inside $\widehat{\Omega}$ are unambiguously defined in this case.

If $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbf{E}^3)$ and the three vectors $\partial_i \boldsymbol{\Theta}(x)$ are linearly independent at all $x \in \Omega$, the set $\widehat{\Omega}$ is again *open* (for a proof, see, e.g., Schwartz [1992] or Zeidler [1986, Section 16.4]), but curvilinear coordinates may be defined only locally in this case: Given $x \in \Omega$, all that can be asserted (by the local inversion theorem) is the existence of an open neighborhood

V of x in Ω such that the restriction of $\boldsymbol{\Theta}$ to V is a $\mathcal{C}^1$-diffeomorphism, hence an injection, of V onto $\boldsymbol{\Theta}(V)$.

1.2 Metric tensor

Let Ω be an open subset of $\mathbb{R}^3$ and let

$$\boldsymbol{\Theta} = \Theta_i \widehat{\boldsymbol{e}}^i : \Omega \to \mathbf{E}^3$$

be a mapping that is *differentiable at a point* $x \in \Omega$. If $\boldsymbol{\delta x}$ is such that $(x + \boldsymbol{\delta x}) \in \Omega$, then

$$\boldsymbol{\Theta}(x + \boldsymbol{\delta x}) = \boldsymbol{\Theta}(x) + \boldsymbol{\nabla\Theta}(x)\boldsymbol{\delta x} + o(\boldsymbol{\delta x}),$$

where the 3×3 matrix $\boldsymbol{\nabla\Theta}(x)$ and the column vector $\boldsymbol{\delta x}$ are defined by

$$\boldsymbol{\nabla\Theta}(x) := \begin{pmatrix} \partial_1\Theta_1 & \partial_2\Theta_1 & \partial_3\Theta_1 \\ \partial_1\Theta_2 & \partial_2\Theta_2 & \partial_3\Theta_2 \\ \partial_1\Theta_3 & \partial_2\Theta_3 & \partial_3\Theta_3 \end{pmatrix}(x) \text{ and } \boldsymbol{\delta x} = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}.$$

Let the three vectors $\boldsymbol{g}_i(x) \in \mathbb{R}^3$ be defined by

$$\boldsymbol{g}_i(x) := \partial_i\boldsymbol{\Theta}(x) = \begin{pmatrix} \partial_i\Theta_1 \\ \partial_i\Theta_2 \\ \partial_i\Theta_3 \end{pmatrix}(x),$$

i.e., $\boldsymbol{g}_i(x)$ *is the* i-th *column vector of the matrix* $\boldsymbol{\nabla\Theta}(x)$. Then the expansion of $\boldsymbol{\Theta}$ about x may be also written as

$$\boldsymbol{\Theta}(x + \boldsymbol{\delta x}) = \boldsymbol{\Theta}(x) + \delta x^i \boldsymbol{g}_i(x) + o(\boldsymbol{\delta x}).$$

If in particular $\boldsymbol{\delta x}$ is of the form $\boldsymbol{\delta x} = \delta t \boldsymbol{e}_i$, where $\delta t \in \mathbb{R}$ and $\boldsymbol{e}_i$ is one of the basis vectors in $\mathbb{R}^3$, this relation reduces to

$$\boldsymbol{\Theta}(x + \delta t \boldsymbol{e}_i) = \boldsymbol{\Theta}(x) + \delta t \boldsymbol{g}_i(x) + o(\delta t).$$

A mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an **immersion at** $x \in \Omega$ if it is differentiable at x and the matrix $\boldsymbol{\nabla\Theta}(x)$ is invertible or, equivalently, if the three vectors $\boldsymbol{g}_i(x) = \partial_i\boldsymbol{\Theta}(x)$ are linearly independent.

Assume that *the mapping* $\boldsymbol{\Theta}$ *is an immersion at* x. Then *the three vectors* $\boldsymbol{g}_i(x)$ constitute the **covariant basis** at the point $\widehat{x} = \boldsymbol{\Theta}(x)$.

In this case, the last relation thus shows that *each vector* $\boldsymbol{g}_i(x)$ *is tangent to the* i-th **coordinate line** *passing through* $\widehat{x} = \boldsymbol{\Theta}(x)$, defined as the image by $\boldsymbol{\Theta}$ of the points of Ω that lie on the line parallel to $\boldsymbol{e}_i$ passing through x (there exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the i-th coordinate line is given by $t \in \,]t_0, t_1[\, \to \boldsymbol{f}_i(t) := \boldsymbol{\Theta}(x + t\boldsymbol{e}_i)$ in a neighborhood of $\widehat{x}$; hence $\boldsymbol{f}_i'(0) = \partial_i \boldsymbol{\Theta}(x) = \boldsymbol{g}_i(x)$); see Figures 1.1-1 and 1.1-2.

Returning to a general increment $\boldsymbol{\delta x} = \delta x^i \boldsymbol{e}_i$, we also infer from the expansion of $\boldsymbol{\Theta}$ about x that (recall that we use the summation convention):

$$|\boldsymbol{\Theta}(x+\boldsymbol{\delta x}) - \boldsymbol{\Theta}(x)|^2 = \boldsymbol{\delta x}^T \boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x)\boldsymbol{\delta x} + o(|\boldsymbol{\delta x}|^2)$$
$$= \delta x^i \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x)\delta x^j + o(|\boldsymbol{\delta x}|^2).$$

Note that, here and subsequently, we use standard notations from matrix algebra. For instance, $\boldsymbol{\delta x}^T$ stands for the transpose of the column vector $\boldsymbol{\delta x}$ and $\boldsymbol{\nabla\Theta}(x)^T$ designates the transpose of the matrix $\boldsymbol{\nabla\Theta}(x)$, the element at the i-th row and j-th column of a matrix $\mathbf{A}$ is noted $(\mathbf{A})_{ij}$, etc.

In other words, the principal part with respect to $\boldsymbol{\delta x}$ of the length between the points $\boldsymbol{\Theta}(x+\boldsymbol{\delta x})$ and $\boldsymbol{\Theta}(x)$ is $\{\delta x^i \boldsymbol{g}_i(x)\cdot\boldsymbol{g}_j(x)\delta x^j\}^{1/2}$. This observation suggests to define a matrix $(g_{ij}(x))$ of order three, by letting

$$g_{ij}(x) := \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) = (\boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x))_{ij}.$$

The elements $g_{ij}(x)$ of this symmetric matrix are called the **covariant components of the metric tensor** at $\widehat{x} = \boldsymbol{\Theta}(x)$.

Note that *the matrix* $\boldsymbol{\nabla\Theta}(x)$ *is invertible* and that *the matrix* $(g_{ij}(x))$ *is positive definite*, since the vectors $\boldsymbol{g}_i(x)$ are assumed to be linearly independent.

The three vectors $\boldsymbol{g}_i(x)$ being linearly independent, *the nine relations*

$$\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j$$

unambiguously define three linearly independent vectors $\boldsymbol{g}^i(x)$. To see this, let *a priori* $\boldsymbol{g}^i(x) = X^{ik}(x)\boldsymbol{g}_k(x)$ in the relations $\boldsymbol{g}^i(x)\cdot\boldsymbol{g}_j(x) = \delta^i_j$. This gives $X^{ik}(x)g_{kj}(x) = \delta^i_j$; consequently, $X^{ik}(x) = g^{ik}(x)$, where

$$(g^{ij}(x)) := (g_{ij}(x))^{-1}.$$

Hence $\boldsymbol{g}^i(x) = g^{ik}(x)\boldsymbol{g}_k(x)$. These relations in turn imply that

$$\boldsymbol{g}^i(x) \cdot \boldsymbol{g}^j(x) = \big(g^{ik}(x)\boldsymbol{g}_k(x)\big) \cdot \big(g^{j\ell}(x)\boldsymbol{g}_\ell(x)\big)$$
$$= g^{ik}(x)g^{j\ell}(x)g_{k\ell}(x) = g^{ik}(x)\delta^j_k = g^{ij}(x),$$

and thus the vectors $\boldsymbol{g}^i(x)$ are *linearly independent* since the matrix $(g^{ij}(x))$ is positive definite. We would likewise establish that $\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x)$.

The three vectors $\boldsymbol{g}^i(x)$ form the **contravariant basis** at the point $\widehat{x} = \boldsymbol{\Theta}(x)$ and the elements $g^{ij}(x)$ of the symmetric positive definite matrix $(g^{ij}(x))$ are the **contravariant components of the metric tensor** at $\widehat{x} = \boldsymbol{\Theta}(x)$.

Let us record for convenience the fundamental relations that exist between the vectors of the covariant and contravariant bases and the covariant and contravariant components of the metric tensor at a point $x \in \Omega$ where the mapping $\boldsymbol{\Theta}$ is an immersion:

$$g_{ij}(x) = \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x) \quad \text{and} \quad g^{ij}(x) = \boldsymbol{g}^i(x) \cdot \boldsymbol{g}^j(x),$$
$$\boldsymbol{g}_i(x) = g_{ij}(x)\boldsymbol{g}^j(x) \quad \text{and} \quad \boldsymbol{g}^i(x) = g^{ij}(x)\boldsymbol{g}_j(x).$$

A mapping $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an **immersion** if it is an immersion at each point in Ω, i.e., if $\boldsymbol{\Theta}$ is differentiable in Ω and the three vectors $\boldsymbol{g}_i(x) = \partial_i \boldsymbol{\Theta}(x)$ are linearly independent at each $x \in \Omega$.

If $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ is an immersion, the vector fields $\boldsymbol{g}_i : \Omega \to \mathbb{R}^3$ and $\boldsymbol{g}^i : \Omega \to \mathbb{R}^3$ respectively form the **covariant**, and **contravariant bases**.

To conclude this section, we briefly explain in what sense the components of the "metric tensor" may be *"covariant"* or *"contravariant"*.

Let Ω and $\widetilde{\Omega}$ be two domains in $\mathbb{R}^3$ and let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ and $\widetilde{\boldsymbol{\Theta}} : \widetilde{\Omega} \to \mathbf{E}^3$ be two $\mathcal{C}^1$-diffeomorphisms such that $\boldsymbol{\Theta}(\Omega) = \widetilde{\boldsymbol{\Theta}}(\widetilde{\Omega})$ and such that the vectors $\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$ and $\widetilde{\boldsymbol{g}}_i(\widetilde{x}) = \widetilde{\partial}_i \widetilde{\boldsymbol{\Theta}}(\widetilde{x})$ of the covariant bases at the *same* point $\boldsymbol{\Theta}(x) = \widetilde{\boldsymbol{\Theta}}(\widetilde{x}) \in \mathbf{E}^3$ are linearly independent. Let $\boldsymbol{g}^i(x)$ and $\widetilde{\boldsymbol{g}}^i(\widetilde{x})$ be the vectors of the corresponding contravariant bases at the same point $\widehat{x}$. A simple computation then shows that

$$\boldsymbol{g}_i(x) = \frac{\partial \chi^k}{\partial x_i}(x)\widetilde{\boldsymbol{g}}_k(\widetilde{x}) \text{ and } \boldsymbol{g}^i(x) = \frac{\partial \widetilde{\chi}^i}{\partial \widetilde{x}_k}(\widetilde{x})\widetilde{\boldsymbol{g}}^k(\widetilde{x}),$$

where $\boldsymbol{\chi} = (\chi^j) := \widetilde{\boldsymbol{\Theta}}^{-1} \circ \boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \widetilde{\Omega})$ (hence $\widetilde{x} = \boldsymbol{\chi}(x)$) and $\widetilde{\boldsymbol{\chi}} = (\widetilde{\chi}^i) := \boldsymbol{\chi}^{-1} \in \mathcal{C}^1(\widetilde{\Omega}; \Omega)$.

Let $g_{ij}(x)$ and $\widetilde{g}_{ij}(\widetilde{x})$ be the covariant components, and let $g^{ij}(x)$ and $\widetilde{g}^{ij}(\widetilde{x})$ be the contravariant components, of the metric tensor at the *same* point $\boldsymbol{\Theta}(x) = \widetilde{\boldsymbol{\Theta}}(\widetilde{x}) \in \mathbf{E}^3$. Then a simple computation shows that

$$g_{ij}(x) = \frac{\partial \chi^k}{\partial x_i}(x)\frac{\partial \chi^\ell}{\partial x_j}(x)\widetilde{g}_{k\ell}(\widetilde{x}) \text{ and } g^{ij}(x) = \frac{\partial \widetilde{\chi}^i}{\partial \widetilde{x}_k}(\widetilde{x})\frac{\partial \widetilde{\chi}^j}{\partial \widetilde{x}_\ell}(\widetilde{x})\widetilde{g}^{k\ell}(\widetilde{x}).$$

These formulas explain why the components $g_{ij}(x)$ and $g^{ij}(x)$ are respectively called "covariant" and "contravariant": *Each* index in $g_{ij}(x)$ "varies like" that of the corresponding vector of the *covariant* basis *under a change of curvilinear coordinates*, while *each* exponent in $g^{ij}(x)$ "varies like" that of the corresponding vector of the *contravariant* basis.

Remark. What is exactly the "second-order tensor" hidden behind its covariant components $g_{ij}(x)$ or its contravariant exponents $g^{ij}(x)$

is beautifully explained in the gentle introduction to tensors given by Antman [1995, Chapter 11, Sections 1 to 3]; it is also shown in *ibid.* that the same "tensor" also has "mixed" components $g^i_j(x)$, which turn out to be simply the Kronecker symbols δ^i_j. □

In fact, analogous justifications apply as well to the components of all the other "tensors" that will be introduced later on. Thus, for instance, the covariant components $v_i(x)$ and $\widetilde{v}_i(x)$, and the contravariant components $v^i(x)$ and $\widetilde{v}^i(x)$ (both with self-explanatory notations), of a vector at the *same* point $\boldsymbol{\Theta}(x) = \widetilde{\boldsymbol{\Theta}}(\widetilde{x})$ satisfy (cf. Section 1.4)

$$v_i(x)\boldsymbol{g}^i(x) = \widetilde{v}_i(\widetilde{x})\widetilde{\boldsymbol{g}}^i(\widetilde{x}) = v^i(x)\boldsymbol{g}_i(x) = \widetilde{v}^i(\widetilde{x})\widetilde{\boldsymbol{g}}_i(\widetilde{x}).$$

It is then easily verified that

$$v_i(x) = \frac{\partial \chi^j}{\partial x_i}(x)\widetilde{v}_j(\widetilde{x}) \text{ and } v^i(x) = \frac{\partial \widetilde{\chi}^i}{\partial \widetilde{x}_j}(\widetilde{x})\widetilde{v}^j(\widetilde{x}).$$

In other words, the components $v_i(x)$ "vary like" the vectors $\boldsymbol{g}_i(x)$ of the *covariant* basis *under a change of curvilinear coordinates*, while the components $v^i(x)$ of a vector "vary like" the vectors $\boldsymbol{g}^i(x)$ of the *contravariant* basis. This is why they are respectively called *"covariant"* and *"contravariant"*. A vector is an example of a "first-order" tensor.

Note, however, that we shall no longer provide such commentaries in the sequel. We leave it instead to the reader to verify in each instance that any index or exponent appearing in a component of a "tensor" indeed behaves according to its nature.

The reader interested by such questions will find exhaustive treatments of tensor analysis, particularly as regards its relevance to elasticity, in Boothby [1975], Marsden & Hughes [1983, Chapter 1], or Simmonds [1994].

1.3 Volumes, areas, and lengths in curvilinear coordinates

We now review fundamental formulas showing how *volume, area,* and *length elements* at a point $\widehat{x} = \boldsymbol{\Theta}(x)$ in the set $\widehat{\Omega} = \boldsymbol{\Theta}(\Omega)$ can be expressed either in terms of the matrix $\boldsymbol{\nabla\Theta}(x)$, or in terms of the matrix $(g_{ij}(x))$.

These formulas thus highlight the crucial rôle played by the matrix $(g_{ij}(x))$ for computing "metric" notions at the point $\widehat{x} = \boldsymbol{\Theta}(x)$. Indeed, the "metric tensor" well deserves its name!

A **domain** in $\mathbb{R}^d, d \geqslant 2$, is a bounded, open, and connected subset D of $\mathbb{R}^d$ with a Lipschitz-continuous boundary, the set D being locally on one side of its boundary. All relevant details needed here about domains are found in Nečas [1967] or Adams [1975].

Given a domain $D \subset \mathbb{R}^3$ with boundary Γ, we let $\mathrm{d}x$ denote the *volume element* in D, $\mathrm{d}\Gamma$ denote the *area element* along Γ, and $\boldsymbol{n} = n_i \widehat{\boldsymbol{e}}^i$ denote the *unit* ($|\boldsymbol{n}| = 1$) *outer normal vector* along Γ ($\mathrm{d}\Gamma$ is well defined and $\boldsymbol{n}$ is defined $\mathrm{d}\Gamma$-almost everywhere since Γ is assumed to be Lipschitz-continuous).

Note also that the assumptions made on the mapping $\boldsymbol{\Theta}$ in the next theorem guarantee that, if D is a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$, then $\{\widehat{D}\}^- \subset \widehat{\Omega}$, $\{\boldsymbol{\Theta}(D)\}^- = \boldsymbol{\Theta}(\overline{D})$, and the boundaries $\partial\widehat{D}$ of $\widehat{D}$ and ∂D of D are related by $\partial\widehat{D} = \boldsymbol{\Theta}(\partial D)$ (see, e.g., Ciarlet [1988, Theorem 1.2-8 and Example 1.7]).

If $\mathbf{A}$ is a square matrix, $\mathbf{Cof\,A}$ denotes the *cofactor matrix* of $\mathbf{A}$. Thus $\mathbf{Cof\,A} = (\det \mathbf{A})\mathbf{A}^{-T}$ if $\mathbf{A}$ is invertible.

Theorem 1.3-1. *Let Ω be an open subset of $\mathbb{R}^3$, let $\boldsymbol{\Theta} : \Omega \to \mathbf{E}^3$ be an injective and smooth enough immersion, and let $\widehat{\Omega} = \boldsymbol{\Theta}(\Omega)$.*

(a) *The volume element $\mathrm{d}\widehat{x}$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ is given in terms of the volume element $\mathrm{d}x$ at $x \in \Omega$ by*

$$\mathrm{d}\widehat{x} = |\det \boldsymbol{\nabla\Theta}(x)|\,\mathrm{d}x = \sqrt{g(x)}\,\mathrm{d}x, \text{ where } g(x) := \det(g_{ij}(x)).$$

(b) *Let D be a domain in $\mathbb{R}^3$ such that $\overline{D} \subset \Omega$. The area element $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \partial\widehat{D}$ is given in terms of the area element $\mathrm{d}\Gamma(x)$ at $x \in \partial D$ by*

$$\mathrm{d}\widehat{\Gamma}(\widehat{x}) = |\mathbf{Cof}\,\boldsymbol{\nabla\Theta}(x)\boldsymbol{n}(x)|\,\mathrm{d}\Gamma(x) = \sqrt{g(x)}\sqrt{n_i(x)g^{ij}(x)n_j(x)}\,\mathrm{d}\Gamma(x),$$

where $\boldsymbol{n}(x) := n_i(x)\boldsymbol{e}^i$ denotes the unit outer normal vector at $x \in \partial D$.

(c) *The length element $\mathrm{d}\widehat{\ell}(\widehat{x})$ at $\widehat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega}$ is given by*

$$\mathrm{d}\widehat{\ell}(\widehat{x}) = \left\{\boldsymbol{\delta x}^T \boldsymbol{\nabla\Theta}(x)^T \boldsymbol{\nabla\Theta}(x)\boldsymbol{\delta x}\right\}^{1/2} = \left\{\delta x^i g_{ij}(x)\delta x^j\right\}^{1/2},$$

where $\boldsymbol{\delta x} = \delta x^i \boldsymbol{e}_i$.

Proof. The relation $\mathrm{d}\widehat{x} = |\det \boldsymbol{\nabla\Theta}(x)|\,\mathrm{d}x$ between the volume elements is well known. The second relation in (a) follows from the relation $g(x) = |\det \boldsymbol{\nabla\Theta}(x)|^2$, which itself follows from the relation $(g_{ij}(x)) = \boldsymbol{\nabla\Theta}(x)^T\boldsymbol{\nabla\Theta}(x)$.

Indications about the proof of the relation between the area elements $\mathrm{d}\widehat{\Gamma}(\widehat{x})$ and $\mathrm{d}\Gamma(x)$ given in (b) are found in Ciarlet [1988, Theorem 1.7-1] (in this formula, $\boldsymbol{n}(x) = n_i(x)\boldsymbol{e}^i$ is identified with the column vector in $\mathbb{R}^3$ with $n_i(x)$ as its components). Using the relations $\mathbf{Cof}(\mathbf{A}^T) =$