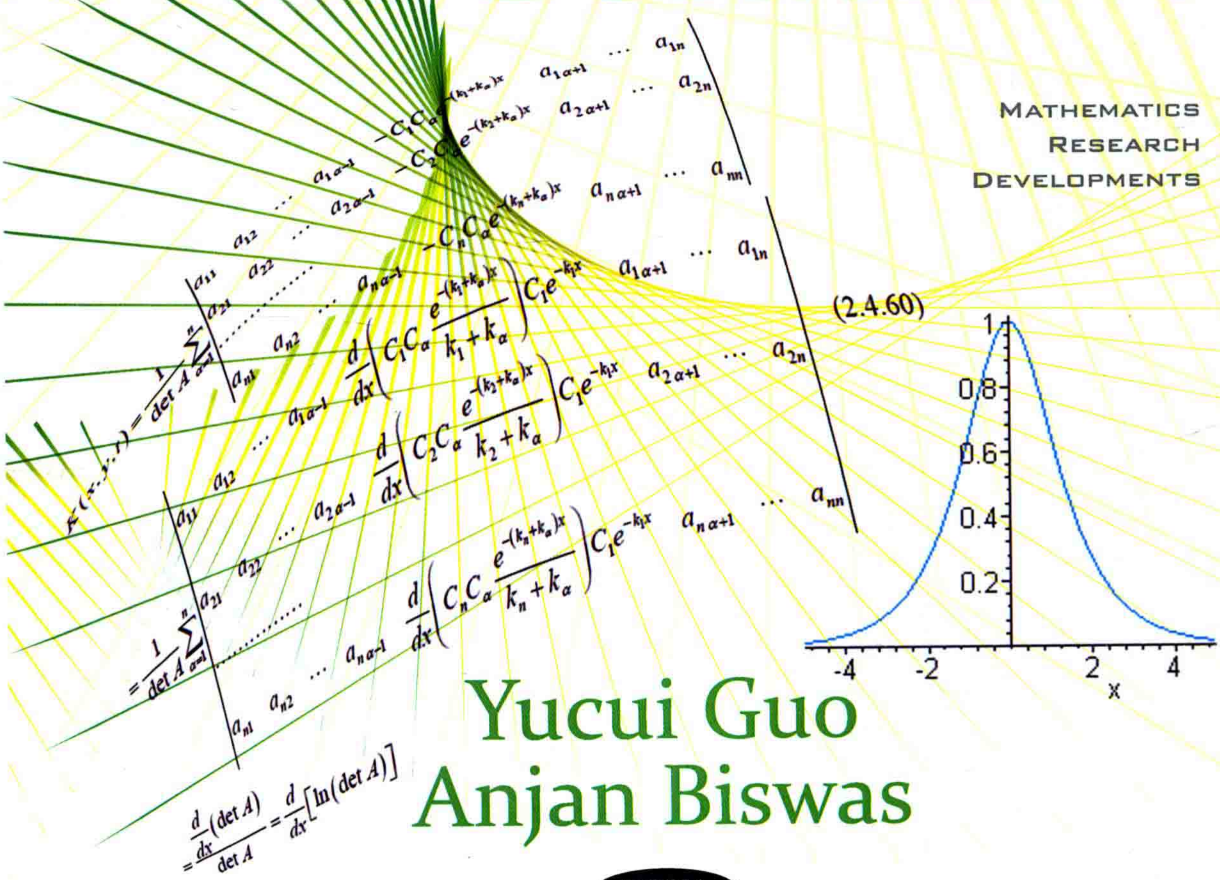
Nonlinear Evolution Equations and Soliton Solutions

MATHEMATICS RESEARCH DEVELOPMENTS



Yucui Guo
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NONLINEAR EVOLUTION EQUATIONS AND SOLITON SOLUTIONS

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AND
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nova publishers
New York

Library of Congress Cataloging-in-Publication Data

Names: Guo, Yucui, 1962- editor. | Biswas, Anjan (Mathematics professor), editor.
Title: Nonlinear evolution equations and soliton solutions / editors, Yucui Guo, and Anjan Biswas
(Beijing University of Post and Telecommunications, Beijing, China, and others).
Description: Hauppauge, New York : Nova Science Publishers, Inc., 2016. |
Series: Mathematics research developments | Includes index.
Identifiers: LCCN 2015041887 (print) | LCCN 2015050010 (ebook) | ISBN 9781634827690 (hardcover) | ISBN 9781634841580 ()
Subjects: LCSH: Evolution equations, Nonlinear. | Solitons. | Transformations (Mathematics) | Differential equations, Nonlinear.
Classification: LCC QC20.7.E88 N66 2016 (print) | LCC QC20.7.E88 (ebook) |
DDC 515/.353--dc23
LC record available at http://lccn.loc.gov/2015041887

Published by Nova Science Publishers, Inc. † New York

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Foreword

I was asked to write a preface for this book in response to the request of Professor Yucui Guo, one of the authors of the book.

Soliton attracts much attention in academia and industry because of its beautiful structure and a wide range of applications. During half a century period of its rapid development, there have been many monographs and related books published. Among them there are a few to clarify the mechanism of formation and propagation of solitons. Professor Guo's book focuses on them through utilizing her recent year publications. In the book, Professor Guo describes several mathematical models in detail to generate soliton solutions. Those mathematical models are usually called nonlinear evolution equations, including the KdV equation, the nonlinear Schrödinger equation, the Sine-Gordon equation, and so on. The principal of formation of solitons is actually caused due to precise balance between dispersion phenomena and nonlinear effect in fluctuations is described clearly in the book.

In general, nonlinear partial differential equations are very difficult to solve for their explicit solutions. They possibly have solitary wave or soliton solutions decaying at infinite boundaries for some practical application problems is one of the reasons of they getting rapid development.

This book will provide some tools for solving nonlinear partial differential equations to get solitary wave solutions in an explicit formula. Those tools include the Inverse Scattering Transformation method, Bäcklund Transformation method, Similarity Transformation methods, and other Special Transformation methods, which are all classic and practical. Basically, they play a fundamental and powerful role in getting soliton solutions for an integrable system.

The book provides the principles and applications for those methods with detailed procedure and accurate computations. The book also covers the integrability and Painlevè property of nonlinear partial differential equations, and describes the amazing application prospect of solitons ---- optical soliton communication. So this book is one of very good sources for graduate students and fresh researchers to grasp some procedure to solve nonlinear partial differential equations with solitary wave/soliton solutions as well as suitable for a graduate class as a textbook or a reference. This book is also one of good reference choice for researchers engaged in the field of nonlinear science.

Within my knowledge, the two authors Professor Yucui Guo and Associate Professor Anjan Biswas are working on their research and book Chapters very hard. I believe that the publication of this book might be very helpful and suggestive to students and teachers/researchers in the study of nonlinear science, integrable systems and solitons.

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09-22-2015

PREFACE

This book focuses on the study of the theories and solving methods of nonlinear partial differential equations or called nonlinear evolution equations or nonlinear mathematical physics equations, especially on an interesting kind of solutions of these equations, so called solitary wave solutions or soliton solutions and their applications. It can be used as textbook and reference book for graduate students and teachers whose research fields are applied mathematics, applied physics and nonlinear science, etc. It is also available for researchers engaging in research on nonlinear science as a reference book. This book emphasizes particularly on clear concepts, rigorous derivation, reasoning through measuring, and logical reasoning. Due to the research boom on nonlinear partial differential equations beginning in 1960, it has been developing only for half a century up to now, and it is still a young discipline corresponding to other classical branches of mathematics. Although there are already some good books published globally, some of them are at the high starting point so that beginners are not easy to understand. At the same time, since the research of nonlinear partial differential equations is interdisciplinary, physicists, mathematicians, and scientists in engineering areas have been studying and paying attention to this field, and their ways must be different. Some works have very strong professional orientation, and are difficult to be understood for persons who don't have too much professional knowledge. The goal of this book is to make it understandable for persons who have only the basic knowledge of college mathematics and physics, and to help readers understand well the content of the text, some contents helpful are concisely described in the appendixes.

In the process of writing this book, the authors' goal is to make a systematical and complete theory and help beginners in this field entry as soon as possible through reading this book. The book also includes some work of the authors as well as graduate students. Here the authors gives credit and thanks to all the authors of references.

Most of contents of the manuscript have been taught in a degree course of graduated students in Beijing University of Posts and Telecommunications, the course named as "Nonlinear Partial Differential Equations and Their Applications." Thanks to the graduate students who had learned the course, it is their enthusiasm and curiosity to motivate the authors to continuously explore in this field, and practice hard, and aspire to write a book for the new students. The purpose and desire is good, and time and effort also spent, but because our academic level is limited, shortcomings and mistakes may still exist. We sincerely hope that readers put forward valuable opinions and suggestions, so that the book will be improved.

The research objects of this book are nonlinear partial differential equations or called nonlinear evolution equations. Due to they are derived from physics and some engineering disciplines and have distinct physical meaning, and they often describe the development of phnomenon with time, so they are also called nonlinear evolution equations or nonlinear mathematical physics equations. As we all know, mathematical physics equations are just mathematical equations with background of physics, including algebraic equations, function equations, ordinary differential equations, partial differential equations, integral equations and differential integral equations, difference equation, and so on. The course named "mathematical physics equations" or "the method of mathematical physics" usually given at the undergraduate level studies linear partial differential equations and their solutions. This book focuses on the research of nonlinear partial differential equations with physical background and related theory.

The mathematical form of a nonlinear partial differential equation containing of time and space variables can usually be expressed as

$$P(t,x;u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots)=0$$

where $u=u(x,t)$ is the target quantity, that is the unknown function; x is spatial coordinate, and sometimes may be 2-dimension, (x,y), or 3-dimension, (x,y,z), or even multidimensional ($x_1,x_2,\cdots,x_n$); t is the time coordinate and sometimes may be generalized time coordinate, because of it may be a variable gone through the coordinate transformation in process for establishing mathematical model. u_x,u_t,u_{xx},u_{tt} denote respectively the first and second order partial derivatives for coordinates x and t. So called nonlinear partial differential equations refer to there are higher power terms of unknown function and (or) derivatives of unknown function contained in them, so they cannot be written as the following linear form. The second-order linear partial differential equation with two independent variables as an example expresses as

$$A(x,y)u_{xx}+2B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=f(x,y)$$

There are a hundred kinds of nonlinear partial differential equations having physical meaning by now. Typical and representative equations are KdV equation

$$u_t+uu_x-\mu u_{xxx}=0 \tag{0.1}$$

Sine - Gordon equation

$$u_{xx}-u_{tt}=\sin u \tag{0.2}$$

and nonlinear Schrodinger equation

$$iu_t + u_{xx} + \beta u|u|^2 = 0 \tag{0.3}$$

and so on.

Although these differential equations have simple forms, the essences of them are very different with that of linear differential equations. For example, the uniqueness, single value, boundedness, and the superposition principle of the solutions, etc. of linear differential equations, are likely not to exist to nonlinear ones. Therefore, for nonlinear partial differential equations, there are no general theory and methods to solve them. But it is interesting that many nonlinear partial differential equations have a kind of special solutions, so called solitary waves or solitons which are very meaningful.

Due to the physical background and meaning of nonlinear partial differential equations and special properties of solitary waves, the methods for solving them and solitary wave theory become a new and vivid branch of nonlinear science and one of the frontier and hot issues of scientific development.

A system is nonlinear if the output is not proportional to the input. The first example is mechanical spring, which generates elongation or displacement under the action of force. When the displacement is small, the force is proportional to the displacement, the relationship between force and displacement is linear relationship, which obeying the Hooke's law $F = kx$. However when displacement goes larger Hooke's law fails and the spring becomes a nonlinear oscillator. The second example is mathematical pendulum, only when the angular displacement of the pendulum is very small, its behavior is linear. And for a dielectric crystal, when the light intensity input is no longer proportional to the output one, it becomes nonlinear dielectric crystals. In fact, almost all of the known systems in natural science or social science are nonlinear, when the input is large enough. Therefore, the numbers of nonlinear systems are much larger than that of linear systems. It can be said that the objective world is nonlinear and linear is only an approximation, and the equations describing these nonlinear behaviors are likely to be nonlinear partial differential equations. Thus, understanding, researching, and applying nonlinear partial differential equations are inevitable in the development of science. With the development of computer science, many nonlinear problems can be solved with computers, which were impossible to solve manually. So, in a sense, nonlinear science is developing with the progression of computer science.

This book mainly studies the solutions of nonlinear evolution equations, soliton theory and its application.

In mathematics, the solutions of nonlinear evolution equations which have the following properties are called the solitary waves or solitons:

1. They are a kind of traveling waves which spread to the single direction, namely in the form of $\varphi(x-at)$ or $\psi(x+at)$;
2. They distribute in a small area of the space, namely $\lim\limits_{x\to\pm\infty} u \to 0$, sometimes also having $\lim\limits_{x\to\pm\infty} u_x \to 0$ and $\lim\limits_{x\to\pm\infty} u_{xx} \to 0$, etc.
3. Wave shapes keep unchanged with the evolution of time;
4. Interactions between solitary waves have elastic properties as similar as the particles do. The solitary wave which has property of elastic collision is called soliton.

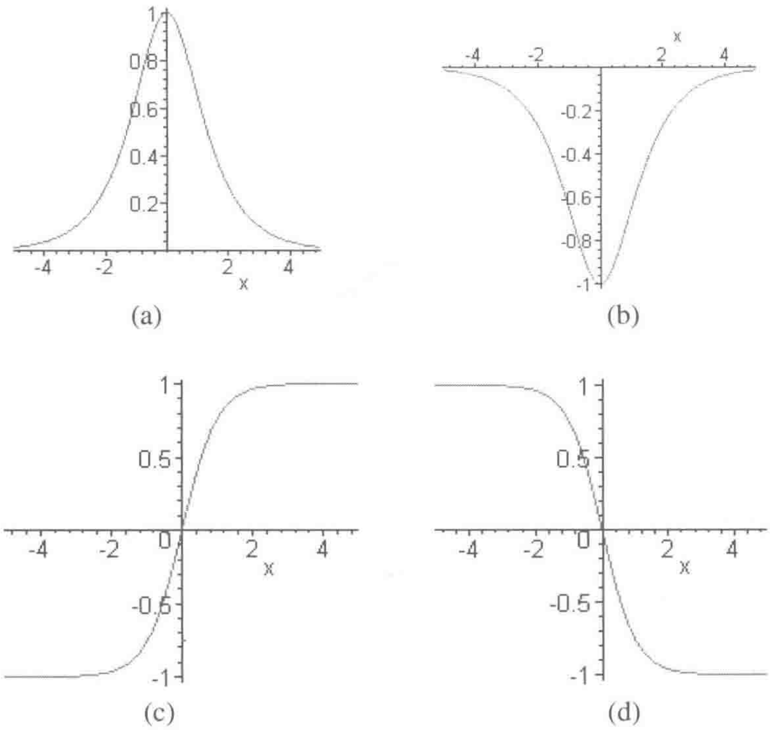

Figure 1.1. Four types of solitary waves.

There are four kinds of solitons commonly: (1) the bell shaped soliton (Figure 1.1 (a)), (2) the ante- bell shaped soliton (Figure 1.1 (b)), (3) the kink soliton (Figure 1.1 (c)) and (4) the ante - kink soliton (Figure 1.1 (d)).

The basic steps of solving practical problems using mathematical method are shown as follows. First step is to simplify the actual problem and form a physical model; Then to quantify the physical model, namely, to establish the mathematical model--the nonlinear partial differential equations; The third step is to solve the partial differential equation, that is to find out the solutions of them, sometimes with certain boundary conditions; The final procedure is to analyze the results obtained, which is used to explain phenomena in nature and guide practice, namely to solve actual problem.

It is generally to go through several processes from the physical model to the mathematical model: (1) To set up time and space coordinates, different problems need to choose different coordinate systems; (2) To determine the target variables and their coordinates of physical process studied, which is a very important. Choosing characterization parameters correctly is sometimes the beginning to establish a new discipline; (3) To find out law of physical process by supposing, guessing, and using the existing laws, namely to establish the physics axiom, which is the hard task. The mathematical models established in the course of mathematical physics equations only involve some simple physical processes. Through which the general methods to quantify physical models have been shown. In this book, more attention is paid to the derivation of mathematical models from the physical processes, for to research the mathematical physics problems, it is impossible to consider only mathematical problems with not thinking the physical pictures. The physical image and geometry image often play a simple and clear key role to understand fully the mathematical problems.

Once the mathematical model has been established from the physical process, the following task is to seek the right method to solve the models, which are nonlinear partial differential equations in this book. With the continuous development of science, it is paid more and more attention to solve nonlinear partial differential equations. A large number of nonlinear partial differential equations are deduced from astronomy, physics, mechanics, earth science and life science and all kinds of engineering sciences, especially in different branches of physics, such as fluid mechanics, nonlinear optics, plasma theory and quantum field theory, etc. Majority of these equations have solutions in the form of a solitary wave which was first noticed by J. S. Russell, a British scientist, in 1830. The nonlinear science has been getting rapid development since the 1960 s. Many achievements have been also obtained in the aspect of solving nonlinear partial differential equations. Some of the important and typical methods will be discussed in detail in this book.

Due to the complexity of nonlinear mathematical physics equations themselves, there is no general method for solving them, the basic thoughts of various methods are to use some transformations or decompositions to simplify the complex equations into simple ones to be dealt with. The forms of transformation and decomposition are also varied, which sometimes need to try and guess mathematically and physically. These conjecture or assumption themselves may not be universal, but the thought supporting them has universal significance. Math skills and physical intuition will get perfect embodiment here as far as possible.

It is to point out once again that the book is accomplished on the basis of the lecture notes of the course, named for nonlinear partial differential equations and their applications, for graduate students in Beijing University of Posts and Telecommunications. Authors wish to thank all of the students who had learned over the course. It is that their constantly thinking and insightful questions that make the book more profound and perfect. Special thanks to the students who participated in the course the firt time it was taught, Wang Xin, Xu Shujiang, Ye Peng, Li Huaying and Li Juan, etc. who input the manuscript into computer.

Chapters 1 through 6 of the this book and all the appendixes were written by Prof. Yucui Guo, and Chapter 7 was written by Dr. Anjan Biswas.

Finally, the Prof. Yucui Guo would like to thank Dr. Fengshan Liu, professor of Applied Mathematics, Delaware State University for his consistent support when the author is writing this book during her visit to Delaware State University. Prof. Yucui Guo would also like to thank Dr. Guoping Zhang, associate Professor of Mathematics, Morgan State University, who carefully revised the English of the whole book. The author appreciates the support from both Dr. Fengshan Liu and Dr. Guoping Zhang.

Yu-cui Guo

CONTENTS

Chapter 1

TYPICAL EQUATIONS AND THEIR SOLITARY WAVE SOLUTIONS

This book focuses on the investigation of solutions and characters of solutions of nonlinear partial differential equations which appear as mathematical models in fields of physics, chemistry, information science, life science, space science, geographical science and environmental science, etc. As a subject, "nonlinear partial differential equations" is an important part of nonlinear science and mathematics. The methods and approaches used to solve nonlinear partial differential equations came from mathematical community, and the backgrounds of these equations were usually physical applications. Therefore the collection of these methods is often referred to as "nonlinear mathematical physics method," which is a discipline having properties of interdisciplinary and fusion of disciplines.

The investigation of nonlinear partial differential equations began from the late 19th century. At the beginning, it seems every single equation had different types of solutions due to the complexity of the equation itself. It is difficult to find a common method or technique to solve them, which was considered as a strong abstruse problem to be dealt with. However the critical change happened in the 1960s. Both mathematics and physics communities found that many different nonlinear partial differential equations can actually be solved by some common method and the solutions obtained have some common properties, which were shared by so-called "soliton" solutions. This fact makes "nonlinear partial differential equations" into a new applied interdisciplinary research field where nonlinear phenomena have been studied extensively. Today, the research on nonlinear partial differential equations has merged into mathematics, engineering science, and almost every discipline of natural science and social science.

Now hundreds of nonlinear partial differential equations discovered possess the physical significance. A large number of new nonlinear partial differential equations are springing up from various disciplines. These equations are roughly divided into two categories: one is called integrable system or the weak non-integrable system which has some good properties, such as solvability with inverse scattering method, existence of solitary wave and similar solitary wave solutions, possession of Bäcklund transformation, Darboux transformation, Hirota bilinear form, Painlevè property and infinite conservation laws, etc. The solitary wave solutions attracted extensive concern due to its wide applications. Another category is called the non-integrable system, which has some dissipative structures and possesses some solutions describing chaotic phenomenon. In this book we focus mainly on the integrable systems or the weak non-integrable systems. In this chapter we introduce several typical equations and their solitary wave solutions.

1.1. THE HISTORICAL REVIEW

Soliton is a solitary wave having elastic scattering property. This unusual phenomenon was firstly observed by John Scott Russell, a famous British scientist, who was also a shipbuilding engineer. In September 1844, Russell reported his unusual findings at Fourteenth Meeting of the British Assoc. for the Advancement of Science in the topic "On waves." He said: "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses. When the boat suddenly stopped, not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation...." Russel recognized in that time that this isolated water peak is not the usual wave. Regular water waves have two parts: one part is over the surface of the water and the other part is under the water. The solitary wave is clearly a complete wave which is located wholly above the surface of the water entirely. It is different also from the shock wave that has a singularity in front of the wave, because it has a rounded and smooth waveform. That is solitary wave is regular everywhere. Solitary wave is different from any kind of usual wave packet constituted of plane waves, because it is a kind of stable solution of fluid mechanics. Russell's study on solitary waves had a lot of insights, but it did not have much influence in the scientific community at that time, because he didn't set up a persuasive mathematical model.

Until more than 60 years later, in 1895, the famous Dutch mathematician d. Korteweg and his student G.de Vries studied the motion of shallow water waves, and discovered the famous KdV equation, under the assumption of long wave approximation and small amplitude, by which the solitary wave phenomenon observed by Russell got a reasonable explanation.

The KdV equation is,

$$\frac{\partial \eta}{\partial t}=\frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{2}{3}\alpha\eta+\frac{1}{2}\eta^2+\frac{\sigma}{3}\frac{\partial^2\eta}{\partial x^2}\right) \tag{1.1.1}$$

where $\eta=\eta(x,t)$ is the height of the wave higher than the equilibrium water level, x is the coordinate along the direction of wave propagation on the balanced surface, t denotes time, l is the depth of water, g is the acceleration of gravity, α is a constant related to the liquid uniform motion and σ is a constant made from

$$\sigma = \frac{1}{3}l^3 - \frac{Tl}{\rho g}$$

in which T is the surface tension of the capillarity, and ρ is the density of the fluid.

Applying the following mathematical transformation

$$t' = \frac{1}{2}\sqrt{\frac{g}{l\sigma}}t, \qquad x' = -\frac{1}{\sqrt{\sigma}}x, \qquad u = \pm\frac{1}{2}\eta - \frac{1}{3}\alpha,$$

and removing "'" on the variables, one gets

$$u_t \pm 6uu_x + u_{xxx} = 0 \tag{1.1.2}$$

which is the general form of KdV equation.

From equation (1.1.1) or (1.1.2), Korteweg and de Vries found out the solitary wave solution (1.1.3), which is shown as in Figure 1.1.1, the same as the solitary wave phenomenon Russell deserved, which is a kind of pulse shape of solitary wave with invariant profile.

$$u(x,t) = \frac{1}{2}c\operatorname{sec}h^2[\frac{1}{2}\sqrt{c}(x - ct + x_0)] \tag{1.1.3}$$

where $c > 0$ expresses the velocity of propagation of a solitary wave which relates to the nature of the wave itself (some concepts and properties relating to the fluctuation are given in Appendix C), and x_0 is an arbitrary constant.

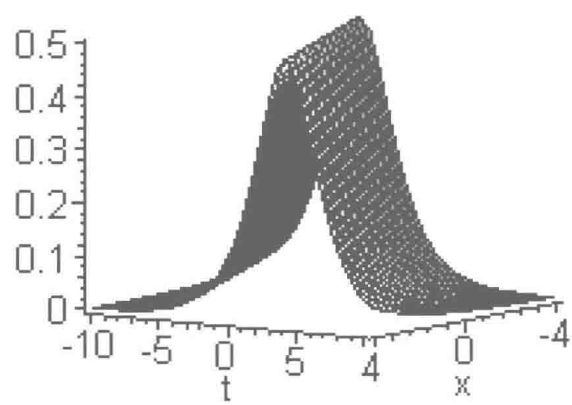

Figure 1.1.1. Pulse shape of solitary wave.

From formula (1.1.3), it can be seen that the speed, denoted with c, of solitary wave propagation is proportional to the amplitude of the wave, so the larger solitary wave will move more quickly than the small one. If a small solitary wave move ahead and a larger one behind, the latter will catch up with the former. Two solitary waves will inevitably overlap or crash. The questions put forward in that time were: Is the solitary wave a kind of stable wave? What will be the result of collision between two solitary waves? Whether or not they will deform? These problems couldn't be solved for a long time. Some people thought that solitary waves are nonlinear waves as solutions of nonlinear partial differential equations, so their overlap will not move together as simple as the linear wave, and the configurations will likely be damaged after the collision of two solitary waves, even fragmented. Another question was: in addition to the fluid mechanics, are there some solitary waves in other areas?

In 1952, it was Enrico Fermi (the father of the atomic bomb) with his scientific team, including John Pasta and Stan Ulam, calculated the vibration of harmonic oscillator which was a string made up of a nonlinear spring coupling 64 particles by using the numerical method to verify the energy equipartition theorem in statistical mechanics. They inspired a few particles and hoped that initial excitation energy should be evenly distributed on each particle in fluctuation after a long time, accordance with the energy equipartition theorem and the weakly nonlinear interaction. But the results were surprised for it was found that after tens of thousands of cycle experiment for a long time, almost all of the energy was back to the initial distribution. That is to say most of the energy had focus to a few original harmonic oscillators that their initial energy state was not zero. This is the famous FPU experiment which result indicated that there might be a solitary wave in another nonlinear system besides water waves. Later, Toda considered the nonlinear vibration of the crystal to approximately simulate the situation of FPU experiment and got finally the solitary wave solution. These results answer correctly the second question mentioned above. So one can imagine there are solitary waves in other fields besides hydrodynamics.

In 1965, M. D. Kruskal and N. Zabusky, two applied mathematicians in Princeton university U.S, had studied nonlinear interaction process of solitary waves collision in the plasma by numerical simulation method, and obtained the important result of the waveform and speed of the solitary waves keeping invariant after the crash, which removed thoroughly the doubts about the stability of solitary wave.

According to the feature of invariant shape of solitary wave after collision, which is similar as the particles, Kruskal and Zabusky called the solitary wave having this character as soliton. Since then, study interest and enthusiasm of scientists on solitons are unstoppable. So far, motion forms of solitons have been found in many scientific fields including laser self-focusing in the medium, the sound waves and electromagnetic waves in plasma, domain wall motion in liquid crystal, vortex in fluid, crystal dislocation, magnetic flux in superconductors, the transmission of signals in the nervous system, etc. It should also be pointed out that one of the important reasons of the soliton research is that solitons can likely be used in the large capacity, high speed optical fiber communication. The research in this area is more and more deeply. It has been confirmed that solitary waves can be used to improve the quality and efficiency of signal transmission system. An amazing characteristic of artificial light soliton pulses is that neither shape nor speed is influenced by collisions with similar wave packets in transmission. These properties make them the high fidelity and good secrecy. Now the study of optical soliton communication has become a new discipline.