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The Finite Element Method

THIRD EDITION

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(The third, expanded and revised edition of *The Finite Element Method in Engineering Science*)

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Preface

The present volume may be regarded as the third edition of the *Finite Element Method on Structural and Continuum Mechanics* first published in 1967. Although the size is now some three times that of the original edition—it is written with identical objectives; first to teach and second to provide a 'state of the art' reference base of the subject—which is now recognized as one of considerable importance to both practicing engineers and physicists as well as researchers.

Since the first volume was written the number of research publications on the finite element method has been increasing almost exponentially. Close to 8000 references are recorded and many more are available as internal reports, etc.† While in early days the contributors have been almost exclusively engineers, today a large number of these come from the field of mathematics which has now adopted the method and made a great contribution to the understanding of finite element method. Clearly, at this stage, a book doing justice to all points of view is impracticable—and in this volume much selection and filtering had to be done representing the viewpoints of the author. This acknowledges both the mathematical basis and the need for intuitive creative thought. Thus although the book starts with the basis of a physical discrete system—and introduces the finite element approximations via well understood elasticity examples—in Chapter 3 the concepts of fundamental mathematical approximation are presented (in a manner avoiding, with apology to mathematicians, some of the jargon and pedantry so as to make it suitable for engineers or physicists). In some later chapters we show, however, how some of the usually accepted criteria can be modified and violated with success. In particular Chapter 11 provides some of the recent developments in this context, showing how a cancellation of errors can occur through inexact integration etc.

† An excellent bibliography compiled by D. Norrie and G. de Vries (IFI/PLENUM 1976) shows the following rate of publication with figures in parentheses giving number of papers in the year: 1961 (10); 1962 (15); 1963 (25); 1964 (33); 1965 (67); 1966 (134); 1967 (162); 1968 (303); 1969 (531); 1970 (510); 1971 (844); 1972 (1004); 1973 (1169); 1974 (1377); 1975 (880 incompl.).

The general definitions of the finite element method can today be made so wide (vide Chapter 3) as to include other useful approximation processes. In particular finite difference methods will now be recognized as a subclass of the procedure and (with some imagination) the boundary integral method, which has been lately used with much success for certain classes of problem, can be brought under the general definition. This generalization is made with two-fold object. First to improve our understanding—second to incorporate selective advantages of the alternatives in a unified manner. Chapter 23 is devoted to a recent development by which the boundary integral and finite element methodologies are combined.

The application of the finite element method is today so wide that it is impossible to present an exhaustive picture in one volume. The reader will find, however, that the main fields of solid mechanics both in linear and non-linear phases, fluid mechanics, heat transfer and electro-magnetism have received some attention—and depending on his interest he can direct his selection appropriately. Clearly the study of the complete volume in one course is not recommended and a teacher using the text will make an appropriate selection of chapters. It is hoped, however, that the wider coverage will prove its use by providing a reasonably self-contained reference to many fields of activity into which sooner or later every one of us is thrown. The notes of the text have been successfully used at many levels of presentation, ranging in Chapters 1-3 from undergraduate courses through postgraduate teaching to courses including practitioners involved in development of the method. The prerequisite knowledge of mathematics and mechanics does not go much beyond a reasonable undergraduate engineering or physics course and some more abstract topics—such as matrices and vectors—are expounded in appendices.

The finite element process is essentially dependent for its success on skillful use of computers and efficient numerical techniques. Emphasis on the latter is made throughout the book but in the concluding chapter, written by Professor R. L. Taylor, much of the programming experience of the University of California, Berkeley, and of the University of Wales College, Swansea, is incorporated in a fairly complete computer system which the reader can use immediately for a variety of problems or extend readily to suit his own needs. For simplicity the system is limited in capacity. This at the same time avoids machine dependency—but its expansion to a larger size can readily be made.

List of symbols

Below a list of principal symbols used in this book is presented for easy reference, although all are defined in the text as they occur. On many occasions, additional ones have to be used in a minor context and a *non-uniqueness* arises. It is hoped that appropriate text explanation will avoid confusion.

The symbols are listed roughly in the order of occurrence in chapter sequence.

Matrices and column vectors are denoted by bold symbols, e.g., **K** and **a** and $\mathbf{K}^T$ stands for transpose of **K**. Dots are used to denote differentiation with respect to one variable, e.g., $\frac{d}{dt} \equiv \dot{\mathbf{a}}$, etc.

Chapter	*Symbol*	
1	$\mathbf{a}_i$, $\mathbf{a}$	nodal or global displacements
	$\mathbf{q}_i^e$	nodal force at i due to element e
	$\mathbf{K}^e$, $\mathbf{K}$	stiffness matrix (element/global)
	$\mathbf{f}_{pi}^e$	nodal element force at i due to p, etc.
	$\mathbf{r}_i$	external nodal force
	$\boldsymbol{\sigma}$	stress (vector)
	$\mathbf{L}$, $\mathbf{T}$	transformation matrices
	$\mathbf{b}$	alternative parameters
	$\mathbf{u}$	displacement vector (components u, v and w)
2, 4, 5, 6	$\boldsymbol{\varepsilon}$	strain (vector)
	$\mathbf{L}$	strain operator
	$\mathbf{N}$	(displacement) shape function
	$\mathbf{B} = \mathbf{LN}$	strain shape function
	$\mathbf{D}$	elasticity matrix
	$\mathbf{b}$	body force (vector)
	E	Young's modulus
	ν	Poisson's ratio
	$\boldsymbol{\varepsilon}_0$, $\boldsymbol{\sigma}_0$	initial strain or stress

	$\mathbf{t}$	boundary traction
	b_x, t_x, etc.,	x–components of body forces and tractions
	ε_x, γ_{xy}, σ_x, τ_{xy}	x–components of direct and shear strain or stress
	U	strain energy
	W	potential energy of loads
	Π	total potential energy
	$\mathbf{I}$	identity matrix
	h	representative element dimension
	ϕ	body force potential (or other scalar function)
	$\boldsymbol{\phi}$	body force potential nodal values
	$\mathbf{m}^{\mathrm{T}} = [1, 1, 0]$ or $[1, 1, 1, 0, 0, 0]$	matrix equivalent of Kronecker delta for two or three dimensional strain/stress vectors
	$x, y, z, x', y', z', r, z, \theta$	Cartesian or cylindrical co-ordinates
3	$\mathbf{A}(\mathbf{u})$, $\mathbf{B}(\mathbf{u})$, etc.	operators defining governing differential equations and boundary conditions
	$\mathbf{u}$, $\boldsymbol{\phi}$, ϕ	unknown function
	$\mathbf{v}$	'test' function
	$\mathbf{a}$, $\mathbf{b}$, etc.	nodal (or other) parameters defining the trial expansion $\mathbf{u} \simeq \mathbf{Na}$
	$\mathbf{w}_j$	'weight' function
	Π	a stationary functional
	$\mathbf{L}$	a linear differential operator
	$\mathbf{C}(\mathbf{u})$	constraint condition on $\mathbf{u}$
	λ	Lagrangian multiplier
	$\mathbf{n}^{\mathrm{T}} = [n_x, n_y, n_z]$	vector normal to boundary
	α	penalty number
	∇	gradient operator $= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^{\mathrm{T}}$
7, 8, 9	l_k^n	Lagrange polynomials
	ξ, η, (ζ)	element, curvilinear, coordinates two and three dimensions
	L_1, L_2, (L_3)	triangular (area) or tetrahedral (volume) coordinates
	$\mathbf{J}$	Jacobian matrix
	H_i, w_i	quadrature weights
10	w	plate deflection

	M_x, M_y, M_{xy}	generalized stress components (moments)
	$\theta_{xi}\theta_{yi}$	rotations
	H^n_{mi}	Hermitian polynomials
	t	plate thickness
11	K, G	bulk and shear moduli
12	$\mathbf{G}$	operator linking stresses and tractions on boundary
13	$\mathbf{K}^b$, $\mathbf{K}^p$	stiffness matrices in bending and in-plane action respectively
	$\lambda_{x'y}$, etc.	direction cosines of between x' and y axes, etc.
	$\mathbf{V}_{ij}$	vector connecting point i to j
	l_{ij}	length of vector $\mathbf{V}_{ij}$
14	ϕ	angle of tangent to shell and Z axis
	R_s and r	radii of curvature
17	$\mathbf{k}$, k	permeability matrix or coefficient
	$\mathbf{H}$	discretized problem matrix
	p	pressure
	ϕ	potential
18, 19	$\mathbf{\Psi}(\mathbf{a})$	non-linear discrete equation operator
	$\mathbf{K}_T$	tangent matrix
	F	yield function
	Q	plastic potential
	$\mathbf{K}_\sigma$	initial stress matrix
20, 21	$\mathbf{M}$	mass matrix
	$\mathbf{C}$	damping matrix
	ω_i, $\bar{\mathbf{a}}_i$	i-th eigenvalue or eigenvector
	ω	frequency
	y_i	mode participation factor
	λ	characteristic number
	$\mathbf{u}$	velocity vector
22	μ	viscosity
	ρ	density
	R_n	Reynolds number
	α	upwinding parameter
23	H_0	Hankel function
	K_{I}, K_{II}, K_{III}	stress intensity factors

Contents

1. Some Preliminaries: The Standard Discrete System

1.1 Introduction

The limitations of the human mind are such that it cannot grasp the behaviour of its complex surroundings and creations in one operation. Thus the process of subdividing all systems into their individual components or 'elements', whose behaviour is readily understood, and then rebuilding the original system from such components to study its behaviour is a natural way in which the engineer, the scientist, or even the economist proceeds.

In many situations an adequate model is obtained using a finite number of well-defined components. Such problems we shall term *discrete*. In others the subdivision is continued indefinitely and the problem can only be defined using the mathematical fiction of an infinitesimal. This leads to differential equations or equivalent statements which imply an infinite number of elements. Such systems we shall term *continuous*.

With the advent of digital computers, *discrete* problems can generally be solved readily even if the number of elements is very large. As the capacity of all computers is finite, *continuous* problems can only be solved exactly by mathematical manipulation. Here, the available mathematical techniques usually limit the possibilities to oversimplified situations.

To overcome the intractability of the realistic type of continuum problem, various methods of *discretization* had from time to time been proposed both by engineers and mathematicians. All involve an *approximation* which, hopefully, is of such a kind that it approaches, as closely as desired, the true continuum solution as the number of discrete variables increases.

The discretization of continuum problems has been approached differently by mathematicians and engineers. The first have developed general techniques applicable directly to differential equations governing the problem, such as finite difference approximations,[1,2] various weighted residual procedures,[3,4] or approximate techniques of determining the stationarity of properly defined 'functionals'. The engineer, on the other hand, often approaches the problem more intuitively by creating an analogy between real discrete elements and finite portions of a continuum domain. For instance, in the field of solid mechanics McHenry,[5] Hrenikoff,[6] and Newmark[7] have, in the early 1940s, shown that reasonably good solutions to a continuum problem can be obtained by substituting small portions of the continuum by an arrangement of simple elastic bars. Later, in the same context, Argyris[8] and Turner *et al.*[9] showed that a more direct, but no less intuitive, substitution of properties can be made much more directly by considering that small portions or 'elements' in a continuum behave in a simplified manner.

It is from the engineering 'direct analogy' view that the term 'finite element' has been born. Clough[10] appears to be the first to use this term, which implies in it a direct use of *standard methodology applicable to discrete systems*. Both conceptually and from the computational viewpoint, this is of the utmost importance. The first allows an improved understanding to be obtained; the second the use of a unified approach to the variety of problems and the development of standard computational procedures.

Since the early 1960s much progress has been made, and today the purely mathematical and 'analogy' approaches are fully reconciled. It is the object of this text to present a view of the finite element method as *a general discretization procedure of continuum problems posed by mathematically defined statements*.

In the analysis of problems of a discrete nature, a standard methodology has been developed over the years. The civil engineer, dealing with structures, first calculates his force–displacement relationships for each element of the structure and then proceeds to assemble the whole following a well-defined procedure of establishing local equilibrium at each 'node' or connecting point of the structure. From such equations the solution of the unknown displacements becomes possible. Similarly, the electrical or hydraulic engineer, dealing with a network of electrical components (resistors, capacitances, etc.) or hydraulic conduits, first establishes a relationship between currents (flows) and potentials for individual elements and then proceeds to assemble the system by ensuring continuity of flows.

All such analyses follow a standard pattern which is universally adaptable to discrete systems. It is thus possible to define a *standard discrete system*, and this chapter will be primarily concerned with establishing the processes applicable to such systems. Much of what is presented here will be known to engineers, but some reiteration is at this stage advisable. As the treatment of elastic, solid structures has been the most developed area of activity this will be introduced first, followed by examples from other fields, before attempting a complete generalization.

The existence of a unified treatment of 'standard discrete problems' leads us to the first definition of the finite element process as a method of approximation to continuum problems such that

(*a*) the continuum is divided into a finite number of parts (elements), the behaviour of which is specified by a finite number of parameters, and

(*b*) the solution of the complete system as an assembly of its elements follows precisely the same rules as those applicable to *standard discrete problems*.

It will be found that numerous classical mathematical procedures of approximation fall into this category—as well as the various direct approximations used in engineering. It is thus difficult to determine the origins of the finite element method and the precise moment of its invention.

Table 1.1 shows the process of evolution which led to the present-day concepts of finite element analysis. Chapter 3 will give, in more detail, the mathematical basis which evolved from the classical landmarks.[11–20]

1.2 The Structural Element and System

To introduce the reader to the general concept of the discrete system we shall first consider a structural engineering example of linear elasticity.

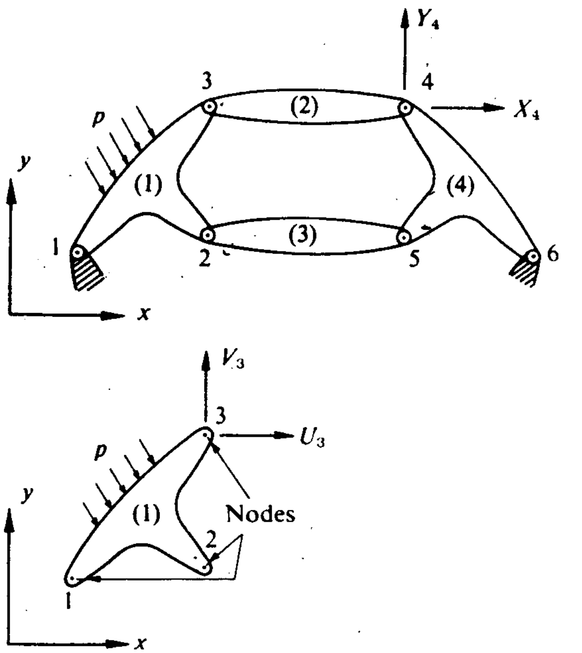
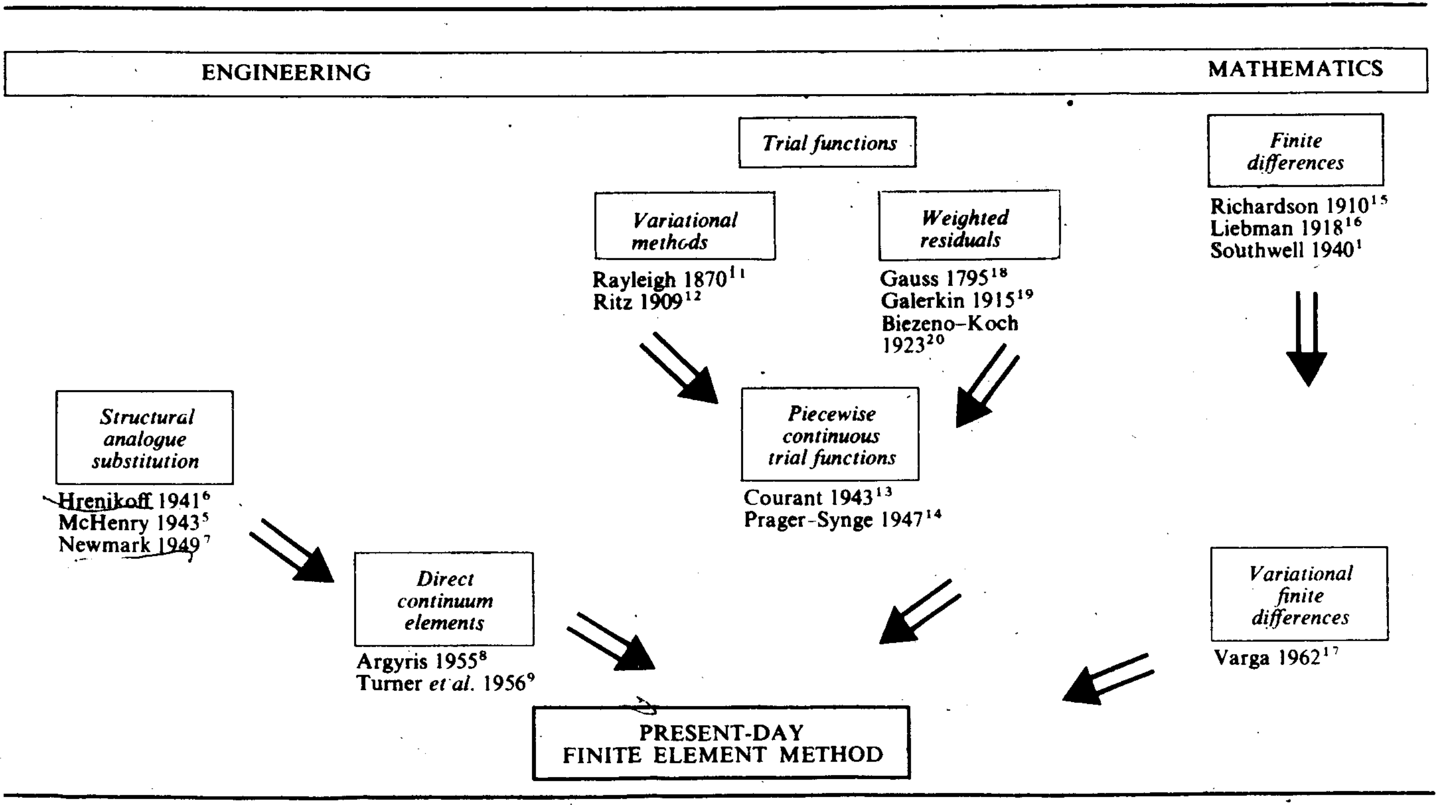


Fig. 1.1 A typical structure built up from interconnected elements

TABLE 1.1
FAMILY TREE OF FINITE ELEMENT METHODS



Let Fig. 1.1 represent a two-dimensional structure assembled from individual components and interconnected at the nodes numbered 1 to n. The joints at the nodes, in this case, are pinned so that moments cannot be transmitted.

As a starting point it will be assumed that by separate calculation, or for that matter from the results of an experiment, the characteristics of each element are precisely known. Thus, if a typical element labelled (1) and associated with nodes 1, 2, 3 is examined, the forces acting at the nodes are uniquely defined by the displacements of these nodes, the distributed loading acting on the element (p), and its initial strain. The last may be due to temperature, shrinkage, or simply an initial 'lack of fit'. The forces and the corresponding displacements are defined by appropriate components (U, V and u, v) in a common co-ordinate system.

Listing the forces acting on all the nodes (three in the case illustrated) of the element (1) as a matrix† we have

$$\mathbf{q}^1 = \begin{Bmatrix} \mathbf{q}_1^1 \\ \mathbf{q}_2^1 \\ \mathbf{q}_3^1 \end{Bmatrix}; \quad \mathbf{q}_1^1 = \begin{Bmatrix} U_1 \\ V_1 \end{Bmatrix}, \text{ etc.} \tag{1.1}$$

and for the corresponding nodal displacements

$$\mathbf{a}^1 = \begin{Bmatrix} \mathbf{a}_1^1 \\ \mathbf{a}_2^1 \\ \mathbf{a}_3^1 \end{Bmatrix}; \quad \mathbf{a}_1^1 = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}, \text{ etc.} \tag{1.2}$$

Assuming linear elastic behaviour of the element, the characteristic relationship will always be of the form

$$\mathbf{q}^1 = \mathbf{K}^1\mathbf{a}^1 + \mathbf{f}_p^1 + \mathbf{f}_{\varepsilon_0}^1 \tag{1.3}$$

in which $\mathbf{f}_p^1$ represents the nodal forces required to balance any distributed loads acting on the element, and $\mathbf{f}_{\varepsilon_0}^1$ the nodal forces required to balance any initial strains such as may be caused by temperature change if the nodes are not subject to any displacement. The first of the terms represents the forces induced by displacement of the nodes.

Similarly, the preliminary analysis or experiment will permit a unique definition of stresses or internal reactions at any specified point or points of the element in terms of the nodal displacements. Defining such stresses by a matrix $\boldsymbol{\sigma}^1$ a relationship of the form

$$\boldsymbol{\sigma}^1 = \mathbf{S}^1\mathbf{a}^1 + \boldsymbol{\sigma}_p^1 + \boldsymbol{\sigma}_{\varepsilon_0}^1 \tag{1.4}$$

† A limited knowledge of matrix algebra will be assumed throughout this book. This is necessary for reasonable conciseness and forms a convenient book-keeping form. For readers not familiar with the subject a brief appendix is included in which sufficient principles of matrix algebra are given to follow intelligently the development. Matrices (and vectors) will be distinguished by bold print throughout.

is obtained in which the last two terms are simply the stresses due to the distributed element loads or initial stresses respectively when no nodal displacement occurs.

The matrix $\mathbf{K}^e$ is known as the element stiffness matrix and the matrix $\mathbf{S}^e$ as the element stress matrix for an element (e).

Relationships Eqs. (1.3) and (1.4) have been illustrated on an example of an element with three nodes and with the interconnection points capable of transmitting only two components of force. Clearly, the same arguments and definitions will apply generally. An element (2) of the hypothetical structure will possess only two points of interconnection, others may have quite a large number of such points. Similarly, if the joints were considered as rigid, three components of generalized force and of generalized displacement would have to be considered, the last corresponding to a moment and a rotation respectively. For a rigidly jointed, three-dimensional structure the number of individual nodal components would be six. Quite generally therefore—

$$\mathbf{q}^e = \begin{Bmatrix} \mathbf{q}_1^e \\ \mathbf{q}_2^e \\ \vdots \\ \mathbf{q}_m^e \end{Bmatrix} \quad \text{and} \quad \mathbf{a}^e = \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{Bmatrix} \tag{1.5}$$

with each $\mathbf{q}_i$ and $\mathbf{a}_i$ possessing the same number of components or *degrees of freedom*.

The stiffness matrices of the element will clearly always be square and of the form

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{ii}^e & \mathbf{K}_{ij}^e & \dots & \mathbf{K}_{im}^e \\ \vdots & \vdots & & \vdots \\ \mathbf{K}_{mi}^e & \dots & \dots & \mathbf{K}_{mm}^e \end{bmatrix} \tag{1.6}$$

in which $\mathbf{K}_{ii}^e$, etc., are submatrices which are again square and of the size $l \times l$, where l is the number of force components to be considered at the nodes.

As an example, the reader can consider a pin-ended bar of a uniform section A and modulus E in a two-dimensional problem shown in Fig. 1.2. The bar is subject to a uniform lateral load p and a uniform thermal expansion strain

$$\varepsilon_0 = \alpha T.$$

If the ends of the bar are defined by the co-ordinates x_i, y_i and x_n, y_n its length can be calculated as

$$L = \sqrt{\{(x_n - x_i)^2 + (y_n - y_i)^2\}}$$

and its inclination from the horizontal as

$$\alpha = \tan^{-1}\frac{y_n - y_i}{x_n - x_i}.$$

Only two components of force and displacement have to be considered at the nodes.

The nodal forces due to the lateral load are clearly

$$\mathbf{f}_p^e = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix}_p = \begin{Bmatrix} -\sin\alpha \\ \cos\alpha \\ -\sin\alpha \\ \cos\alpha \end{Bmatrix} \cdot \frac{pL}{2}$$

and represent the appropriate components of simple beam reactions, $pL/2$. Similarly, to restrain the thermal expansion ε_0 an axial force $(E\alpha TA)$ is needed, which gives the components

$$\mathbf{f}_{\varepsilon 0}^e = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix}_{\varepsilon 0} = \begin{Bmatrix} -\cos\alpha \\ -\sin\alpha \\ \cos\alpha \\ \sin\alpha \end{Bmatrix} (E\alpha TA).$$

Finally, the element displacements

$$\mathbf{a}^e = \begin{Bmatrix} u_i \\ v_i \\ u_n \\ v_n \end{Bmatrix}$$

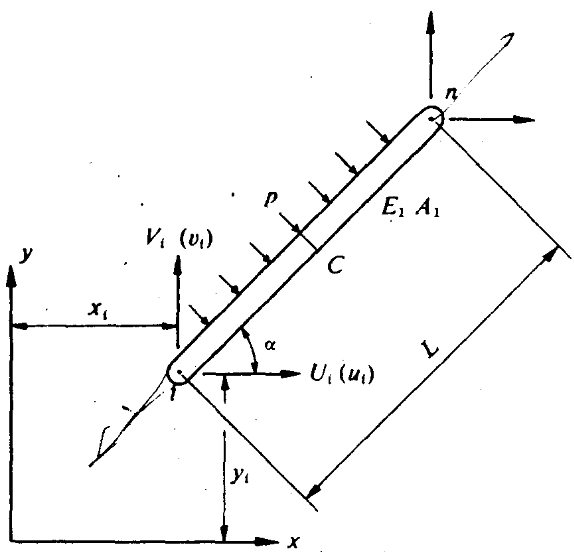


Fig. 1.2 A pin-ended bar