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Some Topics on Value Distribution and Differentiability in Complex and P-adic Analysis

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(复与P进位分析中有关值分布及微分性的一些论题)

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Preface

Complex analysis and p-adic analysis are two closely linked, old branches of mathematics that have played a prominent role in the development of modern mathematics. The recent advancements, new results and applications of complex analysis and p-adic analysis are rather extensive. In this volume, the focus centers on those topics which pertain to two intrinsic properties of analytic functions; namely,

1. Value distribution and
2. Complex differentiability.

In the contemporary parlance, value distribution theory is also known as *Nevanlinna theory*; a theory that R. Nevanlinna introduced in the 1920s in conjunction with his investigation of the growth of entire and meromorphic functions. Since then, analogous theories have been propounded and developed for algebroid functions, subharmonic functions, holomorphic and meromorphic mappings. These, combined with geometric analysis, led to extensions and generalizations of Picard's Little Theorem for holomorphic and meromorphic mappings over complex and p-adic fields.

Differentiability is a fundamental notion which has important ramifications in analysis. It has led to many significant concepts such as the Fréchet and Gâteaux differentials and subdifferentials and it has also provided the framework, for example, via the Cauchy-Riemann system, for studying certain partial differential equations. With the aid of the universal Cauchy kernel, the theory of generalized analytic or pseudo-holomorphic functions (in the sense of I. N. Vekua and L. Bers, respectively) allows one to solve, by complex methods, general (uniformly elliptic) linear or non-linear first-order systems.

During the past several decades, value distribution theory (of complex or p-adic meromorphic functions) has been the main tool in the investigation of factorization theory, value sharing (unicity) or growth of meromorphic functions, and the existence of an admissible meromorphic solution of a functional equation over a constant field or a small function field. In addition, several international conferences and the numerous research articles published on these topics further underscore the importance of value distribution theory in several branches of mathematics.

The main aim of the present collection is two-fold: 1. To provide a forum for timely surveys, new results, techniques, generalizations, extensions, and new trends or applications in areas related to value distribution and differentiability of complex or p-adic meromorphic functions. 2. To include (in a separate part), as an enhancement of the collection, English translations of relevant contributions that were written in Russian around 1990s and were heretofore available only in Russian as internal publications of the Tbilisi State University.

The book consists of twenty chapters arranged in three parts, Part I, II and III. Part I consists mainly of six surveys about the second main theorem in generalized parabolic manifolds; uniqueness polynomials; p-adic value distribution; recent developments of Petrenko's theory of growth of meromorphic functions; linear operators, Fourier transforms and the Riemann ξ-function (a function closely related to the Riemann zeta-function). Part I concludes with two research notes; one is about the hyperbolic hypersurfaces of lower degrees and the other is about the admissible solutions of functional equations of diophantine type.

Using the techniques of Bers' pseudo-holomorphic functions, Part II starts with a contribution constructing complete systems of solutions to the stationary Schrödinger equation. Moreover, one chapter of Part II applies value distribution theory to subfunctions of the time-independent Schrödinger operator. Generally speaking, Part II and Part III deal with applications of complex analysis to differential equations. This includes also ordinary differential equations (in the complex plane), as Chapter 12 shows. Chapter 13 applies value distribution theory also to partial differential equations. Using the Cauchy-Riemann operator of Clifford analysis, complex methods can also be applied to higher dimensional problems. Whereas the usual basis vectors of Clifford analysis satisfy the relations

$$e_j^2 = \mp 1, \ e_i e_j + e_j e_i = 0, \ i, j = 1, \cdots, n, \ i \neq j,$$

the generalized Clifford algebras depend on parameters. Their basis vectors satisfy the relations

$$e_j^2 = -\alpha_j, \ e_i e_j + e_j e_i = 2\gamma_{ij}, \ i, j = 1, \cdots, n, \ i \neq j,$$

(see Chapter 14 in Part II). The Cauchy-Riemann operator in such parameter-dependent Clifford algebras leads to more general systems of partial differential equations. Also G. C. Wen's hyperbolic numbers j (with $j^2 = +1$) are included in this concept (see the Chapters 15 and 16).

Part III deals with boundary value problems for generalized analytic vectors in the complex plane. However, in light of the ongoing research, it is expected that the techniques developed in Part III will also be useful when investigating systems of partial differential equations in higher dimensions.

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Contents

Part I

Value Distribution of Complex and P-adic Functions

Chapter 1

The Second Main Theorem on Generalized Parabolic Manifolds

by Pit-Mann Wong and Philip P. W. Wong

Introduction

The Second Main Theorem for holomorphic maps from the complex Euclidean space into the complex projective space was first established by Ahlfors [1] in 1941 (see also Cartan [4], Weyl and Weyl in [28]). The theory was greatly extended in 1973 by Griffiths-King [15] where the domain is allowed to be affine algebraic varieties. Stoll [26] (see also Wong [29]) further extended the result to the case of meromorphic maps on parabolic manifolds in 1977. Variations of these results (mostly in the 1980 and the 1990) can be found in Chen [7, 8] (in which references for the contribution of Nochka can be found) Lang [17], Wong [31], Wong [32], Cherry-Lang [9], Stoll-Wong [27], Hu-Yang [16]. Most recently, there are the articles by Li [19] and Ru [22]. Indeed the literature is so vast that it would be impossible to give an exhaustive list. The readers are encouraged to look into the short list here for further references.

In some of the recent works in hyperbolic geometry (see [11], [12], [3], [5], [6], [33], [34], [35] and [36]) the projectivization of a holomorphic vector bundle, $\mathbb{P}(E)$, over a complex manifold plays a very important role in establishing a certain type of Schwarz Lemma (see [6, section 5], [33] and [35, section 9]). This means that, under some reasonable assumptions, one should be able to establish a Second Main Theorem (abbrev. SMT) for meromorphic maps from $\mathbb{P}(E)$ into a complex projective space. However, $\mathbb{P}(E)$ is NOT parabolic, hence none of the known results in the literature is applicable. The concept of parabolicity is based on the existence of a non-negative plurisubharmonic exhaustion function τ such that $\phi = \log \tau$ satisfies the complex Monge-Amperé Equation:

$$(dd^c\phi)^m \equiv 0$$

on $M_* = M \setminus \{\tau = 0\}, m = \dim M$. It is well known that the equation above is equivalent to the equation:

$$\sum_{a,b} \tau^{\bar{b}a} \tau_{a\bar{b}} = \tau.$$

The readers are referred to [30], [21], [18], [34] for the theory (and further references) concerning this equation.

In this paper we introduce (see section 2) the concept of p-parabolicity. A Kähler manifold (M, ω) is said to be p-parabolic $(1 \leqslant p \leqslant m)$ if there exists a non-negative plurisubharmonic exhaustion function τ such that $\phi = \log \tau$ satisfies the equation:

$$(dd^c \phi)^p \wedge \omega^{m-p} = 0,$$

where ω is the Kähler metric. Note that m-parabolicity is the classical notion of parabolicity. In section 2 we show that if E is a holomorphic vector bundle of rank $p \leqslant m$ over a Stein manifold, then $\mathbb{P}(E)$ is p-parabolic. A brief discussion of projectivized bundle can be found in section 2. The purpose of this article is to extend the main results in the references (these results are special cases, $p = m$, of our results) mentioned in the first paragraph of this introduction. There are currently much activities in Nevanlinna Theory and related topics, indeed words in the street indicate that a number of new results begin to surface by various people just as this article is completed.

In section 1 we show that the analogue of the classical Green-Jensen Formula and the First Main Theorem (abbrev. FMT) hold on a p-parabolic manifold M. These strongly suggested that a Second Main Theorem (abbrev. SMT), in the spirit of Ahlfors, should be possible. For this, it is necessary to construct the so called associate maps of a linearly non-degenerate meromorphic map $f : M \to \mathbb{P}^n$. If M is the Euclidean space these are the Wronskians of the derivatives of f. In the case of a parabolic manifold, these are defined by Stoll (see [25], [26], [27] and [31]), where the derivatives are defined via a "sufficiently general" global holomorphic form B of type $(m-1, 0)$. For example, on a Stein parabolic manifold, the existence of such a general form is guaranteed. Unfortunately, this is no longer valid for p-parabolic $(p \neq m)$ manifolds (e.g., the fibers of a projectivized bundle is a projective space which does not admit any non-trivial holomorphic form). For this reason, we extend Stoll's approach by using "sufficiently general" *meromorphic* global forms. This is carried out in section 3.

The two main ingredients in the proof of the SMT are (1) the Plücker formulas and (2) the Ahlfors-Stoll estimate. These are carried out in sections 5 and 6. The proof of these formulas are quite technical in higher dimensions, so we include in section 4 a simple proof in $\mathbb{P}^2$ (as well as some related results) as motivation. The final step of the proof of the SMT is carried out in section 7. It is necessary to compensate for the fact that the associate maps are defined via a *meromorphic*, rather than *holomorphic*, form B. The idea is to multiple the form B by a holomorphic section μ of some holomorphic line bundle $\mathcal{L}$ vanishing at the singular set of B (we cannot do this using holomorphic *function* as this would mean that we may choose B to be holomorphic in the first place). The SMT takes the following form (see Theorem 7.1):

$$N_{\mathrm{ram}f}(r,s) + N([\mu = 0]; r, s) + \sum_{j=1}^{q} m_0(A_j; r, s)$$
$$\cdot \leqslant \cdot (n+1)T_f(r,s) + T_1(\mathcal{L}; r, s) + \frac{n(n+1)}{2} R_p(r,s) + O(\log(rG(r)T_f(r))),$$

where $N([\mu = 0]; r, s])$ is the counting function,

$$T_1(\mathcal{L}; r, s) = \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} c_1(\mathcal{L}, h) \wedge (dd^c\tau)^p \wedge \omega^{m-p},$$

where $c_1(\mathcal{L}, h)$ is the first Chern form of $\mathcal{L}$ and $R_p(r,s)$ is the p-th curvature integral of the p-parabolic exhaustion function τ:

$$R_p(r,s) = \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} dd^c[\log A_p] \wedge (dd^c\tau)^p \wedge \omega^{m-p}$$

where A_p is defined by:

$$(dd^c\tau)^p \wedge \omega^{m-p} = A_p(\sqrt{-1})^m \, d\zeta^1 \wedge d\bar{\zeta}^1 \wedge \cdots \wedge d\zeta^m \wedge d\bar{\zeta}^m.$$

If $p = 1, A_1$ is the trace of the Levi form of τ with respect to the Kähler metric ω. In the case of parabolic manifolds, $p = m = \dim M, A_m = \det(\tau_{a\bar{b}})$ and $R_m(r,s)$ is denoted by $R_\tau(r,s)$ in the literature. Moreover, in the case of parabolic Stein manifolds, B is holomorphic and we may set $N([\mu = 0], r, s) = T_1(\mathcal{L}; r, s) = 0$ in the SMT above.

In the special case where E is an algebraic vector bundle over an affine algebraic manifold, the SMT for $\mathbb{P}(E)$ takes a very simple form (see Corollary 7.2):

$$N_{\mathrm{ram}f}(r,s) + \sum_{j=1}^{q} m_0(A_j; r, s) \cdot \leqslant \cdot (n+1)T_f(r,s) + O\Big(\log\big(rT_f(r)\big)\Big).$$

1.1 Monge-Amperé equations and generalized parabolic manifolds

Let M be a connected complex manifold of complex dimension m. A real-valued function ϕ is said to satisfy the complex homogeneous Monge-Amperé equation if

$$(dd^c\phi)^m \equiv 0, \tag{1.1}$$

and we say that the equation is *non-degenerate* at a point x if $(dd^c\phi)^{m-1} \not\equiv 0$ at x. Here $d^c = \sqrt{-1}(\bar{\partial} - \partial)$, hence $dd^c = 2\sqrt{-1}\partial\bar{\partial}$. For $m = 1, dd^c$ is simply the

Laplace operator and the solutions of (1.1) are the harmonic functions. Classically, a manifold M is said to be *parabolic* if there exists a plurisubharmonic exhaustion $\tau : M \to [0, \infty)$ of class C^4 such that $\phi = \log \tau$ is plurisubharmonic satisfying (1.1) on $M_* = M \setminus [\tau = 0]$. Parabolic manifolds are important in Nevanlinna theory due to the fact that a natural generalization of Nevanlinna's Second Main Theorem in one complex variable is, to a large extent, valid on such manifolds. An affine algebraic manifold (that is, a submanifold of $\mathbb{C}^N$ defined as common zeros of polynomials) is parabolic and, on such a manifold the SMT is very much a perfect extension of Nevanlinna's classical result. A parabolic exhaustion on an affine algebraic manifold may be constructed as follows (see [26] for further examples).

Example 1.1 An affine algebraic variety M of dimension m may be exhibited as a finite branched covering map $\pi : M \to \mathbb{C}^m$. For each $a \in \mathbb{C}^m$, let

$$\tau = \pi^{-1}(||z - a||^2) \tag{1.2}$$

be the pull-back of the standard Euclidean exhaustion on $\mathbb{C}^n$ and let $\phi = \log \tau$. Then τ is an exhaustion function on M such that:

(i) $dd^c\tau \geqslant 0$,

(ii) $dd^c\tau > 0$ outside the ramification divisor $R = [\det \pi_* = 0]$,

(iii) $dd^c\phi \geqslant 0$ and $(dd^c\phi)^m = 0$ on $M_* = M \setminus \{\tau = 0\}$ and the equation is non-degenerate on $M_* \setminus \pi(R)$. □

Recent results in Nevanlinna theory and complex hyperbolic geometry indicate that the classical theory of parabolic manifolds is not quite general enough. Indeed, much of the recent works on the theory of holomorphic maps into affine varieties make use of certain (projectivized) vector bundles (more generally, $\mathbb{C}^*$-bundles) over these varieties. The bundle space of a projectivized vector bundle over an affine algebraic variety is not parabolic in the sense defined above. For this reason, we begin by generalizing the concept of parabolicity.

Definition 1.2 A Kähler manifold (X, ω) of dimension m is said to be a *p-parabolic manifold*, $1 \leqslant p \leqslant m$, if there is a plurisubharmonic exhaustion ϕ such that

(i) $\{\phi = -\infty\}$ is a closed subset of strictly lower dimension,

(ii) ϕ is smooth outside $\{\phi = -\infty\}$ and

$$(dd^c\phi)^p \wedge \omega^{m-p} = 0 \tag{1.3}$$

on $X_* = X \setminus \{\phi = -\infty\}$ for some integer $1 \leqslant p \leqslant m$. The exhaustion ϕ is said to be a *k-parabolic exhaustion*. The equation is said to be *k-non-degenerate* at a point x if

$$(dd^c\phi)^{p-1} \wedge \omega^{m-p} \neq 0$$