Classical Electrodynamics

经典电动力学

Tung Tsang

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CLASSICAL ELECTRODYNAMICS

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Preface

Classical electrodynamics, together with classical mechanics, quantum mechanics and mathematical physics, form the central core of both undergraduate and graduate physics curriculum. The electrodynamics course is also required for all students in electrical engineering. At present, there are many textbooks available at the graduate level. Nevertheless, the typical two-semester electrodynamics course has been feared by many graduate students. The inherent nature of the subject (with many differential equations, complicated mathematics, Green's functions, boundary conditions, numerous vectors and even more numerous vector components, tensors, etc.) is very intimidating to many students.

The present book is designed to be a "user-friendly",compact and comprehensive two-semester classical electrodynamics textbook at the graduate level for both physicists and electrical engineers. The book should also be useful for other scientists and engineers, such as physical chemists and materials scientists. The "user-friendly" emphasis is on simple and direct solutions of electrodynamics problems without complications. Intuitive explanations have been used as much as possible. Numerous examples have been solved in the text. Many examples have important and useful applications. Attempts have been made to have simple and direct approaches with explanations on why a particular problem should be solved by a particular method. A few key diagrams with proper choices of coordinate systems or simple symmetry considerations can be helpful. Do not put a round peg into a square hole. Similarly, do not solve a spherical problem in Cartesian coordinates. If explanations are clear, then equations can be short (hopefully). Easy-to-understand reviews have been given for the required mathematics and the undergraduate level electrodynamics.There are many excellent mathematical physics textbooks (such as the various editions of Arfken) which can provide much more detailed discussions on the mathematics.

The book has followed the standard table of contents used by most electrodynamics texts. In Chap. 1, a brief review on mathematics and undergraduate electrodynamics is given for electrostatics, coordinate systems and vector formulas. Intuitive explanations are given for gradient, divergence, curl, divergence theorem and Stokes theorem. There are discussions on how to choose the proper coordinate systems, how to take advantage of the geometries, and on how to avoid the singularities. In Chap. 2, important mathematical concepts such as Fourier series and transform, Legendre polynomials, spherical harmonics, Bessel functions and relaxation methods are introduced in electrostatic applications. Emphasis is on basic concepts such as orthogonality. Similarly, matrices and tensors are introduced as applications to multipole moments.Many complications (such as associated Legendre polynomials) are avoided by symmetry arguments. Simple arguments are used for the dielectrics. Magnetostatics are introduced in Chap.3. Maxwell equations are introduced on page 87 in Chap. 4, followed by the applications to plane waves in Chap.

4 and 5, the applications to wave guides in Chap. 6, the applications to radiating systems in Chap. 7, and the applications to scattering and diffraction in Chap. 8. Detailed discussions have been given on many standard topics and important applications such as antennas, nonlinear media, fiber optics, X-ray scattering, etc. Complete solutions have been given to these problems without omitting steps between two equations. Relativity is introduced in Chap. 9. Emphasis is on the simplifications by using four-vectors and covariance. Relativistic effects are important for Chap. 10 and 11 which deal with relativistic dynamics and radiations from accelerated charges. Useful topics such as magnetic resonance, synchrotron radiation, bremsstrahlung, magnetic mirror and Van Allen radiation belt are also included. Spherical waves and method of partial waves are discussed in Chap. 12. These methods are commonly used in quantum mechanics in addition to electrodynamics. In Chap. 13-15, discussions are given on many important applications in plasma physics, laser and holography, and superconductivity. Hopefully, the book provides comprehensive yet easy-to-understand coverage of all the major topics in a graduate-level textbook. I have also tried to avoid the statement "It can be easily shown that···". About a dozen problems have been given at the end of each chapter. Hints are given to the more difficult problems. To keep the book relatively comprehensive, relatively short, easily readable (hopefully), and within managable size (about 400 pages) for a two-semester course, I have tried to be intuitive and to simplify the mathematics and algebra as much as possible. Gaussian units have been used throughout the book. The conversion between Gaussian and S. I. units is discussed in Appendix A. For convenience, a table of frequently used symbols is provided in Appendix B.

I wish to express my special gratitude to my wife, Dolly, for her constant encouragement and her tolerance for this book project. I am grateful to many students and colleagues at Department of Physics and Astronomy of Howard University over the years for numerous useful discussions. The editorial staffs at World Scientific Publishing Company have been very helpful and encouraging. No textbook was ever written in vacuum. I should like to give a collective acknowledgment to the authors of many excellent textbooks on classical electrodynamics and related subjects.

Tung Tsang

Washington, D. C., U. S. A., 1997

Contents

CHAPTER 1. INTRODUCTION AND REVIEW

1.1. Coulomb's Law, Electric Field and Potential

Electric charge is a characteristic property of the fundamental particles of physics. All matters are composed of protons, electrons and neutrons; the first two particles carry equal and opposite (+ and -) charges. An object will become charged when it carries an excess or deficiency of electrons.

Coulomb's law concerns the force between two charged particles at rest in vacuum. The results of many experiments may be summarized as follows: (a) Like charges repel each other, while opposite charges attract each other; (b) Force varies directly with the magnitude of each charge; (c) Force varies inversely with the square of the distance between the charges; and (d) Force is directed along the line joining the charges.

It is convenient to place the first charge Q at the origin of Cartesian coordinate system. The second charge q is placed at the point (x, y, z). The position vector (or radius vector) $\mathbf{r}$ of the charge q is $\mathbf{r} = x\mathbf{e}_x+y\mathbf{e}_y+z\mathbf{e}_z$, where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors along the x, y and z axes. The length of the vector $\mathbf{r}$ is $r=|\mathbf{r}|=(x^2+y^2+z^2)^{1/2}$. The symbol $\mathbf{e}$ will often be used for unit vectors. Vectors will be denoted by boldface letters.

From Coulomb's law, the force $\mathbf{F}$ on the charge q is

$$\mathbf{F} = \frac{qQ}{r^2}\mathbf{e}_r = \frac{qQ}{r^2}\,\frac{\mathbf{r}}{r} = \frac{qQ}{r^3}(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \tag{1.1}$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector along the direction of $\mathbf{r}$. The Gaussian system of units will be used throughout this book. The unit of charge is statcoulomb (statcoul). The units of length, mass and time are centimeter (cm), gram (gr) and second (s). The units of force and energy are dyne and erg. Discussion will be given in Appendix A on the conversion between Gaussian and S. I. (Systeme International or rationalized MKSA) units.

The small displacement from (x,y,z) to the point $(x+dx,y+dy,z+dz)$ will be denoted as the line element $d\mathbf{r} = \mathbf{e}_x dx+\mathbf{e}_y dy+\mathbf{e}_z dz$. When such a displacement is made on the charge q, the work done by the particle is:

$$\mathbf{F}\cdot d\mathbf{r} = (qQ/r^3)(xdx+ydy+zdz)=(qQ/r^3)(rdr) \tag{1.2}$$

This work can be identified with the decrease in the potential energy U. Hence we have $dU= -\mathbf{F}\cdot d\mathbf{r}$. Eq. (1.2) can be readily integrated to give

$$U=qQ/r \tag{1.3}$$

where the intgration constant is chosen so that U=0 at infinite distance r.

For the standard charge of q=1 statcoul, the standard force and standard potential energy will be defined as the electric field $\mathbf{E}$ (or "electric intensity vector") and electric potential (or "scalar potential") Φ. The unit for $\mathbf{E}$ is dyne/statcoul or statvolt/cm. The unit for Φ is statvolt. For a point charge Q located at the origin, then we have:

$$\mathbf{E} = Q\mathbf{e}_r/r^2 = Q\mathbf{r}/r^3 \quad \text{and} \quad \Phi = Q/r \tag{1.4}$$

at the point (x,y,z). This point is known as the "field point" (where the measurements of **E** and Φ are made).This point can also be denoted by the radius vector **r** (sometimes also written as **x**) from the origin to the field point. For a point charge q placed at the field point, we have $\mathbf{F}=q\mathbf{E}$ and $U=q\Phi$.

Example 1.1. In the S. I. units, the fundamental units of length, mass, time and charge are meter, kilogram, second and coulomb (coul). The Coulomb law is $\mathbf{F}=qQ\mathbf{e}_r/(4\pi\varepsilon_o r^2)$,where $\varepsilon_o=8.85\text{x}10^{-12}$. Let us define 1 statcoul= x coul and 1 statvolt= y volt. What are the conversion factors x and y?
Solution: (a) Let us place two 1 statcoul charges at the distance of 1 cm apart. The force is one dyne in Gaussian units. In S. I. units, $F=10^{-5}$ Newtons and $r=10^{-2}$ meters, Q=q=x. Hence:

$$10^{-5}=(x^2)/[(4\pi)(8.85\text{x}10^{-12})(10^{-2})^{-2}].$$

We get $x=1/(3\text{x}10^9)$ or 1 coul=$3\text{x}10^9$ statcoul.
(b) We have (statvolt)(statcoul)=erg, (coul)(volt)=joule and 1 erg=10^{-7} joule. Hence we have $xy=10^{-7}$. We get y=300, or 1 statvolt=300 volt.

Example 1.2. Let $\mathbf{A}=y\mathbf{e}_x+2x\mathbf{e}_y$. In the two-dimensional xy-plane, what are the values of the integral $\int\mathbf{A}\cdot d\mathbf{r}$ from the points (0,0) to (1,1) for: (a) along the straight line y=x, (b) along the parabola $y=x^2$?
Solution: Since $d\mathbf{r}=\mathbf{e}_x dx+\mathbf{e}_y dy+\mathbf{e}_z dz$, we have $\mathbf{A}\cdot d\mathbf{r} = ydx+2xdy$.
(a) Replacing y by x, we get: $\int\mathbf{A}\cdot d\mathbf{r} = \int(xdx+2xdx) = 3/2$.
(b) Replacing y by x^2, we get $\int\mathbf{A}\cdot d\mathbf{r} = \int(x^2dx+4x^2dx) = 5/3$.
In both (a) and (b), the integration limits are x=0 to x=1. It is clear that the results are different for the two paths of integration. In general, we have $\int\mathbf{A}\cdot d\mathbf{r}\neq 0$ for integration around closed loops.

If $\int\mathbf{A}\cdot d\mathbf{r} = 0$ for any closed path of integration, then it is necessary to have $\mathbf{A}\cdot d\mathbf{r}=d\psi$, where $\psi(x,y,z)$ is a well-defined function of the position. It is necessary to have $\mathbf{A}\cdot d\mathbf{r}$ as a total differential. Since $\mathbf{A}=\mathbf{e}_xA_x+\mathbf{e}_yA_y+\mathbf{e}_zA_z$ and $d\mathbf{r}=\mathbf{e}_x dx+\mathbf{e}_y dy+\mathbf{e}_z dz$, we have:

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz = \mathbf{A}\cdot d\mathbf{r} = A_x dx + A_y dy + A_z dz \tag{1.5}$$

It follows that $A_x=\partial\psi/\partial x$, etc., and

$$\mathbf{A} = \mathbf{e}_x\frac{\partial\psi}{\partial x} + \mathbf{e}_y\frac{\partial\psi}{\partial y} + \mathbf{e}_z\frac{\partial\psi}{\partial z} \tag{1.6}$$

In Cartesian coordinates, we introduce the gradient operator

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{1.7}$$

Then we have $\mathbf{A}=\nabla\psi$ and

$$d\psi = (\nabla\psi)\cdot d\mathbf{r} \tag{1.8}$$

Eq. (1.8) is used as the general definition of the gradient.

From eqs. (1.1) and (1.2), we get

$$\mathbf{E} = -\nabla\Phi \tag{1.9}$$

It follows that $\oint \mathbf{E}\cdot d\mathbf{r}=0$ and $\int_{A\to B} \mathbf{E}\cdot d\mathbf{r} = -(\Phi_B-\Phi_A)$. This is known as the "conservative field". For a particle with charge q, we have $\mathbf{F}=q\mathbf{E}= -\nabla U$ and $U=q\Phi$.

Conversely, in example 1.2, $\oint \mathbf{A}\cdot d\mathbf{r} \neq 0$, and the field is non-conservative. It is impossible to define the function ψ such that $\mathbf{A} =\nabla\psi$ in example 1.2.

When several charges ($Q_1,Q_2,\cdots$) are present, then the total electric field **E** is the vector sum of individual electric fields ($\mathbf{E}_1,\mathbf{E}_2,\cdots$). This is known as the "linear superposition principle". However, it is no longer possible to place all the charges at the origin. We will still denote the field point as **r**. The position of charge Q_i (the source point) will be denoted as $\mathbf{r}_i'$. The vector $\mathbf{r}-\mathbf{r}_i'$ from the source point to the field point will be written as $\mathbf{R}_i$. Then eq. (1.4) may be generalized as:

$$\mathbf{E}(\mathbf{r}) = \sum_i Q_i\mathbf{R}_i/R_i^3 = \sum_i Q_i(\mathbf{r}-\mathbf{r}_i')/|\mathbf{r}-\mathbf{r}_i'|^3$$

$$\Phi(\mathbf{r}) = \sum_i Q_i/R_i = \sum_i Q_i/|\mathbf{r}-\mathbf{r}_i'| \tag{1.10}$$

For continuous distributions of charges, then Q_i may be replaced by $\rho_e(\mathbf{r}')d^3x'$, where ρ_e is the electric charge density (in statcoul/cm^3) at the source point **r'** and $d^3x'=dx'dy'dz'$ is the volume element (sometimes also written as $d^3\mathbf{r}'$) as shown in Fig. 1.1. Hence we have

$$\mathbf{E}(\mathbf{r}) = \int \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\rho_e(\mathbf{r}')d^3x' \qquad \Phi(\mathbf{r}) = \int \frac{\rho_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3x' \tag{1.11}$$

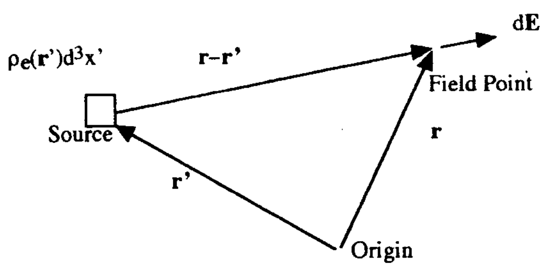

Fig. 1.1. Geometry for eq. (1.11)

In eq. (1.10), we sum over individual discrete charges. In eq. (1.11), the summation is replaced by the integration over the continuous charge distributions.

The magnitude of electric charge is 4.8×10^{-10} statcoul = 1.6×10^{-19} coul for both protons and electrons. Most matters are mixtures with nearly equal numbers of protons and electrons. If there is even a very slight unbalance in the numbers, the Coulomb's law force would be incredibly large. If two persons have 0.001% excess electrons and are standing at arms length apart, the repulsive force would be far greater than the combined weight of all the naval and merchant ships of the entire world!

1.2. Gauss Law

We will begin our discussion of Gauss law with the introduction of the concept of solid angle (in steradians).

In two dimensions, angles are measured in radians. We draw a circle of radius r around the angle. Then the angle (in radians) is defined as s/r, where s is the circular arc length. Similarly, the solid angle may be defined as $d\Omega=da_s/r^2$, where da_s is the spherical surface element shown as shaded area in Fig. 1.2(a), and r is the radius of the sphere. In general, we use any surface which may not be spherical. The area element is defined as $\mathbf{da}=\mathbf{n}da$, where da is the magnitude of the area element and $\mathbf{n}$ is the outward-pointing unit vector perpendicular to the area. From Fig. 1.2(a), it can be seen that $da_s=(da)\cos\theta$ and $\mathbf{e_r}\cdot\mathbf{n}=\cos\theta$. Hence we have $da_s=\mathbf{e_r}\cdot\mathbf{da}$ and $d\Omega=\mathbf{e_r}\cdot\mathbf{da}/r^2$. ($\theta$ is angle between $\mathbf{e_r}$ and $\mathbf{n}$)

For convenience, we will consider only one point charge Q at the origin enclosed by an arbitrary egg-shaped surface S as shown in Fig. 1.2(b). From eq.(1.4),we have $\mathbf{E}=Q\mathbf{e_r}/r^2$, hence $\mathbf{E}\cdot\mathbf{da}=Qd\Omega$. On integration over the entire surface S, we get:

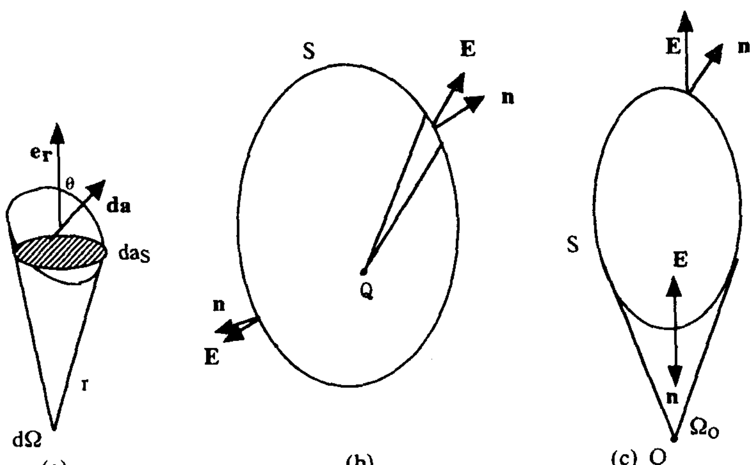

Fig. 1.2. Gauss law. (a) Solid angle dΩ, spherical surface element da_S and area element **da** for arbitrary surface S; (b) Charge Q inside closed surface S; (c) Charge Q outside S.

$$\int_S \mathbf{E}\cdot d\mathbf{a} = Q\int_S d\Omega = Q(4\pi) \quad \text{for Q inside the surface S} \tag{1.12}$$

since $\int_S d\Omega = 4\pi r^2/r^2 = 4\pi$ steradians. We note that $\mathbf{E}\cdot d\mathbf{a}$ is positive for the entire surface S. We also note that the value of the integral is independent of the shape of the surface S. As a simple analogy, we can consider the charge Q as a point source of light and **E** as the light emitted by the source (the light intensity is also inversely proportional to r^2). The integral in eq. (1.12) is the total light emitted by the source which will be captured by any surface S enclosing the source.

We will consider a point charge Q at the origin again. However, the surface S does not enclose the charge Q as shown in Fig. 1.2(c). Over the top side of surface S, $\mathbf{E}\cdot d\mathbf{a}$ is positive, hence $\int d\Omega = \Omega_0$, where Ω_0 is the solid angle of the tangential cone. Over the bottom side of S, the vectors **E** and **n** are in opposite directions, hence $\mathbf{E}\cdot d\mathbf{a}$ is negative, and $\int d\Omega = -\Omega_0$. It follows that $\int_S d\Omega = 0$ over the entire closed surface S:

$$\int_S \mathbf{E}\cdot d\mathbf{a} = 0 \quad \text{for Q outside the surface S} \tag{1.13}$$

In the light source analogy, the light is simply passing through the surface S, entering on one side while exiting on the other side, hence no light is captured.

Eqs. (1.12) and (1.13) are the Gauss law for a single point charge. For several point charges, we have

$$\int_S \mathbf{E}\cdot d\mathbf{a} = 4\pi \Sigma_i Q_i \tag{1.14}$$

where the summation is over all charges inside the closed surface S. For continuous charge distributions, the summation is replaced by integration. The Gauss law is

$$\int_S \mathbf{E}\cdot d\mathbf{a} = 4\pi \int_V \rho_e(\mathbf{r}) d^3x \tag{1.15}$$

where V is the volume inside the closed surface S. The surface integral $\int_S \mathbf{E}\cdot d\mathbf{a}$ is known as the electric flux through the surface S.

1.3. Divergence Theorem

Divergence is the net outflow rate of a vector field. Let us consider fluids flowing at the velocity $\mathbf{u}(x,y,z) = u_x\mathbf{e}_x + u_y\mathbf{e}_y + u_z\mathbf{e}_z$ throughout the space as shown in Fig. 1.3(a). The net outflow rate from the small cube is $\int \mathbf{u}\cdot d\mathbf{a}$, where the surface integral is taken over the three pairs of opposite faces of the cube. We will start with the pair of shaded faces perpendicular to the x-axis. For the left face, $d\mathbf{a} = -\mathbf{e}_x dydz$, hence we have $\mathbf{u}\cdot d\mathbf{a} = -u_x(x)dydz$. For the right face, $d\mathbf{a} = \mathbf{e}_x dydz$, hence we have $\mathbf{u}\cdot d\mathbf{a} = u_x(x+dx)dydz$.For the pair of faces,

$$\int \mathbf{u}\cdot d\mathbf{a} = [u_x(x+dx) - u_x(x)]dydz = [\frac{\partial u_x}{\partial x} dx]dydz = \frac{\partial u_x}{\partial x} d^3x \tag{1.16}$$

where $d^3x = dxdydz$ is the volume of the cube.Combining all six faces (three pairs of opposite faces), we get:

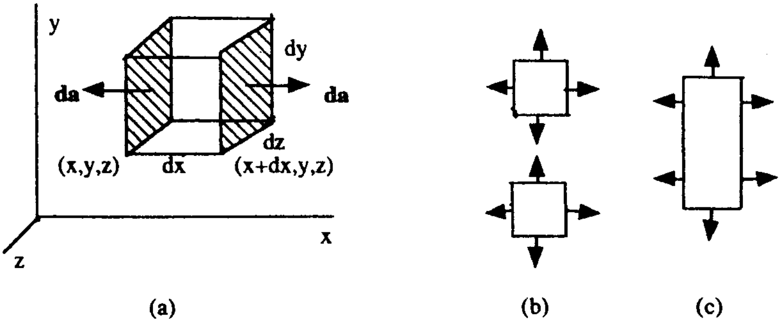

Fig. 1.3. (a) Outflow from two opposite faces of the small cube dxdydz; (b) Outflow from two neighboring small cubes; (c) Outflows from the two cubes after the cancellations of the outflows from the common interior surface.

$$\int \mathbf{u}\cdot d\mathbf{a} = (\nabla\cdot\mathbf{u})d^3x \qquad (1.16)$$

where:

$$\nabla\cdot u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \qquad (1.17)$$

$\nabla\cdot\mathbf{u}$ is known as the divergence of $\mathbf{u}$. It is the net rate of outflow per unit volume (flux) from the point (x,y,z) in space. The operator ∇ is defined by eq. (1.7).

In Fig. 1.3(b), we have shown the outflow from two neighboring small cubes. The outflows on the common surface of the two cubes will cancel each other. The total outflow from the two cubes is shown in Fig. 1.3(c) after the cancellation.

For a finite volume V, the total outflow will eventually come out on the outer surface S of the volume V. All the outflows on the interior surfaces will cancel out. Hence eq. (1.16) may be generalized to the case of finite volume:

$$\int_V (\nabla\cdot\mathbf{u})d^3x = \int_S \mathbf{u}\cdot d\mathbf{a} \qquad (1.18)$$

This is the divergence theorem.

In electrostatics, we may combine the Gauss law with the divergence theorem. By choosing $\mathbf{u}$ as the electric field $\mathbf{E}$, the left side of eq. (1.15) (Gauss law) can be written as:

$$\int_S \mathbf{E}\cdot d\mathbf{a} = \int_V (\nabla\cdot\mathbf{E})d^3x = \int_V 4\pi\rho_e d^3x$$

Since the volume V can have any arbitrary shape, it follows that

$$\nabla\cdot\mathbf{E} = 4\pi\rho_e \qquad (1.19)$$

This is the differential form of Gauss law for electrostatics in vacuum.

It is often easier to work with the potential Φ (scalar function with only one component) rather than the electric field $\mathbf{E}$ (vector with three components). Combining eq. (1.19) with $\mathbf{E}=-\nabla\Phi$ (eq. 1.9), we get

$$\nabla\cdot(\nabla\Phi) = -4\pi\rho_e \qquad (1.20)$$

The divergence of the gradient is known as the Laplacian operator ∇^2:

$$\nabla\cdot(\nabla\Phi)=\nabla^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2} \tag{1.21}$$

Eq. (1.20) is known as the Poisson equation. When $\rho_e=0$, then we have

$$\nabla^2\Phi=0 \tag{1.22}$$

Eq. (1.22) is known as the Laplace equation.

1.4. Curl and Stokes Theorem

We will return to the fluid flow problem of the last section and consider the circulation around a small rectangle in the xy-plane as shown in Fig.1.4(a). Area of the small rectangle is dxdy. The normal is along the z-direction. Hence the area element of the rectangle is $\mathbf{da}=\mathbf{e_x}dxdy$. The fluid velocity is denoted as $\mathbf{u}$. The curl is denoted as $\nabla\times\mathbf{u}$ and is defined as the net circulation per unit area around the closed path as shown by the arrows:

$$(\nabla\times\mathbf{u})\cdot\mathbf{da}=(\nabla\times\mathbf{u})_z dxdy=\Sigma\,\mathbf{u}\cdot\mathbf{dr} \tag{1.23}$$

Starting from the lower left corner and adding up the four sides, we get

$$\Sigma\,\mathbf{u}\cdot\mathbf{dr}=\mathbf{u}(y)\cdot\mathbf{e_x}dx+\mathbf{u}(x+dx)\cdot\mathbf{e_y}dy+\mathbf{u}(y+dy)\cdot(-\mathbf{e_x}dx)+\mathbf{u}(x)\cdot(-\mathbf{e_y}dy)$$
$$=u_x(y)dx+u_y(x+dx)dy-u_x(y+dy)dx-u_y(x)dy.$$

By combining the first term with the third term and by combining the second term with the fourth term, we get the z-compnent of the curl as:

$$(\nabla\times\mathbf{u})_z dxdy=[(-\partial_y u_x)dy]dx+[(\partial_x u_y)dx]dy \tag{1.24}$$

where we have introduced the more compact notation

$$\partial_x=\frac{\partial}{\partial x},\quad \partial_y=\frac{\partial}{\partial y};\quad \partial_z=\frac{\partial}{\partial z} \tag{1.25}$$

From eq. (1.24), we get $(\nabla\times\mathbf{u})_z=\partial_x u_y-\partial_y u_x$. The x and y-components can be obtained by similar calculations. The complete result for $\nabla\times\mathbf{u}$ can be expressed in the determinant form

$$\nabla\times\mathbf{u}=\begin{vmatrix}\mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z}\\ \partial_x & \partial_y & \partial_z\\ u_x & u_y & u_z\end{vmatrix} \tag{1.26}$$

where the operator ∇ is defined in eq. (1.7).

In Fig. 1.4(b), we have shown the circulations from two neighboring small rectangles. The circulations on the common boundary between the two rectangles will cancel out. The total circulation from the two rectangles is shown in Fig. 1.4(c) after the cancellation. Over a surface S of finite size, the total circulation is $\int\mathbf{u}\cdot\mathbf{dr}$ around the boundary C of the surface S, since all interior circulations would cancel out. Eq. (1.23) may be generalized to:

$$\int_C \mathbf{u}\cdot\mathbf{dr}=\int_S(\nabla\times\mathbf{u})\cdot\mathbf{da} \tag{1.27}$$

This is Stokes theorem.

Combining eq. (1.26) with eq. (1.17) (the definition of the divergence operator), it can