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（影印版）67

S. P. Novikov　V. A. Rokhlin (Eds.)

Topology II
Homotopy and Homology, Classical Manifolds

拓扑学 II
同伦与同调，经典流形

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

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List of Editors, Authors and Translator

Editor-in-Chief

R.V. Gamkrelidze, Russian Academy of Sciences, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow; Institute for Scientific Information (VINITI), ul. Usievicha 20a, 125219 Moscow, Russia, e-mail: gam@ipsun.ras.ru

Consulting Editors

S. P. Novikov, Department of Mathematics, Institute for Physical Sciences and Technology, University of Maryland at College Park, College Park, MD 20742-2431, USA, e-mail: novikov@ipst.umd.edu

V. A. Rokhlin†

Authors

D. B. Fuchs, Department of Mathematics, University of California, Davis, CA 95616-8633, USA, e-mail: fuchs@math.ucdavis.edu

O. Ya. Viro, Department of Mathematics, Uppsala University, P.O. Box 480, 75106 Uppsala, Sweden, e-mail: oleg.viro@math.uu.se

Translator

C. J. Shaddock, 39 Drummond Place, Edinburgh EH3 6NR, United Kingdom

Preface*

Algebraic topology, which went through a period of intense development from the forties to the sixties of the last century, has now reached a comparatively stable state. A body of concepts and facts of general mathematical interest has been clearly demarcated, and at the same time the area of applications of topology has been significantly widened to include theoretical physics and a number of applied disciplines, as well as geometry and analysis.

The subject matter of the two parts of this volume can be characterized as "elementary topology". This term has a quite precise meaning and denotes those parts of topology in which only comparatively simple algebra is used. The most important topics in this volume are: homotopy groups, bundles, cellular spaces, homology, Poincare duality, characteristic classes, and Steenrod squares. In most cases proofs are omitted, but they are not difficult as a rule, and the reader can reconstruct them if desired, obtaining all the necessary ideas from the text. Thus the book may be regarded as the synopsis of a textbook on topology.

The textbook itself has been written only in part: we have in mind *Beginner's course in topology: geometric chapters* by D.B. Fuks and V.A. Rokhlin. In writing the present work we have used not only this book, but also the numerous drafts of its second part, on homology, work on which was broken off on the death of V.A. Rokhlin in December 1984.

It was originally intended that V.A. Rokhlin would be one of the authors of both parts of this volume (as well as of other volumes in the *Encyclopaedia of Mathematical Sciences*). He played an active part in preparing the detailed plan of this volume and in discussions of some of its key sections. While writing this book the authors have continually referred to his texts, both published and unpublished. Unfortunately for purely formal reasons V.A. Rokhlin cannot be considered to be our coauthor; indeed, we very much doubt that our text would meet with his approval. In spite of this, we dedicate this volume with gratitude to the memory of Vladimir Abramovich Rokhlin.

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* Publisher's note: This is the Preface to the Russian edition of Enc. Math. Sc. 24, thus referring only to Parts I and II of the present volume. For organisational reasons the second part of the Russian edition of Enc. Math. Sc. 12 was added to this volume as Part III.

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I. Introduction to Homotopy Theory

O.Ya. Viro, D.B. Fuchs

Translated from the Russian
by C.J. Shaddock

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Chapter 1
Basic Concepts

§1. Terminology and Notations

1.1. Set Theory.

In addition to the standard set-theoretical terminology and notations, whose use is unambiguous, we shall use the following.

If A is a subset of a set X, the inclusion of A in X may be regarded as the map defined by $x \mapsto x$. Notation: in : $A \to X$. If there is no ambiguity about A and X, we simply write in.

If A is a subset of X and B a subset of Y, then to each mapping $f : X \to Y$ such that $f(A) \subset B$, there corresponds the map $f|_{A,B} : A \to B$ defined by $x \mapsto f(x)$, called a *submap* of f. If there is no ambiguity about A and B, we may just write $f|$ instead of $f|_{A,B}$. If $B = Y$, then $f|_{A,B}$ is also called the *restriction* of f to A and denoted by $f|_A$.

The quotient (or factor) set of X under a partition S is denoted by X/S. The map $X \to X/S$ that takes each point to the element of the partition containing it is called the *projection*, denoted by pr.

If S and T are partitions of sets X and Y, and $f : X \to Y$ is a map that maps the elements of S to the elements of T, then there is a corresponding map $X/S \to Y/T$, taking an element A of S to the element of T that contains $f(A)$. This map is denoted by $f/S, T$, and is called the *quotient map* of f. In particular, it is defined when T is the partition into single points, and f is constant on the elements of S. Thus, to each map $f : X \to Y$ constant on the elements of a partition S of X, there corresponds a map $X/S \to Y$; it is denoted by f/S. If there is no ambiguity about S and T, we simply write $f/$ instead of $f/S, T$.

The *sum of a family of sets* $\{X_\mu\}_{\mu \in M}$ is the union of disjoint copies of the sets X_μ, that is, the set of pairs (x_μ, μ) such that x_μ is an element of the set X_μ. Notation: $\coprod_{\mu \in M} X_\mu$. The map of X_ν $(\nu \in M)$ into $\coprod_{\mu \in M} X_\mu$ defined by $x \mapsto (x, \nu)$ is denoted by in_ν Each family of maps $\{f_\mu : X_\mu \to Y_\mu\}_{\mu \in M}$ determines a map $\coprod_{\mu \in M} X_\mu \to \coprod_{\mu \in M} Y_\mu$ in a natural way; it is called the *sum of the maps* f_μ and denoted by $\coprod_{\mu \in M} f_\mu$. If M consists of the numbers $1, \dots, n$, then we write $X_1 \coprod \dots \coprod X_n$, $f_1 \coprod \dots \coprod f_n$ as well as $\coprod X_\mu$ and $\coprod f_\mu$.

The map $X_1 \times \dots \times X_n \to X_i$: $(x_1, \dots, x_n) \to x_i$ is called the ith *projection*, denoted by pr_i. If we have maps $f_1 : X_1 \to Y_1, \dots, f_n : X_n \to Y_n$, then there is a map $X_1 \times \dots \times X_n \to Y_1 \times \dots \times Y_n : (x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$, called the *product of the maps* $f_1, \dots, f_n$ and denoted by $f_1 \times \cdots \times f_n$

1.2. Logical Equivalence.

We shall use the expression "iff" to mean "if and only if".

1.3. Topological Spaces.

A. If A is a subset of a topological space X, then its interior will be denoted by $\operatorname{Int} A$, or more precisely $\operatorname{Int}_X A$, its closure by $\operatorname{Cl} A$, or $\operatorname{Cl}_X A$, and finally, its frontier, that is, $\operatorname{Cl} A \setminus \operatorname{Int} A$ by $\operatorname{Fr} A$, or $\operatorname{Fr}_X A$.

B. Our notations for the standard topological spaces will follows those of D.B. Fuchs in Part III of the present volume. In particular, the fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, the skew field of quaternions by $\mathbb{H}$, and the algebra of Cayley numbers by Ca. The corresponding n-dimensional spaces, that is, the n-fold products $\mathbb{R} \times \cdots \times \mathbb{R}$, $\mathbb{C} \times \cdots \times \mathbb{C}$, $\mathbb{H} \times \cdots \mathbb{H}$ and $\mathrm{Ca} \times \cdots \times \mathrm{Ca}$ are denoted by $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ and Ca^n. We regard $\mathbb{R}^n$ as a metric space with the distance between $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ defined as $[\sum_{i=1}^n (x_i - y_i)^2]^{1/2}$ The spaces $\mathbb{C}^n$, $\mathbb{H}^n$, and Ca^n can be naturally identified with $\mathbb{R}^{2n}$, $\mathbb{R}^{4n}$, and $\mathbb{R}^{8n}$, and in particular have natural metrics and topologies. The closed ball and sphere in $\mathbb{R}^n$ with centre $(0, 0, \ldots, 0)$ and radius 1 are called simply the *n-ball* and *$(n-1)$-sphere*, and denoted by D^n and S^{n-1}. In particular, d^0 is a point, S^0 a pair of points, and $S^{-1} = \emptyset$. The unit interval $[0,1] \subset \mathbb{R}$ is denoted by I. I^n denotes the *unit n-cube* $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | 0 \leq x_i \leq 1, i = 1, \ldots, n\}$; its frontier (in $\mathbb{R}^n$) is denoted by ∂I_n. Real projective n-dimensional space is denoted by $\mathbb{R}P^n$, complex by $\mathbb{C}P^n$, quaternionic projective space by $\mathbb{H}P^n$, and the Cayley projective line and plane by $\mathrm{Ca}P^1$ and $\mathrm{Ca}P^2$. Recall that $\mathbb{R}P^1$, $\mathbb{C}P^1$, $\mathbb{H}P^1$, and $\mathrm{Ca}P^1$ are canonically homeomorphic to the spheres S^1, S^2, S^4, and S^8. The real Grassmann manifolds are denoted by $\mathbb{R}G(m,n)$ or $G(m,n)$. By definition, $G(m,n)$ is the set of n-dimensional (vector) subspaces of the space $\mathbb{R}^{m+n}$. The manifold of oriented n-dimensional subspaces of $\mathbb{R}^{m+n}$ is denoted by $G_+(m,n)$. The complex Grassmann manifold of n-dimensional (complex vector) subspaces of $\mathbb{C}^{m+n}$ is denoted by $\mathbb{C}G(m,n)$. The corresponding quaternionic Grassmann manifold is $\mathbb{H}G(m,n)$.

C. The appearance of the symbol ∞ as a dimensional parameter denotes passage to the inductive limit. Thus $\mathbb{R}^\infty$ is the inductive limit of the sequence of spaces $\mathbb{R}^k$ with the natural inclusion $\mathbb{R}^k \to \mathbb{R}^{k+1} : (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0)$. The points of $\mathbb{R}^\infty$ may be interpreted as infinite sequences $(x_1, x_2, \ldots)$ of real numbers, in which only finitely many terms are non-zero. A topology is introduced into $\mathbb{R}^\infty$ by the rule: a set $U \subset \mathbb{R}^\infty$ is open if all the intersections $U \cap \mathbb{R}^n$ are open in the spaces $\mathbb{R}^n$. The symbols $\mathbb{C}^\infty$, $\mathbb{H}^\infty$, D^∞, S^∞, $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$, $\mathbb{H}P^\infty$, $G(\infty, n)$, $G(\infty, \infty)$ etc. are interpreted similarly. None of these spaces are metrizable.

1.4. Operations on Topological Spaces.

A. The sum $\coprod_{\mu \in M} X_\mu$ of a family of topological spaces is canonically provided with a topology: a subset of the sum is declared to be open if its inverse images under all the maps $\mathrm{in}_\nu : X_\nu \to \coprod_{\mu \in M} X_\mu$ are open. It is clear that each of the maps in_ν is an embedding and that the images $\mathrm{in}_\nu(X_\nu)$ are simultaneously open and closed in $\coprod X_\mu$. It is also clear that if $f_\mu : X_\mu \to Y_\mu$, $\mu \in M$, are continuous maps, then their sum $\coprod_{\mu \in M} f_\mu : \coprod_{\mu \in M} X_\mu \to \coprod_{\mu \in M} Y_\mu$ is continuous.

B. The product $X_1 \times \cdots \times X_n$ of topological spaces $X_1, \ldots, X_n$ is canonically provided with a topology: a basis for the open sets in $X_1 \times \cdots \times X_n$ consists of the sets $U_1 \times \cdots \times U_n \subset X_1 \times \cdots \times X_n$, where $U_1, \ldots, U_n$ are open sets in $X_1, \dots, X_n$. It is clear that the projections $\mathrm{pr}_i : X_1 \times \cdots \times X_n \to X_i$ are continuous open maps for any spaces $X_1, \dots, X_n$. If $Y, X_1, \dots, X_n$ are any sets whatever, then to each map $f : Y \to X_1 \times \cdots \times X_n$ there correspond the maps $\mathrm{pr}_i \circ f : Y \to X_i$, and for any given maps $f_i : Y \to X_i$, there exists a unique map $f : Y \to X_1 \times \cdots \times X_n$ with $\mathrm{pr}_i \circ f = f_i$. It is clear that if $Y, X_1, \dots, X_n$ are topological spaces, then f is continuous iff all the $\mathrm{pr}_i \circ f$ are continuous. It is also evident that the product $f_1 \times \cdots \times f : X_1 \times \cdots \times X_n \to Y_1 \times .. \times Y_n$ of continuous maps $f_1 : X_1 \to Y_1, \dots, f_n : X_n \to Y_n$ is continuous.

C. The quotient space X/S of a topological space X with respect to any partition S has a natural topology: a set is open if its inverse image under the map $\mathrm{pr} : X \to X/S$ is open. This natural topology is called the *quotient topology*, and the set X/S endowed with this topology is called the *quotient space of X with respect to the partition S*. The map $\mathrm{pr} : X \to X/S$ is clearly continuous.

In the special case when all the elements of S are points except for a single set A, the space X/S is called the *quotient of X by A* and denoted by X/A.

It follows from the definition of the quotient topology that if X and Y are any topological spaces with partitions S and T, and $f : X \to Y$ is a continuous map taking the elements of S into elements of T, then the map $f/S, T : X/S \to Y/T$ is continuous.

D. Let X and Y be topological spaces, A a subset of Y, and $\phi : A \to X$ be a continuous map. The quotient space of the sum $X \coprod Y$ with respect to the partition into one-point subsets of $\phi \coprod (Y \setminus A)$ and sets of the form $x \coprod \phi^{-1}(x)$ with $x \in X$ is denoted by $X \cup_\phi Y$; we say that it is obtained by *attaching Y to X by ϕ*. It is clear that the natural injection $X \to X \cup_\phi Y$ is a topological embedding. In the case when X is a point, attaching Y to X by ϕ: $A \to X$ is clearly equivalent to forming the quotient space Y/A.

E. The product of the interval $I = [0, 1]$ with the space X is called the *cylinder over X*. The subsets $X \times 0$ and $X \times 1$ of $X \times I$ are called its (lower and upper) *bases* (they are copies of X), and a subset of the form $x \times I$, $x \in X$, is called a *generator* (it is a copy of I). If all the points of the base $X \times 0$ are identified to each other, we obtain the *cone* $CX = X \times I/X \times 0$ *over* X. The cone CX has a *base*, usually identified with X – the image of the upper base of $X \times I$ – and a *vertex*, the point obtained from the lower base $X \times 0$. The images of the generators of the cylinder under the map $\mathrm{pr} : X \times I \to CX$ are called *generators* of the cone. If we take the quotient of the cone with respect to its base, we obtain the *suspension ΣX over X*; thus $\Sigma X = CX/X$. Alternatively we may say that ΣX is obtained as the quotient space of the cylinder $X \times I$ with respect to the partition whose elements are the bases $X \times 0$ and $X \times 1$ and the one-point subsets of $X \times (0, 1)$. The images of the bases are called the *vertices of the suspension*. The sets $\mathrm{pr}(x \times I)$ are the *generators of the suspension*. The set $\mathrm{pr}(X \times \frac{1}{2})$ is the *base of the suspension*, and is a copy of X. The suspension X may be regarded as two cones over X joined together by their bases. The joined bases form the base of the suspension. It is clear that CS^n and S^n are homeomorphic to D^{n+1} and S^{n+1}.

To each map $f : X \to Y$ there corresponds the map $f \times \mathrm{id} : X \times I \to Y \times I$ and its quotient maps $CX \to CY$ and $\Sigma X \to \Sigma Y$ are continuous if f is continuous. The map $(f \times \mathrm{id})/ : \Sigma X \to \Sigma Y$ is denoted by Σf and called the *suspension of the map* f.

F. It is convenient to regard the *join* $X * Y$ of the spaces X and Y as the union of the line segments joining each point of X to each point of Y. For example, the join of two segments lying on skew lines in $\mathbb{R}^3$ is a tetrahedron. A formal definition of the join is the following: it is obtained as the quotient of $X \times Y \times I$ with respect to the partition whose elements are the sets $x \times Y \times 0$ $(x \in X)$ and $X \times y \times 1$ $(y \in Y)$ and the points of $X \times Y \times (0, 1)$. The set $\mathrm{pr}(x \times y \times I) \subset X * Y$, $x \in X$, $y \in Y$, is called a *generator of the join*; it is just the segment joining $x \in X$ and $y \in Y$. X and Y themselves are embedded in $X * Y$ as follows: $X \to X * Y : x \mapsto \mathrm{pr}(x \times Y \times 0)$ and $Y \to X * Y : y \mapsto \mathrm{pr}(X \times y \times 1)$. Their images under these embeddings are called the *bases of the join*. The generators cover the join. Each of them is determined by the points of the base that they join. Two distinct generators can intersect only in a single point and this point can only lie in one of the bases. Equivalently the join $X * Y$ can be defined as $(X \coprod Y) \cup_\phi (X \times Y \times I)$, where ϕ is the map $X \times Y \times (0 \cup 1) \to X \coprod Y$ with $\phi(x, y, 0) = x$, $\phi(x, y, 1) = y$. The quotient space of $X * Y$ with respect to the partition consisting of the bases X, Y and the points of the complement $(X * Y) \setminus (X \cup Y)$ is the same as the suspension $\Sigma(X \times Y)$.

The operation $*$ (like $\times$) is commutative: there is an obvious canonical homeomorphism $Y * X \to X * Y$. It can be shown that for Hausdorff locally compact spaces the operation $*$ is associative, but this is not true in general. In fact, if X and Y are Hausdorff and locally compact, the maps $X * Y \to CX \times CY : \mathrm{pr}(x, y, t) \mapsto (\mathrm{pr}(x, 1 - t), \mathrm{pr}(y, t))$ is a toplogical embedding with image $\{(\mathrm{pr}(x, s), \mathrm{pr}(y, t)) \in CX \times CY | s + t = 1\}$. By repeating this construction in the case of locally compact Hausdorff spaces $X_1, \dots, X_n$, we can obtain a homeomorphism of $((X_1 * X_2) * \dots) * X_n$ onto the space

$$\{(\mathrm{pr}(x_1, t_1), \dots, \mathrm{pr}(x_n, t_n)) \in CX_1 \times \dots \times CX_n | t_1 + \dots + t_n = 1\}$$

and then use the associativity of the operation $\times$.

The join $X * D^0$ is canonically homeomorphic to the cone CX, and $X * S^0$ to the suspension ΣX. The join $X * S^k$ is canonically homeomorphic to the multiple suspension $\Sigma^{k+1} X$; in particular, $S^p * S^q$ is canonically homeomorphic to S^{p+q+1}.

To each pair of maps $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ there corresponds the map $(f_1 \times f_2 \times \mathrm{id}_I)/ : X_1 * X_2 \to Y_1 * Y_2$, which is continuous if f_1 and f_2 are continuous. It is denoted by $f_1 * f_2$.

G. Let $f : X \to Y$ be a continuous map. The result of attaching the product $X \times I$ to Y by the map $X \times 1 \to Y : (x, 1) \to f(x)$ is called the *mapping cylinder* of f, and denoted by $\mathrm{Cyl} f$. The subsets of $\mathrm{Cyl} f$ obtained from $X \times 0$ and Y are called the *lower and upper bases* of $\mathrm{Cyl} f$. The bases are related to X and Y by obvious canonical homeomorphisms and are usually identified with X and Y. The subsets of $\mathrm{Cyl} f$ obtained from $x \times I$ with $x \in X$ are called the *generators* of $\mathrm{Cyl} f$; they are canonically homeomorphic to I. There is a canonical map $\mathrm{Cyl} f \to Y$, taking each generator to its point of intersection with Y. It is clear that the composition of the inclusion $X \to \mathrm{Cyl} f$ and this map $\mathrm{Cyl} f \to Y$ is equal to f.

The result of attaching the cone CX to Y by the map f of its base is called the *mapping cone* of f, and denoted by $\mathrm{Con} f$. Clearly $\mathrm{Con} f = \mathrm{Cyl} f / X$. The subset of $\mathrm{Con} f$ obtained from Y is called the *base* of $\mathrm{Con} f$; it is obviously canonically homeomorphic to Y and is usually identified with Y.

If $Y = X$ and $f = \mathrm{id}_X$, then $\mathrm{Cyl} f$ is canonically homeomorphic to $X \times I$, and $\mathrm{Con} f$ to CX.

H. If X and Y are topological spaces, let $C(X, Y)$ denote the set of all continuous maps $X \to Y$. If $A_1, \dots, A_n$ are subsets of X, and $B_1, \dots, B_n$ subsets of Y, then $C(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$ denotes the subset of $C(X, Y)$ consisting of the maps ϕ for which $\phi(A_1) \subset B_1, \dots, \phi(A_n) \subset B_n$. The notation $\phi : (X, A_1, \dots, A_n) \to (Y, B_1, \dots, B_n)$ is used for such maps.

The set $C(X, Y)$ is endowed with the *compact-open topology* – the topology of uniform convergence on compact sets (that is, the topology with a basis of sets of the form $C(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$, where $A_1, \dots, A_n$ are compact and $B_1, \dots, B_n$ are open). As well as $C(X, Y)$ all the sets $C(X, A_1, \dots, A_n; Y, B_1, \dots, B_n)$ also become topological spaces.

It is clear that if X is a point, then $C(X, Y) = Y$; if X is a discrete space with n points (that is, a collection of n isolated points), then $C(X, Y) = Y \times \dots \times Y$ (n factors). This is the reason for denoting the space $C(X, Y)$ by Y^X.

Let X, Y, Z be topological spaces. To each continuous map $\phi : X \times Y \to Z$ there corresponds the continuous map $\phi^\vee : X \to C(X, Z)$ defined by $[\phi^\vee(x)](y) = \phi(x, y)$. It can be shown that the map $C(X \times Y, Z) \to C(X, C(Y, Z)) : \phi \mapsto \phi^\vee$ is continuous, and is a homeomorphism if Y is regular and locally compact. This relation between $C(X \times Y, Z)$ and $C(X, C(Y, Z))$ makes the notation Y^X for $C(X, Y)$ even more attractive: it takes the form of the equation $Z^{X \times Y} = (Z^Y)^X$, and is called the *exponential law*. For any topological spaces $X, Y_1, \dots, Y_n$, there is a canonical homeomorphism $C(X, Y_1 \times \dots \times Y_n) \to C(X, Y_1) \times \dots \times C(X, Y_n) : f \mapsto (\mathrm{pr}_1 \circ f, \dots, \mathrm{pr}_n \circ f)$.

1.5. Operations on Pointed Spaces.

In homotopy theory we often have to consider not merely topological spaces, but *pointed spaces*, that is, each space considered contains a distinguished point, or *base point*, and all maps considered take base points to base points; identical spaces with different base points are regarded as though they were different spaces. The transition to pointed spaces shows itself to a greater or lesser extent in all operations on spaces. For some operations the modification just consists in providing the resulting space with a base point. For example, the base point in the product $X \times Y$ is (x_0, y_0), where x_0, y_0 are the base points of the factors. Some operations need to be modified more significantly. Thus in the cone CX, all the points of the generator corresponding to the base point of X are identified to each other. Similar modifications are made to the suspension, join (in which the points of the generator joining the base points of the factors are identified), and the mapping cylinder and cone (where it is of course assumed that the base point is mapped to the base point). In each case the point to which the generator is shrunk is taken as the base point. With these modifications, we still have the homeomorphisms $CS^n = D^{n+1}$, $\Sigma S^n = \Sigma S^{n+1}$ and $S^p * S^q = S^{p+q+1}$, if $(1, 0, \dots, 0)$ is taken as base point in the spheres and balls.

The space of mappings reduces to the space of mappings that take base point to base point. The base point of $C(X, Y)$ is the map taking the whole of X to the base point of Y. In homotopy theory a special role is played by the space of continuous maps of the circle with base point into a pointed space; notation: $\Omega(X, x_0) := C(S^1, (1, 0); X, x_0)$, abbreviated to ΩX; it is called the *loop space of* X *with origin at* x_0. The same terminology and notation is also used for the space $C(I, 0 \cup 1; X, x_0)$, related to $C(S^1, (1, 0); X, x_0)$ by the canonical homeomorphism $C(S^1, (1, 0); X, x_0) \to C(I, 0 \cup 1; X, x_0) : f \mapsto [t \to f(e^{2\pi i t})]$.

Finally we shall describe two operations that are specific to pointed spaces. Let $\{X_\mu\}_{\mu \in M}$ be a family of pointed topological spaces with base points x_μ. The quotient space of the sum $\coprod_{\mu \in M} X_\mu$ by the subset $\coprod_{\mu \in M} x_\mu$ is called the *bouquet of spaces* X, denoted by $\bigvee_{\mu \in M} X_\mu$, or, more precisely, $\bigvee_{\mu \in M}(X_\mu, x_\mu)$. The point $\mathrm{pr}(\coprod_{\mu \in M} x_\mu)$ is called the *centre* of the bouquet, and is naturally taken as base point. The bouquet $\bigvee_{\mu \in M} X_\mu$ is covered by copies of the spaces X_μ (usually identified with X_μ), which intersect each other only in the centre of the bouquet. Figure 1 shows a bouquet of two circles ("figure of eight").

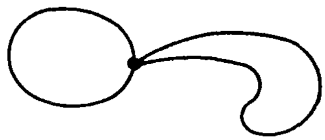

Fig. 1

Let $X_1, \dots, X_n$ be topological spaces with base points $x_1, \dots, x_n$. The canonical embeddings $X_1 \to X_1 \times \cdots \times X_n : x \mapsto (x, x_2, \dots, x_n), \dots, X_n \to X_1 \times \cdots \times X_n : x \mapsto (x_1, \dots, x_{n-1}, x)$ determine a canonical embedding $(X_1, x_1) \vee \cdots \vee (X_n, x_n) \to X \times \cdots \times X_n$, which allows us to regard the bouquet $X_1 \vee \cdots \vee X_n$ as a subspace of $X_1 \times \cdots \times X_n$. The quotient space $X_1 \times \cdots \times X_n / X_1 \vee \cdots \vee X_n$ is called the *smash product* or *tensor product* of $X_1, \dots, X_n$, and is denoted by $X_1 \otimes \cdots \otimes X_n$, or, more precisely, by $(X_1, x_1) \otimes \cdots \otimes (X_n, x_n)$. The notations $X_1 \wedge \cdots \wedge X_n$ and $X_1 \# \cdots \# X_n$ are also used. The point $\mathrm{pr}(X_1 \vee \cdots \vee X_n) \in X_1 \otimes \cdots \otimes X_n$ is called the *centre* of the tensor product and is taken as base point. It is not hard to see that $S^p \otimes S^q = S^{p+q}$, and that for any pointed space X, $\Sigma X = X \otimes S^1$.

If $\{X_\mu\}_{\mu \in M}$ and $\{Y_\mu\}_{\mu \in M}$ are families of pointed spaces and $\{f_\mu : X_\mu \to Y_\mu\}_{\mu \in M}$ is a family of continuous maps (taking base points to base points), then this gives rise to a continuous map $(\coprod f_\mu)/ : \bigvee_{\mu \in M} X_\mu \to \bigvee_{\mu \in M} Y_\mu$, denoted by $\bigvee_{\mu \in M} f_\mu$. The map $f_1 \otimes \cdots \otimes f_n : X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_n$ is defined similarly.

§2. Homotopy

2.1. Homotopies.

A. A continuous map $g : X \to Y$ is said to be *homotopic* to a continuous map $f : X \to Y$ if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for all $x \in X$. Any such mapping is called a *homotopy* connecting f and g. We also say that H is a *homotopy of* f.

Homotopy is clearly an equivalence relation. It divides the space $C(X, Y)$ of continuous maps of X into Y into equivalence classes called *homotopy classes*. The set of these classes is denoted by $\pi(X, Y)$.

B. As an example of homotopy we may take rectilinear homotopy. Let f and g be continuous maps of X into a subspace Y of $\mathbb{R}^n$. If for each point $x \in X$, the line segment joining $f(x)$ to $g(x)$ lies entirely in Y, then the formula $H(x, t) = (1 - t)f(x) + tg(x)$ defines a homotopy between f and g. Such a homotopy is called *rectilinear*. Hence any two maps of an arbitrary space into a convex subspace of a Euclidean space are homotopic.

C. If $f, f' : X \to Y$, $g : Y \to Y'$, $h : X' \to X$ are continuous maps and $F : X \times I \to Y$ is a homotopy between f and f', then $g \circ F \circ (h \times \mathrm{id}_I)$ is a homotopy between $g \circ f \circ h$ and $g \circ f' \circ h$. Hence the mapping $C(g, h) : C(X, Y) \to C(X', Y')$, induced by the maps g and h, takes homotopy classes to homotopy classes. The mapping $\pi(X, Y) \to \pi(X', Y')$ arising in this way is denoted by $\pi(g, h)$. It is determined by the homotopy classes of g and h. The mapping $\pi(g, \mathrm{id}) : \pi(X, Y) \to \pi(X, Y')$ is also denoted by g_*, and the mapping $\pi(\mathrm{id}, h) : \pi(X, Y) \to \pi(X', Y)$ by h^*.

D. Let A be a subset of X. A homotopy $H : X \times I \to Y$ is said to be *relative to* A, or, briefly, to be an *A-homotopy*, if $(H(x, t) = H(x, 0)$ for all $x \in A, t \in I$. Two maps that can be connected by an A-homotopy are said to be *A-homotopic*. Clearly, A-homotopic maps coincide on A. If we want to emphasize that a homotopy is not relative, we call it *free*.

Like ordinary homotopy, A-homotopy is an equivalence relation. The classes into which it divides the set of continuous maps $X \to Y$ that agree on A with a given map $f : A \to Y$, are called *A-homotopy classes*, or, more precisely, *homotopy classes of continuous extensions of f to X*.

Note that the rectilinear homotopy between f and g is relative to the set on which f and g coincide.

2.2. Paths.

A continuous mapping of the interval I into X is called a *path* in the space X. The points $s(0)$ and $s(1)$ are called the *origin* and *end* of the path s. If $s(0) = s(1)$ the path s is called *closed*. Closed paths are also called *loops*.

If s is a path, the path defined by $t \mapsto s(1 - t)$ is called the *inverse* of S, and denoted by s^{-1}. The path defined in terms of paths s_1 and s_2 with $s_1(1) = s_2(0)$ by the formula

$$t \mapsto \begin{cases} s_1(2t), & \text{if } t \leq \frac{1}{2}, \\ s_2(2t - 1), & \text{if } t \geq \frac{1}{2}, \end{cases}$$