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The Resolution of Singular Algebraic Varieties

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Obergurgl
Tyrolean Alps, Austria
June 3–30, 2012

David Ellwood
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The Resolution of Singular Algebraic Varieties

A tribute to Shreeram Abhyankar and Heisuke Hironaka

Preface

A remote place in the Tyrolean Alps at 2000 meters of altitude, a venue with perfect working facilities and a clear cut goal: to excite young mathematicians in the resolution of singularities of algebraic varieties by offering them a four week program of classes and problem sessions—this was the scene of the 12th Clay Summer School at Obergurgl, Austria.

It was previewed from the outset that such a school should go beyond mere mathematical education: it should represent a decisive step in the career of participants by teaching how to grasp and incorporate the main features of a complicated theory, to evaluate and combine the many ideas involved in its proofs, and to develop an overall picture of what mathematics can be good for with respect to intellectual, cultural and personal development. This scientific intention was to be matched by social effects: the communication with colleagues and teachers, group work, mutual respect and stimulation, the dialectic of modesty versus ambition.

The topic: Resolution of singularities consists in constructing for a given algebraic variety X an algebraic manifold $\widetilde{X}$ together with a surjective map $\pi : \widetilde{X} \longrightarrow X$. This map gives a parametrization of the singular variety by a smooth variety. Algebraically, this means to find, for a given system of polynomial equations, a systematic transformation of the polynomials by means of blowups which transform the system into one that satisfies the assumption of the implicit function theorem, so that certain variables can be expressed as functions of the remaining variables. This transformation allows one to interpret the solution set of the given system as the projection of a graph to the singular variety.

The existence of resolutions is instrumental in many circumstances since it allows one to deduce properties of the variety from properties of the parametrizing manifold. Applications abound.

The pioneer in this problem was Oscar Zariski. He introduced abstract algebraic ideas and techniques to the field, and proved many important cases, both in small dimensions and, for more restrictive assertions, in arbitrary dimension. His perspective was mostly based on varieties defined over fields of characteristic zero. He recommended to his student Shreeram Abhyankar to abandon, after several vain attempts, the difficult positive characteristic case of surfaces. As a matter of protest and stubbornness, Abhyankar intensified his efforts and succeeded in his thesis to settle this case[1].

At that time, another of Zariski's many students, Heisuke Hironaka, was a friend of Abhyankar, and together they were discussing this subject at the end of the fifties. It seems that these conversations produced the key idea for the characteristic

[1] Shreeram Abhyankar, Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$, Ann. Math. (2) 63 (1956), 491–526.

zero case of arbitrary dimension, the so called notion of hypersurfaces of maximal contact. In a technically enormously challenging though rather elementary tour de force Hironaka established this result[2].

It took some thirty years to really understand this proof and to put it on a transparent logical fundament: the systematic use of a local resolution invariant which measures the complexity of a singularity. It serves two purposes: Firstly, the stratum where it attains its maximum (i.e., where the worst singularities occur) is smooth and can be taken as the center of the next blowup. Secondly, under this blowup, the invariant drops at each point above points of the center. In this way, induction can be applied, and there will exist (a possibly very long) sequence of blowups which makes the invariant eventually drop to zero. At that time, it is shown that the variety has become smooth.

The last years have seen further simplifications of the proof and strengthening of the result. However, the case of positive characteristic in arbitrary dimension still resists. It is one of the main challenges of modern algebraic geometry.

The participants: Around eighty students were selected for the school from about 250 applicants. They were recommended to acquaint themselves with the basic prerequisites in advance of the school to ensure a common framework for the presentation of the courses. They were also advised that the four weeks of the school would be an intensive experience, demanding a strong personal dedication to learning the material of the classes, solving the daily problems and exercises, as well as sharing their insights and doubts with the other participants. Special emphasis was laid on the respectful contact among peers which is essential for the success of such a four week event.

The lecturers: Herwig Hauser developed the various contexts in which resolution may appear (algebraic, analytic, local, global, formal, ...), exposed the main concepts and techniques (singularities, order functions, transversality, blowups, exceptional divisors, transforms of ideals and varieties, hypersurfaces of maximal contact, resolution invariants), and presented the logical and technical structure of the characteristic zero proof. His article is reminiscent of the engaging style of his lectures at the school, with a vast number of exercises and examples. It gives a concise yet comprehensive overview of resolution techniques including much of the required basics from commutative algebra and/or algebraic geometry. Blowups (in various forms) and transforms of subvarieties are introduced in lectures 4–6. Lecture 7 describes the precise statements of resolution of singularities. Lecture 8 introduces the order invariant of singularities which is used in one of the main inductions in the proof of resolution. This invariant is then refined via the use of a hypersurface of maximal contact and coefficient ideals which are discussed in lectures 9 and 10 respectively. Lecture 11 gives a detailed outline of the proof of strong resolution of a variety in characteristic zero. Finally lecture 12 concerns positive characteristic: it discusses problems that arise and gives references to topics of recent progress.

Orlando Villamayor concentrated on the commutative algebra side of resolution: local multiplicity and Hilbert-Samuel functions, integral ring extensions, finite morphisms, Rees algebras, actions of symmetry groups and a replacement of hypersurfaces of maximal contact by means of generic projections. This gives a new proof of resolution in characteristic zero and has very good chances to be applicable in

[2]Heisuke Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. Parts I. & II., Ann. Math. 79 (1964): 109–203, 205–326.

arbitrary characteristic. His comprehensive set of notes have been compiled jointly with Ana Bravo. The final results provide a natural smooth stratification of the locus of maximal multiplicity (or maximal Hilbert function) on a variety X (valid in all characteristics) and use this stratification to give a strategy for reducing the maximal value of multiplicity (or Hilbert function) by blowing up smooth centers in characteristic zero. The motivating ideas come from Hironaka's idealistic exponents (of which the Rees algebra gives a generalization) and the classical local interpretation of multiplicity as the degree of a (formal) realization of the singularity as a finite extension over the germ of a regular local ring.

Josef Schicho presented an axiomatic approach to the resolution of singularities in the form of a parlor game. Instead of the various constructions of the classical proof one distills their essential properties and uses them to define a game between two players. The existence of a winning strategy is then a purely combinatorial problem which can be solved by logical arguments. The second part consists in showing that there do exist constructions fulfilling the many axioms or moves of the game. This works up to now only in zero characteristic. Only elementary algebra is needed. In principle, the game also prepares a prospective approach for the case of positive characteristic. Schicho's lecture notes describe the precise rules of the game, the reduction of the resolution problem to the game, and the winning strategy. As an aside, several other games with a mathematical flavor were discussed, and FlipIt was one of them. Together with Jaap Top (Schicho learned about FlipIt from his webpage), a formulation of the game in terms of linear algebra over the field with two elements is provided.

The last week of the school culminated in a series of mini-courses in which invited experts reported on their latest research on resolution, several of whom also contributed to this volume.

Steven Cutkovsky's paper gives an overview of the resolution problem in positive characteristic, enriched with some significant computations on a range of important topics of current interest. The first part of the paper provides the central results, illustrated by an exposition of the main ideas of resolution of surfaces. The second part deals with the problem of making an algebraic map monomial, in appropriate local coordinates, by sequences of blowups. The application by the author and Piltant to a general form of local ramification for valuations is explained. Some results on monomialization in positive characteristic and on global monomialization in characteristic zero conclude the paper.

The article by Santiago Encinas provides an informative survey about resolution of toric varieties, with an emphasis on the resolution algorithm of Blanco and the author. The procedure uses the binomial equations defining the toric variety as an embedded subvariety in an affine space. The article contains an introduction to toric varieties as well as several worked examples.

Anne Frühbis-Krüger gives an introduction to the computational applications of desingularization. In the first part, she details the data that is actually computed by the resolution algorithm. This is a priori not clear because theoretically the result arises by compositions of blowups and is not naturally embedded in an affine or projective space; rather it comes as a union of affine charts. The applications described in the second part include the dual graph of an isolated surface singularity, the log canonical threshold, the topological Zeta function, and the Bernstein-Sato polynomial.

Hiraku Kawanoue gives an overview of his Idealistic Filtration Program, a program for resolution of singularities. The presentation gives a clear overview of the main ideas, and it contains several examples that illustrate the general strategy through clever examples both in characteristic zero and in characteristic p. A new result concerning the monomial case of the Idealistic Filtration Program using the radical saturation is also included.

Takehiko Yasuda presents his geometric approach to the resolution of singularities (higher Semple-Nash blowups, considered mainly in characteristic zero, and F-blowups, defined only in positive characteristic). In this method a series of blowups is constructed directly from the given variety, using not only first-order data, but also higher order ones. Each blowup is the parameter space of some geometric object on the given variety. The final section provides some open problems associated with this approach.

The venue: The Obergurgl Center is owned by the University of Innsbruck. It serves as a conference and research venue for up to 120 participants. Lecture halls, seminar rooms, library and meeting lounges are of the highest standards. They are complemented by cosy rooms in a traditional alpine style, various leisure facilities, excellent food and a very attentive personnel. Above all, the natural surroundings are spectacular, culminating in an inspiring skyline of alpine peaks.

Our thanks go to the former staff of the Clay Mathematics Institute, particularly Julie Feskoe, Vida Salahi and Lina Chen, as well as all the personnel of the Obergurgl Center. We are also grateful for the support of the University of Innsbruck, the Obergurgl tourist office, and the referees of this volume, who participated in an unusually detailed refereeing process.

David Alexandre Ellwood, Herwig Hauser, Josef Schicho, Shigefumi Mori
Boston, Vienna, Linz, Kyoto
August 2014

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Blowups and Resolution

Herwig Hauser

To the memory of Sheeram Abhyankar with great respect

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References

This manuscript originates from a series of lectures the author[1] gave at the Clay Summer School on Resolution of Singularities at Obergurgl in the Tyrolean Alps in June 2012. A hundred young and ambitious students gathered for four weeks to hear and learn about resolution of singularities. Their interest and dedication became essential for the success of the school.

The reader of this article is ideally an algebraist or geometer having a rudimentary acquaintance with the main results and techniques in resolution of singularities. The purpose is to provide quick and concrete information about specific topics in the field. As such, the article is modelled like a dictionary and not particularly suited to be read from the beginning to the end (except for those who like to read dictionaries). To facilitate the understanding of selected portions of the text

2010 *Mathematics Subject Classification.* Primary: 14B05, 14E15, 14J17, 13H05, 32S45.

Key words and phrases. Blowups, resolution, singularities, algebraic varieties.

[1]Supported in part by the Austrian Science Fund FWF within the projects P-21461 and P-25652.

without reading the whole earlier material, a certain repetition of definitions and assertions has been accepted.

Background information on the historic development and the motivation behind the various constructions can be found in the cited literature, especially in [Obe00, Hau03, Hau10a, FH10, Cut04, Kol07, Lip75]. Complete proofs of several more technical results appear in [EH02].

All statements are formulated for algebraic varieties and morphism between them. They are mostly also valid, with the appropriate modifications, for schemes. In certain cases, the respective statements for schemes are indicated separately.

Each chapter concludes with a broad selection of examples, ranging from computational exercises to suggestions for additional material which could not be covered in the text. Some more challenging problems are marked with a superscript $^+$. The examples should be especially useful for people planning to give a graduate course on the resolution of singularities. Occasionally the examples repeat or specialize statements which have appeared in the text and which are worth to be done personally before looking at the given proof. In the appendix, hints and answers to a selection of examples marked by a superscript $^\triangleright$ are given.

Several results appear without proof, due to lack of time and energy of the author. Precise references are given whenever possible. The various survey articles contain complementary bibliography.

The Clay Mathematics Institute chose resolution of singularities as the topic of the 2012 summer school. It has been a particular pleasure to cooperate in this endeavour with its research director David Ellwood, whose enthusiasm and interpretation of the school largely coincided with the approach of the organizers, thus creating a wonderful working atmosphere. His sensitiveness of how to plan and realize the event has been exceptional.

The CMI director Jim Carlson and the CMI secretary Julie Feskoe supported very efficiently the preparation and realization of the school.

Xudong Zheng provided a preliminary write-up of the lectures, Stefan Perlega and Eleonore Faber completed several missing details in preliminary drafts of the manuscript. Faber also produced the two visualizations. The discussion of the examples in the appendix was worked out by Perlega and Valerie Roitner. Anonymous referees helped substantially with their remarks and criticism to eliminate deficiencies of the exposition. Barbara Beeton from the AMS took care of the TeX-layout. All this was very helpful.

1. Lecture I: A First Example of Resolution

Let X be the zeroset in affine three space $\mathbb{A}^3$ of the polynomial

$$f = 27x^2y^3z^2 + (x^2 + y^3 - z^2)^3$$

over a field $\mathbb{K}$ of characteristic different from 2 and 3. This is an algebraic surface, called *Camelia*, with possibly singular points and curves, and certain symmetries. For instance, the origin 0 is singular on X, and X is symmetric with respect to the automorphisms of $\mathbb{A}^3$ given by replacing x by $-x$ or z by $-z$, and also by replacing y by $-y$ while interchanging x with z. Sending x, y and z to t^3x , t^2y and t^3z for $t \in \mathbb{K}$ also preserves X. See figure 1 for a plot of the real points of X. The intersections of X with the three coordinate hyperplanes of $\mathbb{A}^3$ are given as the zerosets of the equations

$$x = (y^3 - z^2)^3 = 0,$$
$$y = (x^2 - z^2)^3 = 0,$$
$$z = (x^2 + y^3)^3 = 0.$$

These intersections are plane curves: two perpendicular cusps lying in the xy- and yz-plane, respectively the union of the two diagonals in the xz-plane. The singular locus $\mathrm{Sing}(X)$ of X is given as the zeroset of the partial derivatives of f inside X. This yields for $\mathrm{Sing}(X)$ the additional equations

$$x \cdot [9y^3z^2 + (x^2 + y^3 - z^2)^2] = 0,$$
$$y^2 \cdot [9x^2z^2 + (x^2 + y^3 - z^2)^2] = 0,$$
$$z \cdot [9x^2y^3 - (x^2 + y^3 - z^2)^2] = 0.$$

Combining these equations with $f = 0$, it results that the singular locus of X has six irreducible components, defined respectively by

$$x = y^3 - z^2 = 0,$$
$$z = x^2 + y^3 = 0,$$
$$y = x + z = 0,$$
$$y = x - z = 0,$$
$$x^2 - y^3 = x + \sqrt{-1} \cdot z = 0,$$
$$x^2 - y^3 = x - \sqrt{-1} \cdot z = 0.$$

The first four components of $\mathrm{Sing}(X)$ coincide with the four curves given by the three coordinate hyperplane sections of X. The last two components are plane cusps in the hyperplanes given by $x \pm \sqrt{-1} \cdot z = 0$. At points $a \neq 0$ on the first two singular components of $\mathrm{Sing}(X)$, the intersections of X with a plane through a and transversal to the component are again cuspidal curves.

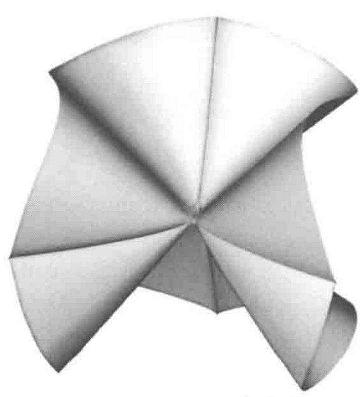

FIGURE 1. The surface *Camelia*: $27x^2y^3z^2 + (x^2 + y^3 - z^2)^3 = 0$.

Consider now the surface Y in $\mathbb{A}^4$ which is given as the cartesian product $C \times C$ of the plane cusp $C : x^2 - y^3 = 0$ in $\mathbb{A}^2$ with itself. It is defined by the equations $x^2 - y^3 = z^2 - w^3 = 0$. The singular locus $\mathrm{Sing}(Y)$ is the union of the two cusps $C_1 = C \times 0$ and $C_2 = 0 \times C$ defined by $x^2 - y^3 = z = w = 0$, respectively $x = y = z^2 - w^3 = 0$. The surface Y admits the parametrization

$$\gamma : \mathbb{A}^2 \to \mathbb{A}^4, (s, t) \mapsto (s^3, s^2, t^3, t^2).$$

The image of γ is Y. The composition of γ with the linear projection

$$\pi : \mathbb{A}^4 \to \mathbb{A}^3, (x, y, z, w) \mapsto (x, -y + w, z)$$

yields the map

$$\delta = \pi \circ \gamma : \mathbb{A}^2 \to \mathbb{A}^3, (s, t) \mapsto (s^3, -s^2 + t^2, t^3).$$

Replacing in the polynomial f of X the variables x, y, z by s^3, $-s^2 + t^2$, t^3 gives 0. This shows that the image of δ lies in X. As X is irreducible of dimension 2 and δ has rank 2 outside 0 the image of δ is dense in (and actually equal to) X. Therefore the image of Y under π is dense in X: This interprets X as a contraction of Y by means of the projection π from $\mathbb{A}^4$ to $\mathbb{A}^3$. The two surfaces X and Y are not isomorphic because, for instance, their singular loci have a different number of components. The simple geometry of Y as a cartesian product of two plane curves is scrambled up when projecting it down to X.

The point blowup of Y in the origin produces a surface Y_1 whose singular locus has two components. They map to the two components C_1 and C_2 of $\mathrm{Sing}(Y)$ and are regular and transversal to each other. The blowup X_1 of X at 0 will still be the image of Y_1 under a suitable projection. The four singular components of $\mathrm{Sing}(X)$ will become regular in X_1 and will either meet pairwise transversally or not at all. The two regular components of $\mathrm{Sing}(X)$ will remain regular in X_1 but will no longer meet each other, cf. figure 2.

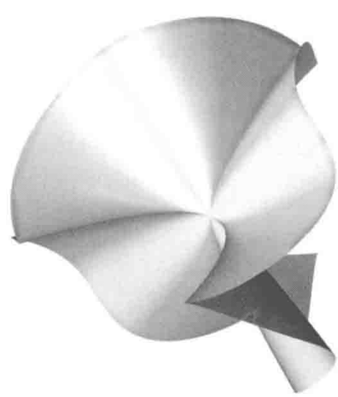

FIGURE 2. The surface X_1 obtained from *Camelia* by a point blowup.

The point blowup of Y_1 in the intersection point of the two curves of $\mathrm{Sing}(Y_1)$ separates the two curves and yields a surface Y_2 whose singular locus consists of two disjoint regular curves. Blowing up these separately yields a regular surface Y_3 and thus resolves the singularities of Y. The resolution of the singularities of X is more complicated, see the examples below.

Examples

EXAMPLE 1.1. ▹ Show that the surface X defined in $\mathbb{A}^3$ by $27x^2y^3z^2 + (x^2 + y^3 - z^2)^3 = 0$ is the image of the cartesian product Y of the cusp $C : x^3 - y^2 = 0$ with itself under the projection from $\mathbb{A}^4$ to $\mathbb{A}^3$ given by $(x, y, z, w) \mapsto (x, -y + w, z)$.

EXAMPLE 1.2. ▹ Find additional symmetries of X aside from those mentioned in the text.