METHODS OF MATHEMATICAL PHYSICS

Third Edition

数学物理方法

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METHODS
OF
MATHEMATICAL PHYSICS

by

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Preface to the Third Edition

In the present edition we have made changes in Chapter 1, mainly as a result of comments by Professor A. S. Besicovitch. Some theorems are stated more explicitly, a few proofs are added, and some are shortened. We are indebted to him for an elementary proof of the theorem of bounded convergence for Riemann integrals, which appears in the notes. In Chapter 6 the proof of Poisson's equation has been improved. In Chapter 17 we have discussed the Airy integral for complex argument in more detail, and have given conditions for uniformity of approximation for asymptotic solutions of Green's type for complex argument. In Chapter 23 we have added some remarks on the analytic continuation of the solutions, and a note applies them to the parabolic cylinder functions.

We should like to express our thanks to several readers for drawing our attention to errors and misprints.

HAROLD JEFFREYS
BERTHA JEFFREYS

April 1953

Preface to the Second Edition

As a second edition of this book has been called for, we have taken the opportunity of making considerable revisions. Most of the notes at the end have been incorporated in the text. Otherwise the principal changes are as follows. In Chapter 1, the Heine-Borel theorem and Goursat's modification have been placed early, and used to derive several theorems that had been proved by separate applications of methods that could be used to prove the general theorems. In other respects, notably the theory of the Riemann integral, the theory has been given more fully. In Chapter 4 an account of block matrices has been added, and the theorem on characteristic solutions of commuting matrices has been more fully discussed. Chapter 5 (multiple integrals) has been almost completely rewritten, and now includes an account of the theory of functions of several variables, part of which was given in Chapter 11. In Chapter 9 the treatment of relaxation methods has been extended, and should now serve as an adequate introduction to the special works on the subject. Many improvements have been made in Chapters 11 and 12, including an important correction to the proof of Cauchy's theorem, a proof of the Osgood-Vitali theorem, and a complete revision of the theory of inverse functions. In Chapter 17 the conditions for the truth of Watson's lemma have been somewhat relaxed, so that they are now wide enough to cover almost all physical applications, and the method of stationary phase is more fully treated. In Chapter 24 the treatment of multipole radiation has been extended.

Where possible the proofs have been either replaced by shorter ones or generalized. Some new examples have been added.

We are indebted to numerous correspondents for pointing out errata. The two most serious corrections were given by Professor J. E. Littlewood and Dr M. L. Cartwright. We are particularly grateful for comments by Professor Littlewood (Chapters 1, 5, 11 and 12), Mr P. Hall (Chapter 4), Professor A. S. Besicovitch and Dr J. C. Burkill (Chapter 5).

HAROLD JEFFREYS
BERTHA JEFFREYS

15 *November* 1948

Preface to the First Edition

This book is intended to provide an account of those parts of pure mathematics that are most frequently needed in physics. The choice of subject-matter has been rather difficult. A book containing all methods used in different branches of physics would be impossibly long. We have generally included a method if it has applications in at least two branches, though we do not claim to have followed the rule invariably. Abundant applications to special problems are given as illustrations. We think that many students whose interests are mainly in applications have difficulty in following abstract arguments, not on account of incapacity, but because they need to 'see the point' before their interest can be aroused.

A knowledge of calculus is assumed. Some explanation of the standard of rigour and generality aimed at is desirable. We do not accept the common view that any argument is good enough if it is intended to be used by scientists. We hold that it is as necessary to science as to pure mathematics that the fundamental principles should be clearly stated and that the conclusions shall follow from them. But in science it is also necessary that the principles taken as fundamental should be as closely related to observation as possible; it matters little to pure mathematics what is taken as fundamental, but it is of primary importance to science. We maintain therefore that careful analysis is more important in science than in pure mathematics, not less. We have also found repeatedly that the easiest way to make a statement reasonably plausible is to give a rigorous proof. Some of the most important results (e.g. Cauchy's theorem) are so surprising at first sight that nothing short of a proof can make them credible. On the other hand, a pure mathematician is usually dissatisfied with a theorem until it has been stated in its most general form. The scientific applications are often limited to a few special types. We have therefore often given proofs under what a pure mathematician will consider unnecessarily restrictive conditions, but these are satisfied in most applications. Generality is a good thing, but it can be purchased at too high a price. Sometimes, if the conditions we adopt are not satisfied in a particular problem, the method of extending the theorem will be obvious; but it is sometimes very difficult, and we have not thought it worth while to make elaborate provision against cases that are seldom met. For some extensive subjects, which are important but need long discussion and are well treated in some standard book, we have thought it sufficient to give references.

We consider it especially important that scientists should have reasonably accessible statements of conditions for the truth of the theorems that they use. One often sees a statement that some result has been rigorously proved, unaccompanied by any verification that the conditions postulated in the proof are satisfied in the actual problem—and very-often they are not. This misuse of mathematics is to be found in most branches of science. On the other hand, many results are usually proved under conditions that are sufficient but not necessary, and scientists often hesitate to use them, under the mistaken belief that they are necessary. We have therefore often given proofs under more general conditions than are usually taught to scientists, where the usual sufficient conditions are often not satisfied in practice but less stringent ones are satisfied. Both troubles are due chiefly to the fact that the theorems are scattered through many books and papers, and the scientist does not know what to look for or where to look.

The book can be read consecutively, but some parts are independent of much that precedes them, and it is possible, and indeed desirable, to study different chapters concurrently. In some cases we have given special cases of a theorem before the general form where the latter involves more elaborate treatment, especially where the student is likely to meet applications to several instances of the special cases before he needs the general theorem.

We hesitated before including a chapter on the theory of functions of a real variable. This is far from a complete treatment, but fuller works are mostly longer than the theoretical physicist has time to read; and unfortunately they sometimes relegate theorems that are frequently needed to small type or unworked examples, or omit them altogether. We have aimed at giving accounts of the principal methods of the theory but not at proving every result in detail; but we think that students will benefit by filling in some of the details for themselves. If a student has difficulty in achieving the degree of abstraction needed in most of this chapter, we advise him to read as much as he can stand and then proceed to a later chapter, referring back when necessary. He will find that he has covered the whole of it before finishing Chapter 14, and that he knows both what is there and why it is there. We have not succeeded in avoiding forward references altogether, but the most serious, the proof in Chapter 12 of the theorem that an algebraic equation of degree n has n roots, used in Chapter 4, is so time-honoured that a few smaller transgressions may, we hope, be forgiven.

The notation of special functions has grown up haphazard, and is inconvenient in several respects. Quantum theorists are making wholesale changes of definition to ensure normalization, but we consider that this replaces the old complications by new ones. We have modified the usual definitions of the Legendre functions, with the result that a more symmetrical treatment becomes possible and the relation to Bessel functions becomes free from complicated numerical factors. We have returned to Heaviside's definition of the function K_n but denoted it by Kh_n. Among other advantages, this simplifies the relation to Legendre functions of the second type. We have also dropped the Γ notation for the factorial function, which seems to have no recommendations whatever.

The immediate stimulus for the book was the announcement that the second edition of *Operational Methods in Mathematical Physics* by one of us was out of print. Most of this tract has been incorporated and later developments have been added. The chapter on dispersion was somewhat out of place in the tract, as it was largely independent of the operational method, but was included because the notion of group velocity had not previously been discussed in relation to the method of steepest descents. It now finds a more natural place in a chapter on asymptotic expansions, in which some methods widely used but hitherto accessible only in scattered papers are also described. Most of *Cartesian Tensors* has also been incorporated. The applications of thermodynamics in it to hydrodynamics and elasticity would be more suitably treated in textbooks of the latter subjects.

We have not tried to give a detailed account of any branch of physics; that is a matter for the special text-books.

We are deeply indebted to many friends for their encouragement during the writing of this book. Above all we must thank Dr F. Smithies, who placed his great knowledge freely at our disposal, and generously helped in the proof reading. His suggestions have

been invaluable. It is only fair to him to say that in some places we have persisted in our ways in spite of his vigorous protests. Dr J. C. P. Miller gave us special help with Chapters 9 and 23, and Mr H. Bondi with Chapter 24. We have also had valuable suggestions at various points from Professors M. H. A. Newman, A. C. Offord, L. Rosenhead and H. W. Turnbull, and from Mr A. S. Besicovitch, Miss M. L. Cartwright and Mr D. P. Dalzell.

We also thank the Universities of Cambridge, London and Manchester for permission to use examination questions as examples, and the staff of the Cambridge University Press for their care in the printing and their readiness to meet the wishes of a rather exacting pair of authors.

HAROLD JEFFREYS
BERTHA JEFFREYS

1946

The main sections of each chapter are numbered decimally at intervals of 0·01; subsections are indicated by further decimals. When the argument of a section or subsection continues that of the previous one, the numbering of the equations also continues.

Notes at the end are numbered according to the subsection referred to; references to them are indicated by a small index letter in heavy type in the text; for instance, the [a] on p. 52, in subsection 1·134, refers to note **1·134** *a*, which will be found on p. 692.

Sources of examples are indicated by the following abbreviations:

M. T.	Mathematical Tripos, Part II and Schedule A.
M. T., Sched. B.	Mathematical Tripos, Part III and Schedule B.
Prelim.	Preliminary Examination in Mathematics.
M/c, III.	Manchester, Final Honours in Mathematics.
I.C.	Imperial College, London.

Authors' Notes

In this *second impression* of the Third Edition, the following notes have been added: **5·051***a* on differentiation under the integral sign, **10·11***a* on a method used in planetary theory **23·07***a* giving references for work on Coulomb wave functions. Paragraphs **10·01**, **10·013** on the Calculus of Variations have been revised. Some minor corrections and addenda have been made in the text and examples.

July 1961

In this *third impression* of the Third Edition, the following notes have been added: **9·041***a* on interpolation when first derivatives are given and **9·181***a* on the advance in automatic computation. The treatment of orthogonal transformations in Chapter 4 has been extended and an amendment has been made to the proof of Watson's lemma in **17·03**. Further minor corrections and addenda have been made in the text and examples.

March 1966

In this *paperback edition* of the Third Edition, the following alterations have been made: **23·07** on Schrödinger's equation for the hydrogen-like atom has been revised and the note **23·07***a* expanded, and in the Addenda there are references to work on Isotropic Tensors in the note **3·031***a*. Further minor corrections have been made in the text.

January 1972

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Chapter 1

THE REAL VARIABLE

'In dem days dey wuz monstus fon' er minners.'

JOEL CHANDLER HARRIS, *Uncle Remus*

1·01. The relation of mathematics to physics. The simplest mathematical notion is that of the number of a class. This is the property common to the class and to any class that can be matched with it by pairing off the members, one from each class, so that all members of each class are paired off and none left over. In terms of the definition we can give meanings to the fundamental operations of addition and multiplication. Consider two classes with numbers a, b and no common member. The sum of a and b is the number of the class consisting of all members of the two classes taken together. The product of a and b is the number of all possible pairs taken one from each class. We cannot always give meanings to subtraction and division, because, for instance, we cannot find a class whose number is $2-3$ or $7/5$. But it is found to be a great convenience to extend the notion of number so as to include negative numbers, ratios of numbers irrespective of whether they are positive or negative, and even irrational numbers. When this is done we can define all the four fundamental operations of arithmetic, and the result of carrying them out will always be a number within the system. We need trouble no more about whether an operation is *possible* with a particular set of numbers, since we know that it is, once we have given sufficient generality to what we mean by a number. So long as we keep to the fundamental operations we can use algebra; that is, we can prove formulae that will be correct when any numbers whatever are substituted for the symbols in them, with only one exception, namely, that we must not divide by 0.

Now the formulae may still be correct when we replace the letters in them by something other than numbers, and it is to this fact that the possibility of mathematical physics is due. It is therefore useful to know just what conditions have to be satisfied if we are to take over the rules of algebra into any subject that does not deal entirely with numbers. We may then have to find new meanings for the fundamental operations (or have them found for us) and for the sign $=$, but can still manipulate the symbols with their new meanings in the old way. A suitable set of conditions is as follows.* We say that they are to hold in a *field* F consisting of all elements of the system considered:

(1) For any a, b of F, $a+b$ and ab are uniquely determined elements of F.

(2) $b+a = a+b$. (Commutative law of addition.)

(3) $(a+b)+c = a+(b+c)$. (Associative law of addition.)

(4) $ba = ab$. (Commutative law of multiplication.)

(5) $a(bc) = (ab)c$. (Associative law of multiplication.)

(6) $a(b+c) = ab+ac$. (Distributive law.)

(7) There are two elements 0 and 1 in F, such that $a+0 = a$, $a1 = a$.

(8) For any element a of F there is an element x of F such that $a+x = 0$.

(9) For every element a of F, other than 0, there is an element y of F such that $ay = 1$.

* Stated first by Dedekind for the case where $+$ and $\times$ have their ordinary arithmetic meanings; in general by H. Weber.

It is to be noticed that the first seven rules are true if F consists only of the positive integers and 0, but the last two are false of that F, since there is no positive or zero integer x that makes $a+x = 0$ if $a = 1$, and there is no positive or zero integer y that makes $ay = 1$ if $a = 2$. The eighth rule introduces negative numbers and hence subtraction. The ninth introduces reciprocals and hence division and rational fractions. The rules are true if F consists of all rational numbers, positive or negative.

The rules mention no ordering relation: that is, they suppose a meaning attached to equality and therefore to $\neq$, but do not distinguish between greater and less. We could agree to arrange the numbers in any order, keeping the same correspondences between them according to (1), (7), (8), (9), and the rules would still be true. Algebra and pure geometry can get on to some extent without such a distinction, but higher mathematics cannot, nor can any kind of physics. A measurement is not a statement of exact equality but of equality within a certain range of error. We therefore need new rules concerning inequalities.

(10) For any a, b of F, either $a > b$, $a = b$, or $b > a$. (Law of comparability.)

(11) For given a, b of F, only one of $a > b$, $a = b$, $b > a$ can be true. (Trichotomy.)

(12) If $a > b$ and $b > c$, then $a > c$. (Transitive property.)

(13) If $a > b$, then $a + c > b + c$ for any c. (Additivity of ordering.)

(14) If $a > b$, $c > 0$, then $ac > bc$. (Multiplicativity of ordering.)

(15) If $a > b$, $b < a$. (Definition of $<$.)

The use of mathematics in science is that of a language, in which we can state relations too complicated to be described, except at inordinate length, in ordinary language. The rules satisfied by the symbols are the grammar of the language. This point of view has been developed greatly in recent years, especially by R. Carnap. But for a language to be suitable it must satisfy two conditions. It must be possible to say in it the things that we need to say; that is, it must have sufficient generality. It must also be self-consistent; that is, starting from the rules themselves it must be impossible to deduce something declared to be false by those rules. It would, for instance, be fatal to the scientific usefulness of mathematics if it was possible to prove by it that for some a and b, a is both greater and less than b. It was always taken for granted until the later nineteenth century that mathematics was consistent. But then an unexpected set of difficulties cropped up, and showed that a complete analysis of the foundations was necessary. The great *Principia Mathematica* of Whitehead and Russell showed that all the propositions asserted in mathematics concerning real numbers (not only ratios of integers, positive or negative) could be restated as propositions about the elementary notion of comparing classes by pairing their members, and demonstrable from the axioms of such comparison and others relating to pure logic. Later workers have modified some of the latter axioms, and the best choice of axioms is still a matter of discussion. Gödel and Carnap, more recently, have shown that the proposition that a given system of axioms for mathematics is consistent cannot be proved by methods using only the rules of the system. But it is found impossible to prove certain propositions that could be proved if the system was inconsistent. We have to come back to something like ordinary language after all when we want to talk *about* mathematics! This work on the boundary between logic and what we usually consider the elements of mathematics has a considerable modern literature, and it is well for physicists to know of its existence, though its detailed study is a matter for specialists.

1·02. Physical magnitudes. Generality requires that, in any particular field, the language shall contain symbols for the things that we need to talk about and for the processes that we carry out. A shepherd would be severely handicapped if he had to do his best with a language containing no words for sheep and shearing; in fact he would make such words, and that is what we habitually do in science. So long as the language is consistent it is none the worse for containing a lot of words that we do not use. A pure mathematician, working entirely on the theory of numbers, can use ordinary algebra freely in spite of the fact that he may not need to use negative numbers or fractions. For him rules (8) and (9) are just an unnecessary generality. Now in physics the fundamental notion of measurement corresponds closely to that of addition, and most physical laws are statements of proportionality, which corresponds to the notions of multiplication and division. This is the ultimate reason why mathematics is useful. Thus, for instance, we can say that if two bars are placed end to end to make one straight bar, the length of the combined bar is the sum of those of the original ones. This is not a theorem or an experimental fact; it is the definition of addition for lengths. Further, it is irrelevant which is taken first; thus the commutative law of addition holds. Again, if we unite three bars, the total length is independent of the order; hence the associative law of addition also holds. These are experimental facts established by actual comparison with other bars. These rules are enough to justify the use of scales of measurement for length, by which any length is compared with a standard one by means of a scale, every interval of which has been compared with a standard object in the process of manufacture. Quantities measurable by some process of physical addition have been called *fundamental magnitudes* by N. R. Campbell.* The most widely important ones are numbers (of classes), length, time, and mass, but physical processes of addition can also be stated for area and volume, for electric charge, potential, and current, and many other quantities.

There is a divergence of practice among physicists at the next stage. A statement that a distance is 3·7 cm. contains a number and a unit. It is often thought that algebra applies only to numbers and therefore that in the mathematical treatment the symbol used for the distance refers only to the 3·7 and not to the centimetres. The unit matters, otherwise we should find ourselves saying that 10 mm. expresses a different length from 1 cm. and that 1 cm. is the same as 1 mile; and this is contrary to physics because the only justification of using measurement at all is in the direct physical comparison by superposition. We avoid this difficulty if we say that the symbol for the length refers to the length itself and not simply to the number contained in its measure. '1 inch = 2·54 cm.' is a useful statement; either symbol, '1 inch' or '2·54 cm.', denotes the *same* length. In general theorems this procedure can always be followed. When a particular application to a measured system is made we naturally give the symbols their actual values in terms of the measures, which will include a statement of the units; but in the general theory the unit is irrelevant. The symbols will then be said to stand, not for numbers, but for ***physical magnitudes***.

The alternative method would be to let the symbols stand for the numbers, but then confusion can occur, and does, between the relations between measures of the same system in different units, which are different ways of saying the same thing, and of different systems in the same units, which say different things. If, however, the numerical values in terms of special units are used for a and b in ab, their product will be the number in the

* 'Elementary' or 'Additive' might be better.

expression of ab in what is usually called the *consistent* unit for ab. The word *germane*, introduced by E. A. Guggenheim, is better because it is not inconsistent to measure distances upward in feet, horizontally in yards, and downward in fathoms; it is merely a nuisance. With adequate care this method can be used correctly, but it has several disadvantages; in particular it then leads to placing too much emphasis on the units and too little on the fundamental physical comparisons without which the units would be useless. It also suggests many comparisons that are physically meaningless, as we shall see in a moment.

If we use the notion of magnitude and retain the processes of algebra the question will at once arise, what do we mean by $a = b$ and $a+b$ if a is a length and b a time or a mass? A meaning could be attached to $a+b$, though it would be very artificial, but no physical process will give one to $a = b$. But a/b would have a meaning, being respectively a velocity or a length per unit mass.

The group of rules (10)–(14) therefore needs modification. Those up to (9) could stand, though they bring in many additions and subtractions and possibly some multiplications and divisions that we shall never have occasion to use; but *in addition to the three possibilities enumerated in* (10) *we must admit a fourth, that a and b may not be comparable and therefore belong to different fields, and their product and ratio may belong to other fields again.* This is a further disadvantage of the use of symbols to denote only the number stated in a measure, since all numbers are comparable, and the language would not exhibit the fact that it is meaningless to say that a time is greater than a density. We can then say also that *if a and b are not comparable, $a+b$ is not a physical magnitude and addition does not arise.* The whole field of physical magnitudes is thus divided into plots. Magnitudes in the same plot will be comparable, but their product will belong to a different plot unless at least one of them is a number.

The language needed for physics is therefore not quite the same as ordinary algebra. Since the latter is self-consistent and the statement that some magnitudes are not comparable cuts out some propositions from it and adds no new ones, the language of magnitude is also self-consistent. It will be seen that the modification corresponds to the notion of *dimensions*. Quantities of different dimensions are not comparable; also some quantities of the same dimensions are not. For instance, according to one pair of definitions in use, electric charge and magnetic pole strength have the same dimensions, and they are both additive magnitudes, but it is meaningless to add them. The field of physical magnitudes can be taken to satisfy the laws of algebra, but is classified; comparable quantities satisfy (10), and are capable of addition at least in calculation; incomparable ones do not. It should be noticed that failure of addition by a physical process is not confined to incomparable magnitudes. For instance, there is no process of combining two substances of density 1 g./cm.^{3} to give one of density 2 g./cm.^{3} Density is not measured directly but calculated from the additive magnitudes mass and length, and is called a *derived magnitude*. Some quantities can be both additive and derived; thus electric current measured by its magnetic effect is an additive magnitude, but regarded as the charge passing per unit time it is derived. Many derived magnitudes are ratios of two magnitudes of the same dimensions; thus we could regard the shape of a triangle as specified by two ratios, those of two sides to the third. These ratios are pure numbers and the rules of algebra can be applied to them without change.*

* A similar treatment was advocated by W. Stroud; for discussion and applications to teaching, cf. Sir J. B. Henderson, *Engineering*, 116, 1923, 409–10.

1·03. Real numbers. Most of the present chapter will be already familiar to those who have studied a good modern book on calculus, and it is not intended to compete with standard works on pure mathematics. We think, however, that some discussion here is not out of place, for several reasons. First, the latter works for the most part do not emphasize *why* the refined arguments that they give have any relevance to physics, and physicists therefore tend to believe that they are irrelevant. Secondly, they are liable to be so long that a physicist can hardly be blamed if he decides that he has not the time to work through them. Thirdly, the attention to very peculiar functions has led the subject to be regarded as the pathology of functions. The reply is that every function, except an absolute constant, is peculiar somewhere, and that by studying where a function is peculiar we can arrive at constructive results about it that would be very hard to obtain otherwise. But we are entitled to regard ourselves as general practitioners and to restrict ourselves to the kinds of peculiarities that occur in physics; rare diseases may be handed over for treatment to a specialist, in this case a professional pure mathematician.

The nature of the problem was foreshadowed in a theorem of Euclid that the ratio of the hypotenuse to one side of an isosceles right-angled triangle is not equal to any rational fraction. Euclid, it must be remembered, made no use of what we should now call numerical measures of physical magnitudes. When he said that two lines were equal he meant that one could be placed on the other so that the two ends of one coincided with the two ends of the other; this is the direct physical comparison and does not require any numerical description of the lengths. When he said that the square on the hypotenuse was twice that on a side he meant that it could be cut into pieces and that the pieces could then be put together so as to make the square on the side twice over. He was working throughout with the quantities themselves, not with the numbers that we choose to associate with them in measurement with regard to any special unit. The use of numbers for this purpose is a choice of a language. What Euclid's theorem showed was that the language of rational numbers was incapable of describing simultaneously the lengths of the side and the hypotenuse of a triangle that could easily be drawn by the rules of his geometry.

Measurement in terms of a unit is too useful a procedure to be lightly abandoned, and it could be retained, consistently with Euclid's theorem, in any of the following ways: (1) Since an infinite number of pairs of integers x, y can be found such that $x^2+y^2=z^2$, where z is another integer, and so that x/y is as near 1 as we like, we could suppose that the sides of a right-angled triangle satisfy $x^2+y^2=z^2$ exactly but that $x=y$ is not true exactly but only within the errors of measurement, and the sides are always exact multiples of some definite length. (2) We might say that x/y can be exact but $x^2+y^2=z^2$ is only approximate. (3) We can say that the language of rational numbers is not enough for what we need to say, and that we need a fuller language in which $x=y$ and $x^2+y^2=z^2$ can be both said consistently. The last alternative is the one that has been universally adopted by the admission to arithmetic of irrational numbers. It does not contradict Euclid's axioms; the first does, since he assumes that a line can have *any* length, and the second contradicts one of their best-known consequences. An experimental proof that it is right is impossible because either (1) or (2) could be true within the errors of measurement even if x, y, z were restricted to be integers. But they would be intolerably complicated, and the adoption of either would require the existence of an unknown and indeterminable standard of length such that all actual lengths are