

ALM<sub>3</sub>

Advanced Lectures in Mathematics

# Computational Conformal Geometry

Xianfeng David Gu • Shing-Tung Yau



高等教育出版社  
HIGHER EDUCATION PRESS

**ALM<sub>3</sub>**

**Advanced Lectures in Mathematics**

# **Computational Conformal Geometry**

Xianfeng David Gu • Shing-Tung Yau



**高等教育出版社**  
HIGHER EDUCATION PRESS



**International Press**

## 图书在版编目 (CIP) 数据

计算共形几何 = Computational Conformal Geometry:

英文/顾险峰, 丘成桐著. — 北京: 高等教育出版社,

2008.1

ISBN 978-7-04-023189-2

I. 计… II. ①顾… ②丘… III. 计算几何: 共形微分几

何—英文 IV. 018

中国版本图书馆 CIP 数据核字 (2007) 第 172585 号

Copyright © 2008 by Xianfeng David Gu and Shing-Tung Yau

The exclusive right of publication owned by

**Higher Education Press**

4 Dewai Dajie, Beijing 100011, P. R. China, and

**International Press**

385 Somerville Ave, Somerville, MA, U.S.A

*All rights reserved. No part of this book may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without permission.*

策划编辑	王丽萍	责任编辑	王丽萍	封面设计	张申申
责任校对	王 雨	责任印制	陈伟光		

出版发行	高等教育出版社	购书热线	010-58581118
社 址	北京市西城区德外大街 4 号	免费咨询	800-810-0598
邮政编码	100011	网 址	<a href="http://www.hep.edu.cn">http://www.hep.edu.cn</a>
总 机	010-58581000		<a href="http://www.hep.com.cn">http://www.hep.com.cn</a>
		网上订购	<a href="http://www.landaco.com">http://www.landaco.com</a>
经 销	蓝色畅想图书发行有限公司		<a href="http://www.landaco.com.cn">http://www.landaco.com.cn</a>
印 刷	涿州市京南印刷厂	畅想教育	<a href="http://www.widedu.com">http://www.widedu.com</a>
开 本	787×1092 1/16	版 次	2008 年 1 月第 1 版
印 张	18.5	印 次	2008 年 1 月第 1 次印刷
字 数	340 000	定 价	49.00 元
插 页	2		

本书如有缺页、倒页、脱页等质量问题, 请到所购图书销售部门联系调换。

版权所有 侵权必究

物 料 号 23189-00

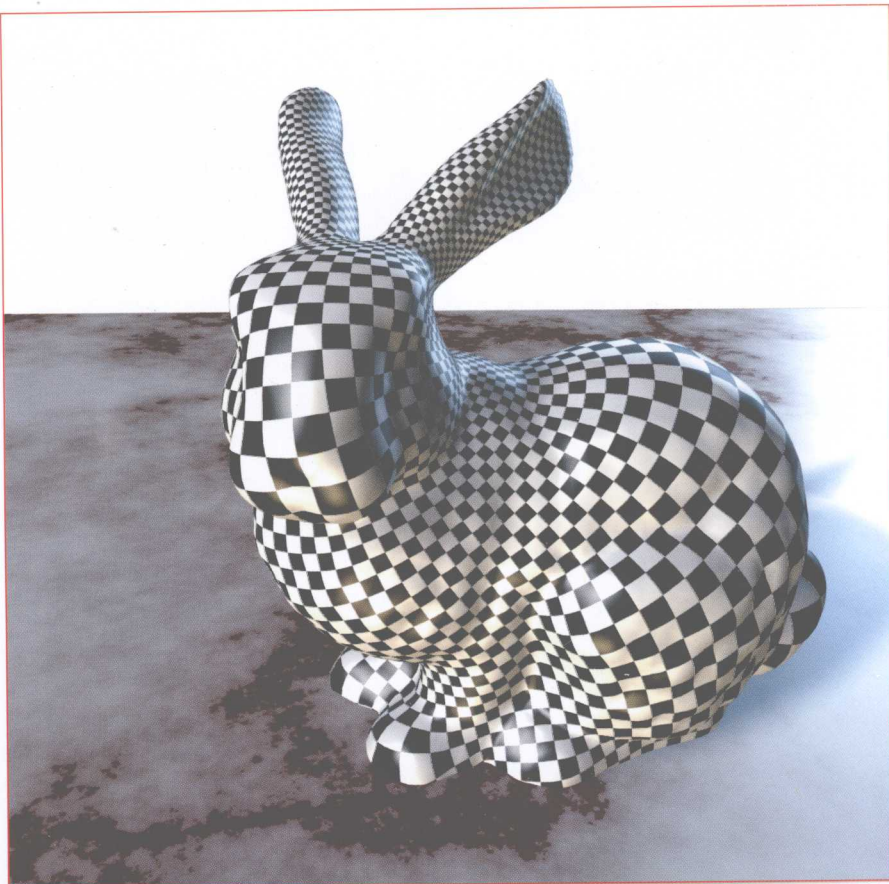
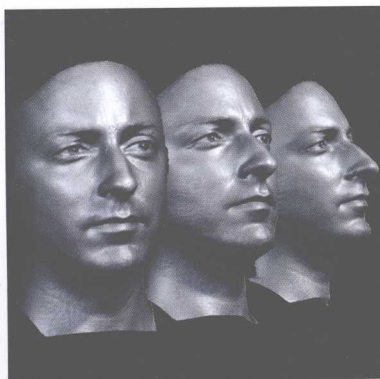


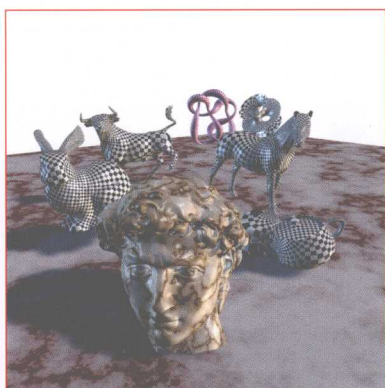
Fig. 1.1. Visualization of the conformal structure of the Stanford bunny surface.



A human face surface scanned by 3D scanner made by Geometric Informatics Inc.



Fig. 1.11. Human face surfaces with different expressions scanned by 3D scanner made by Geometric Informatics Inc.



Visualization of the conformal structures of the surfaces.



Fig. 1.2. Conformal Mappings.

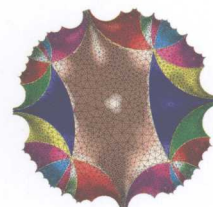
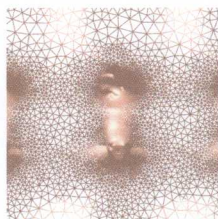
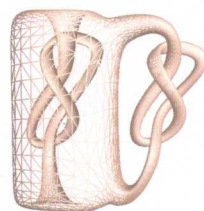
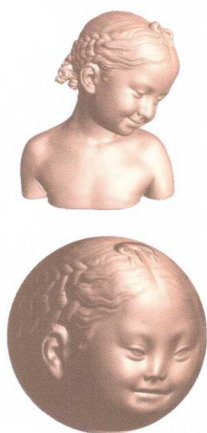


Fig. 1.8. Riemann uniformization theorem. Every closed surface has a Riemannian metric, which is conformal to the original metric and induces constant Gaussian curvature  $+1, 0$  or  $-1$ . Their universal covering spaces can be isometrically embedded into the sphere, the plane or the hyperbolic space.

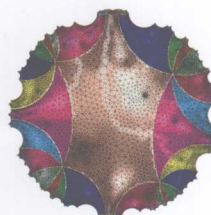
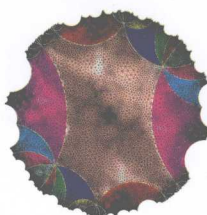
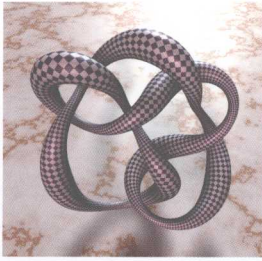


Fig. 1.38. Hyperbolic structures of high genus surfaces.



The conformal structure of a knot surface.



Harmonic 1-form and their conjugates on a genus two surface.

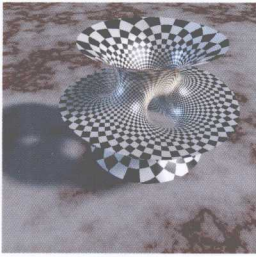


Fig. 5.3. The conformal structure of the Costa minimal surface.



Holomorphic 1-form basis of a genus two surface.

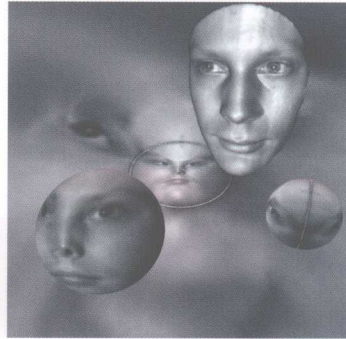


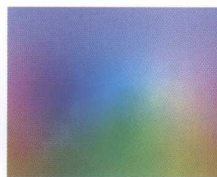
Fig. 10.2. The human face is double covered to form a topological sphere, then the double covering surface is conformally mapped to the unit sphere.



(a) Original Surface



(b) Conformal Parameterization



(c) Geometry Image



(d) Normal map

Fig. 1.30. Geometry image of Michelangelo's David head model.

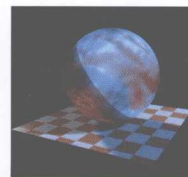
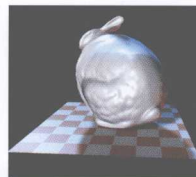


Fig. 1.28. Geometric morphing of the Stanford bunny surface to the unit sphere.

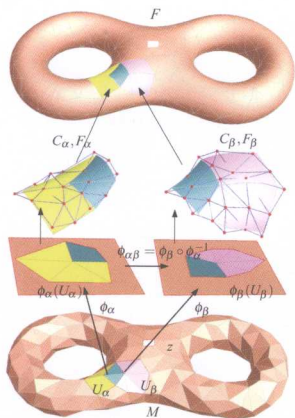


Fig. 1.40. Framework of manifold spline.



Fig. 1.42. Manifold Spline surface.



Spherical harmonic map.



Spherical harmonic map.



Fig. 11.5, 1.6, 1.7., Conformal structures of surfaces.

# ADVANCED LECTURES IN MATHEMATICS

## EXECUTIVE EDITORS

Shing-Tung Yau  
Harvard University  
Cambridge, MA. USA

Lizhen Ji  
University of Michigan  
Ann Arbor, MI. USA

Kefeng Liu  
University of California, Los Angeles  
Los Angeles, CA. USA  
Zhejiang University  
Hangzhou, China

## EXECUTIVE BOARD

Chongqing Cheng  
Nanjing University  
Nanjing, China

Tatsien Li  
Fudan University  
Shanghai, China

Zhong-Ci Shi  
Institute of Computational Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Zhiying Wen  
Tsinghua University  
Beijing, China

Zhouping Xin  
The Chinese University of Hong Kong  
Hong Kong, China

Lo Yang  
Institute of Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Weiping Zhang  
Nankai University  
Tianjin, China

Xiangyu Zhou  
Institute of Mathematics  
Chinese Academy of Sciences (CAS)  
Beijing, China

Xiping Zhu  
Zhongshan University  
Guangzhou, China

**Dedicated to all mathematicians and computer scientists.**

---

# Preface

Conformal geometry is in the intersection of many fields in pure mathematics, such as Riemann surface theory, differential geometry, algebraic curves, algebraic topology, partial differential geometry, complex analysis, and many other related fields. It has a long history in pure mathematics, and is an active field in both modern geometry and modern physics. For example, the conformal fields in super string theory and modular space in theoretic physics are research areas with very fast developments.

Recently, with the rapid development of three dimensional digital scanning technology, computer aided geometric design, bio-informatics, and medical imaging, more and more three dimensional digital models are available. The need for effective methods to represent, process, and utilize the huge amount 3D surfaces has become urgent. Digital geometric processing emerges as an inter-disciplinary field, combining computer graphics, computer vision, visualization, and geometry.

Computational conformal geometry plays an important role in digital geometry processing. It has been applied in many practical applications already, such as surface re-pairing, smoothing, de-noising, segmentation, feature extraction, registration, re-meshing, mesh spline conversion, animation, and texture synthesis. Especially, conformal geometry lays down the theoretic foundation and offers rigorous algorithms for surface parameterizations. Computational conformal geometry is also applied in computer vision for human face tracking, recognition, expression transfer; in medical imaging, for brain mapping, virtual colonoscopy, data fusion; in geometric modeling for constructing splines on manifolds with general topologies.

The fundamental reason why conformal geometry is so useful lies in the following facts:

- Conformal geometry studies conformal structure. All surfaces in daily life have a natural conformal structure. Therefore, the conformal geometric algorithms are very general.
- Conformal structure of a general surface is more flexible than Riemannian metric structure and more rigid than topological structure. It can handle large deformations, which Riemannian geometry cannot efficiently deal with; it preserves a lot of geometric information during the deformation, whereas topological methods lose too much information.
- Conformal maps are easy to control. For example, the conformal maps between two simply connected closed surfaces form a 6-dimensional space, therefore by fixing three points, the mapping is uniquely determined. This fact makes conformal geometric methods very valuable for surface matching and comparison.

- Conformal maps preserve local shapes, therefore it is convenient for visualization purposes.
- All surfaces can be classified according to their conformal structures, and all the conformal equivalent classes form a finite dimensional manifold. This manifold has rich geometric structures, and can be analyzed and studied. In comparison, the isometric classes of surfaces form an infinite dimensional space, which is really difficult to deal with.
- Computational conformal geometric algorithms are based on elliptic partial differential equations, which are easy to solve and the process is stable. Therefore, computational conformal geometry methods are very practical for real engineering applications.
- In conformal geometry, all surfaces in daily life can be deformed to three canonical spaces: the sphere, the plane, or the disk (the hyperbolic space). In other words, any surface admits one of the three canonical geometries: spherical geometry, Euclidean geometry, or hyperbolic geometry. Most digital geometric processing tasks in three dimensional space can be converted to the task in these two dimensional canonical spaces.

The major goals for writing this book are twofold. First, we want to introduce the beautiful theories of conformal geometry to general audiences, and make the elegant conformal structures better appreciated. The major concepts in conformal geometry are profound and abstract, which mainly existed in the imaginations of professional mathematicians. Our conformal geometric methods can compute those concepts explicitly on all kinds of surfaces in daily life, and display them using modern computer graphics and visualization technologies. Therefore, the students can see them, sense them, and accumulate intuition. Professional mathematicians can design experiments and use computers to help their exploration.

Furthermore, we would like to introduce the practical conformal geometric algorithms, and make them easily accessible for computer scientists and engineers. Therefore, the whole book is written to use less abstract mathematical reasoning, but more intuitive explanations and hands on experience. Major concepts and theorems are visualized by figures and computational algorithms are given. Students can implement the algorithms by themselves and see the abstract concepts represented as data structures on computers and create the images reflecting various geometric structures.

The book has two parts. The first part focuses on the theoretical foundations. It covers algebraic topology, differential exterior calculus, differential geometry, Riemann surface theory, surface Ricci flow, and general geometric structures. All of this knowledge is required for doing research in computational conformal geometry. Most of these topics are elementary, and some advanced topics are briefly touched with thorough references.

The second part focuses on computational algorithms, and is completely written in computer science language. It covers the computational algorithms for surfaces, which can be easily generalized to 3-manifolds. Then the algorithms on computing conformal structures for surfaces using various methods are explained in detail. Finally, algorithms for computing hyperbolic structure, and projective structure using Ricci flow method are examined. All algorithms are accompanied by pseudo-code, which is extremely easy to convert to programming language. We hope students can build the software system from scratch, and follow the book to implement various algorithms. The algorithms described in the book have already been applied in industrial applications.

The major content of the book is summarized from our research projects during the last several years. This textbook has been taught in graduate level courses in the Math-

ematics Department at Harvard University and the Computer Science Department at the State University of New York at Stony Brook. The theory part takes one semester, the computer science part takes one semester. The problem sets and programming exercises are valuable for students to improve their understanding and build their practical skill for developing geometric processing software. More teaching materials, coding samples and geometric surface data sets are available from the authors by requests.<sup>1</sup>

The first author is very grateful to all professors in the *Center for Visual Computing* at Stony Brook: Arie Kaufman, Hong Qin, Dimitris Samaras, Klaus Mueller and all faculty members in the Computer Science Department at Stony Brook University and the Computer Information Science and Engineering Department at the University of Florida. The first author wants to thank Steven Gortler, Hugues Hoppy, John Snyder, Julie Dorsey, Leonard McMillan, who led him to the graphics field; Tom Sederberg, Ralph Martin, Shi-Min Hu, Jörg Peters, who led him to the geometric modeling field; Tony Chan, Paul Thompson and Baba Vemuri, who led him to medical imaging field.

Special thanks to Harry Shum, Baining Guo, Dinesh Manocha, Ming Lin, Helmut Pottman, Leif Kobelt for their consistent support and guidance. The first author also wants to thank Craig Gostman, Alla Sheffa, Peter Schröder, Mathieu Desbrun, Bruno Lévy, Myung-Soo Kim, Cindy Grimm, Seungyong Lee, and Kun Zhou for collaborations and support.

The first author also wants to thank the following mathematicians for fruitful collaborations, inspiring discussions, and valuable advices: Feng Luo, Kefeng Liu, Huai-dong Cao. Special thanks for Xiaowei Wang and Wei Luo for the collaborations in applied mathematics.

Special thanks to Yalin Wang, who helped us to conduct the most advanced research in brain mapping, and adapted conformal geometric methods for real industrial applications. Special thanks to Song Zhang, who developed the 3D scanner with high accuracy and high speed, and used our conformal geometric technology for motion capture applications. Special thanks to Junho Kim, who developed an efficient random access mesh compression algorithm, Euclidean Ricci flow and volumetric harmonic forms. The first author also appreciates all the students in the Center for Visual Computing at Stony Brook.

Special thanks to Lance Cong, Alex Mohr, Luke Farrer, and Susan Frank, who contribute to the artwork of this beautiful textbook. Special thanks to Ralph Martin, Feng Luo and Joe Marino, who helped proof reading the manuscript.

Last but not least, we also want to thank both of our families, without their supports, this book could not be accomplished.

Stony Brook, New York,  
Summer 2007

David Gu  
Shing-Tung Yau

<sup>1</sup> The color version of all of the figures, teaching materials, sample codes, and sample data sets can be found at <http://www.cs.sunysb.edu/~gu/>.

## 高等教育出版社自然科学学术出版中心

高等教育出版社是教育部所属的国内最大的教育出版基地,其自然科学学术出版中心下设研究生教育与学术著作分社和自然科学学术期刊分社,正努力成为中国最重要的学术著作出版单位和最大的学术期刊群出版单位。

研究生教育与学术著作分社充分发掘国内外出版资源,为研究生及高层次读者服务,已出版《教育部推荐研究生教学用书》、《当代科学前沿论丛》、《中国科学院研究生院教材》、《中国工程院院士文库》、《长江学者论丛》等一系列研究生教材和优秀学术著作。

自然科学学术期刊分社主要负责教育部大型英文系列学术期刊出版项目 Frontiers in China 中基础科学、生命科学、工程技术类期刊的出版工作,目标是搭建国内学术界与海外交流的平台,以及国内学术期刊界合作的平台。

地 址: 北京市朝阳区惠新东街 4 号富盛大厦 15 层 (100029)

网 址: <http://academic.hep.com.cn/>

购书电话: 010-58581114/1115/1116/1117/1118

---

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Overview of Theories	3
1.1.1	Riemann Mapping	3
1.1.2	Riemann Uniformization	5
1.1.3	Shape Space	6
1.1.4	General Geometric Structure	7
1.2	Algorithms for Computing Conformal Mappings	8
1.3	Applications	13
1.3.1	Computer Graphics	15
1.3.2	Computer Vision	18
1.3.3	Geometric Modeling	23
1.3.4	Medical Imaging	25
	Further Readings	27

---

## Part I Theories

<b>2</b>	<b>Homotopy Group</b>	<b>31</b>
2.1	Algebraic Topological Methodology	31
2.2	Surface Topological Classification	33
2.3	Homotopy of Continuous Mappings	38
2.4	Homotopy Group	38
2.5	Homotopy Invariant	39
2.6	Covering Spaces	40
2.7	Group Representation	43
2.8	Seifert-van Kampen Theorem	44
	Problems	46
<b>3</b>	<b>Homology and Cohomology</b>	<b>49</b>
3.1	Simplicial Homology	49
3.1.1	Simplicial Complex	49
3.1.2	Geometric Approximation Accuracy	51
3.1.3	Chain Complex	53
3.1.4	Chain Map and Induced Homomorphism	56
3.1.5	Simplicial Map	57
3.1.6	Chain Homotopy	57

---

3.1.7	Homotopy Equivalence .....	58
3.1.8	Relation Between Homology Group and Homotopy Group .....	59
3.1.9	Lefschetz Fixed Point .....	59
3.1.10	Mayer-Vietoris Homology Sequence .....	61
3.1.11	Tunnel Loop and Handle Loop .....	62
3.2	Cohomology .....	63
3.2.1	Cohomology Group .....	63
3.2.2	Cochain Map .....	64
3.2.3	Cochain Homotopy .....	65
	Problems .....	67
<b>4</b>	<b>Exterior Differential Calculus .....</b>	<b>69</b>
4.1	Smooth Manifold .....	69
4.2	Differential Forms .....	71
4.3	Integration .....	72
4.4	Exterior Derivative and Stokes Theorem .....	73
4.5	De Rham Cohomology Group .....	73
4.6	Harmonic Forms .....	74
4.7	Hodge Theorem .....	75
	Problems .....	77
<b>5</b>	<b>Differential Geometry of Surfaces .....</b>	<b>79</b>
5.1	Curve Theory .....	79
5.2	Local Theory of Surfaces .....	81
5.2.1	Regular Surface .....	82
5.2.2	First Fundamental Form .....	83
5.2.3	Second Fundamental Form .....	84
5.2.4	Weingarten Transformation .....	84
5.3	Orthonormal Movable Frame .....	86
5.3.1	Structure Equation .....	88
5.4	Covariant Differentiation .....	90
5.4.1	Geodesic Curvature .....	91
5.5	Gauss-Bonnet Theorem .....	92
5.6	Index Theorem of Tangent Vector Field .....	93
5.7	Minimal Surface .....	95
5.7.1	Weierstrass Representation .....	97
5.7.2	Costa Minimal Surface .....	98
	Problems .....	99
<b>6</b>	<b>Riemann Surface .....</b>	<b>103</b>
6.1	Riemann Surface .....	103
6.2	Riemann Mapping Theorem .....	108
6.2.1	Conformal Module .....	108
6.2.2	Quasi-Conformal Mapping .....	109
6.2.3	Holomorphic Mappings .....	110
6.3	Holomorphic One-Forms .....	111
6.4	Period Matrix .....	113
6.5	Riemann-Roch Theorem .....	116
6.6	Abel Theorem .....	119
6.7	Uniformization .....	120

6.8	Hyperbolic Riemann Surface .....	122
6.9	Teichmüller Space .....	125
6.9.1	Quasi-Conformal Map .....	125
6.9.2	Extremal Quasi-Conformal Map .....	127
6.10	Teichmüller Space and Modular Space .....	127
6.10.1	Fricke Space Model .....	128
6.10.2	Geodesic Spectrum .....	130
	Problems .....	131
<b>7</b>	<b>Harmonic Maps and Surface Ricci Flow .....</b>	<b>133</b>
7.1	Harmonic Maps of Surfaces .....	133
7.1.1	Harmonic Energy and Harmonic Maps .....	134
7.1.2	Harmonic Map Equation .....	135
7.1.3	Radó's Theorem .....	135
7.1.4	Hopf Differential .....	136
7.1.5	Complex Form .....	137
7.1.6	Bochner Formula .....	137
7.1.7	Existence and Regularity .....	139
7.1.8	Uniqueness .....	139
7.2	Surface Ricci Flow .....	140
7.2.1	Conformal Deformation .....	141
7.2.2	Surface Ricci Flow .....	142
	Problems .....	143
<b>8</b>	<b>Geometric Structure .....</b>	<b>145</b>
8.1	$(X, G)$ Geometric Structure .....	146
8.2	Development and Holonomy .....	146
8.3	Affine Structures on Surfaces .....	147
8.4	Spherical Structure .....	148
8.5	Euclidean Structure .....	149
8.6	Hyperbolic Structure .....	151
8.7	Real Projective Structure .....	152
	Problems .....	153

---

## Part II Algorithms

---

<b>9</b>	<b>Topological Algorithms .....</b>	<b>159</b>
9.1	Triangular Meshes .....	159
9.1.1	Half-Edge Data Structure .....	160
9.1.2	Code Samples .....	163
9.2	Cut Graph .....	168
9.3	Fundamental Domain .....	169
9.4	Basis of Homotopy Group .....	170
9.5	Gluings Two Meshes .....	171
9.6	Universal Covering Space .....	172
9.7	Curve Lifting .....	174
9.8	Homotopy Detection .....	175
9.9	The Shortest Loop .....	176
9.10	Canonical Homotopy Group Generator .....	178

Further Readings .....	180
Problems .....	181
<b>10 Algorithms for Harmonic Maps .....</b>	<b>183</b>
10.1 Piecewise Linear Functional Space, Inner Product and Laplacian .....	184
10.2 Newton's Method for Open Surface .....	188
10.3 Non-Linear Heat Diffusion for Closed Surfaces .....	190
10.4 Riemann Mapping .....	193
10.5 Least Square Method for Solving Beltrami Equation .....	195
10.6 General Surface Mapping .....	197
Further Readings .....	201
Problems .....	201
<b>11 Harmonic Forms and Holomorphic Forms .....</b>	<b>203</b>
11.1 Characteristic Forms .....	204
11.2 Wedge Product .....	205
11.3 Characteristic 1-Form .....	206
11.4 Computing Cohomology Basis .....	207
11.5 Harmonic 1-Form .....	209
11.6 Hodge Star Operator .....	210
11.7 Holomorphic 1-Form .....	212
11.8 Inner Product Among 1-Forms .....	216
11.9 Holomorphic Forms on Surfaces with Boundaries .....	217
11.10 Zero Points and Critical Trajectories .....	220
11.11 Flat Metric Induced by Holomorphic 1-Forms .....	222
11.12 Conformal Invariants .....	225
11.13 Conformal Mappings for Multi-Holed Annuli .....	227
Further Readings .....	229
Problems .....	230
<b>12 Discrete Ricci Flow .....</b>	<b>233</b>
12.1 Circle Packing Metric .....	234
12.2 Discrete Gaussian Curvature .....	238
12.3 Discrete Surface Ricci Flow .....	240
12.4 Newton's Method .....	243
12.5 Isometric Planar Embedding .....	246
12.6 Surfaces with Boundaries .....	247
12.7 Optimal Parameterization Using Ricci Flow .....	249
12.8 Hyperbolic Ricci Flow .....	252
12.9 Hyperbolic Embedding .....	254
12.9.1 Poincaré Disk Model .....	254
12.9.2 Embedding the Fundamental Domain .....	254
12.9.3 Hyperbolic Embedding of the Universal Covering Space .....	256
12.10 Hyperbolic Ricci Flow for Surfaces with Boundaries .....	259
Further Readings .....	260
Problems .....	261
<b>A Major Algorithms .....</b>	<b>265</b>
<b>B Acknowledgement .....</b>	<b>267</b>