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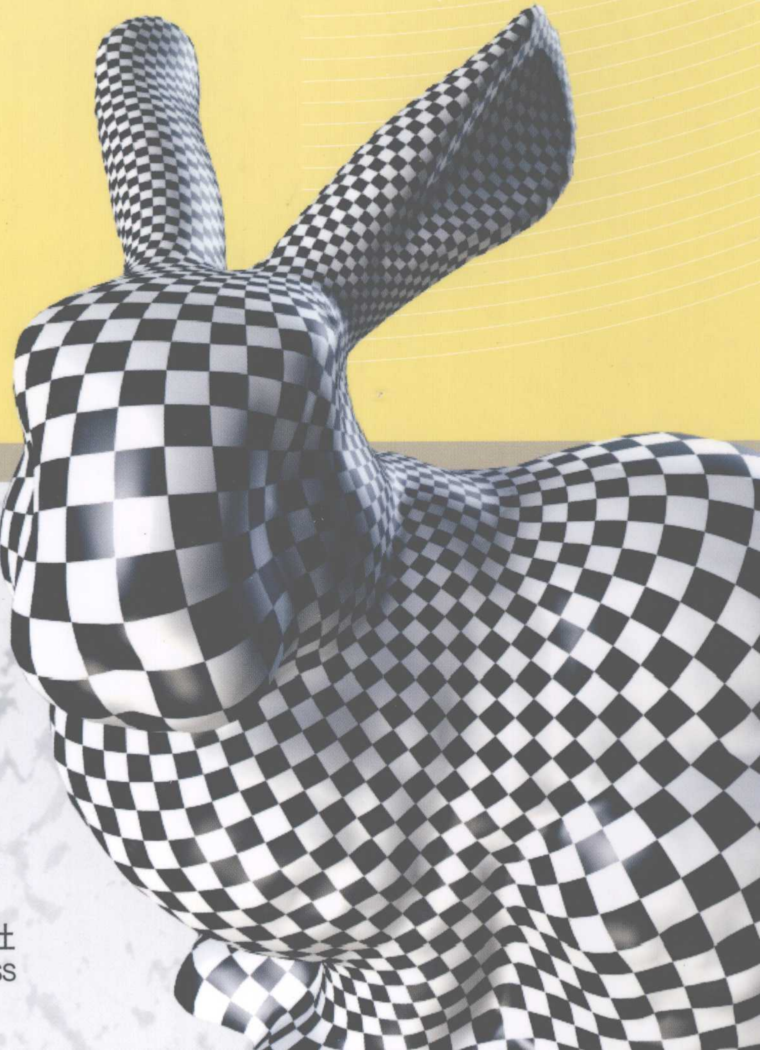
Advanced Lectures in Mathematics

Variational Principles for Discrete Surfaces

Theories and Algorithms

Feng Luo • Xianfeng David Gu • Junfei Dai



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ADVANCED LECTURES IN MATHEMATICS

Dedicated to our parents, wives and kids.

Preface

This book consists of mathematical and algorithmic studies of geometry of polyhedral surfaces based on the variations principles. The part of mathematics is based on a lecture series given by Feng Luo at the Center of Mathematical sciences at Zhejiang University, China, in June and July 2006. The algorithmic theory and applications to computer graphic are based on the work of Xianfeng Gu and are written by him. The task of writing the part of mathematics of the note was done by Junfei Dai who prepared them with great care and made a number of improvements in the exposition.

The aim of this book is to introduce to the students and researchers an emerging field of polyhedral surface geometry and computer graphics based on variation principles. These variational principles are derived from the derivatives of the cosine law for triangles. From mathematical point of view, one of the most fascinating identity in low-dimensional polyhedral geometry is the Schlaefli formula. It relates in a simple and elegant to way the volume, edge lengths and dihedral angles of tetrahedra in the spheres and hyperbolic spaces in dimension 3. The formula can be considered as a foundation of 3-dimensional variational principles for triangulated objects. For a long time, mathematicians have been considering the Gauss-Bonnet formula as the 2-dimensional counterpart of Schlaefli. The recent breakthrough in this area was due to the work of Colin de Verdiere in 1995 who found the first 2-dimensional identity relating edge lengths and inner angles similar to the Schlaefli identity. The mathematical work produced in this book can be considered as establishing all 2-dimensional counterparts of Schaefli formula. It turns out there are continuous families of Schlaefli type identities in dimension 2. These identities produce many interesting variational principles for polyhedral surfaces. In the part of mathematics of the book, we are focusing on a study of the rigidity phenomena on polyhedral surfaces. Some moduli space problems are also discussed in the book.

In the part of algorithm of the book, we introduce discrete curvature flow from both theoretical and practical points of view. Discrete curvature flow is a powerful tool for designing metrics by prescribed curvatures. The algorithm maps general surfaces with arbitrary topologies to three canonical spaces. Therefore, all geometric problems of surfaces in 3D space are converted to 2D ones. This greatly improves the efficiency and accuracy for engineering applications. The discrete Ricci flow algorithm, and Ricci energy optimization algorithm are rigorous, robust, flexible and efficient. They have been applied for surface matching, registration, shape classification, shape analysis and many fundamental applications in practice.

This book is written for senior undergraduate students and graduate students majoring in mathematics or computer science. The mathematical requirements to follow the proofs in the book are some basic knowledge of differential geometry and elementary surface topology. We have not stated and proved theorems in the book in the most general form to avoid the technical details. For computer science majors with basic knowledge in data structure and algorithm, all the algorithms in the book can be implemented straightforwardly by following the pseudo codes and the software system can be built step by step from scratch. Some data sets and source code are also available upon request.

This book should be valuable for researchers in surface geometry, computer graphics, computer vision, geometric modelling, visualization, medical imaging and scientific computation fields. The computational algorithms are also useful for geometric modelers, industrial products designer, digital artists, animators, game developers and anyone who needs digital geometry processing tools.

We owe much to many colleagues and friends with whom we have discussed the subject matter over the years. The first author would like to thank Ben Chow who introduced him the wonderful world of discrete curvature flows and the Ricci flow. The second author is very grateful to all professors in the *Center of Visual Computing* at Stony Brook: Arie Kaufman, Hong Qin, Dimitris Samaras, Klaus Mueller and all faculty members in the Computer Science department in Stony Brook University. The second author deeply appreciate the encouragements and valuable advices from Professor Shing-Tung Yau.

The authors thank National Science Foundation of USA for supporting our collaborative research on discrete curvature flow.

Last but not least, we also want to thank our families, without their supports, this book could not be accomplished.

Piscataway, New Jersey,
Stony Brook, New York,
Hangzhou, China,

Feng Luo
Xianfeng David Gu
Junfei Dai
Summer 2007

Contents

1

Introduction

1.1 Variational Principle and Isoperimetric Problems

The variational approach to characterize a geometric object is to find among geometric shapes of similar properties the one with the least measurement (the least energy). Let us illustrate this theme by the famous isoperimetric problem which asks for a characterization of the circle among all piecewise smooth non-self intersecting loops in the plane. Let $\gamma : S^1 \to \mathbb{C}$ or $\gamma : [0, 2\pi] \to \mathbb{C}$ be a piecewise smooth embedding so that $\gamma(0) = \gamma(2\pi)$. Then the length $l(\gamma)$ of γ and the area $A(\gamma)$ enclosed by γ are given by

$$l(\gamma) = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}dt$$

and

$$A(\gamma) = \int\int_\Omega dxdy = \oint_\gamma xdy = \int_0^{2\pi} x(t)y'(t)dt$$

where $\gamma(t) = (x(t), y(t))$ and Ω is the Jordan domain enclosed by γ. Define the energy of γ, denoted by $E(\gamma)$, by

$$E(\gamma) = l^2/A.$$

Then the isoperimetric inequality (see [Cha] for a proof) states that

$$E(\gamma) \geqslant 4\pi^2\rho^2/\pi\rho^2 = 4\pi^2$$

with equality if only if γ is a round circle.

The variational principle says that circles are the minimizers of energy E among all piecewise smooth Jordan curves. Technically, the difficult part is to prove the existence of minimizes. As long as one knows the existence, it is not hard to see that the minimizer has to be a circle. Indeed, both length l and area A are invariant under rigid motion. It follows that minimizer has to have the rotational symmetry. Thus, it must be a circle.

A counterpart of the isoperimetric problem is that the regular n-sided polygons (or n-gons) are the $E(\gamma)$ energy minimizer among all n-gons. This was first proved by Steinitz. The argument is elementary and elegant. First, due to the finite dimensionality, one can show that the energy minimizer exists. Next, one shows that the minimizer must be convex due to the reflection operation which increases the energy (see Fig. 1.1).

It is known that among all triangles of area 1 with the same base, the isosceles triangle has the smallest circum length (see Fig. 1.2). Using this fact to the minimizer, one sees that the edge lengths of the minimizer must be the same. Finally, Steinitz showed that the

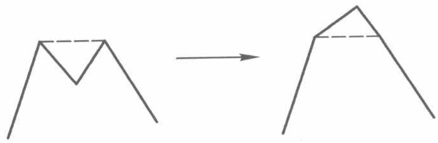

Fig. 1.1. Reflection

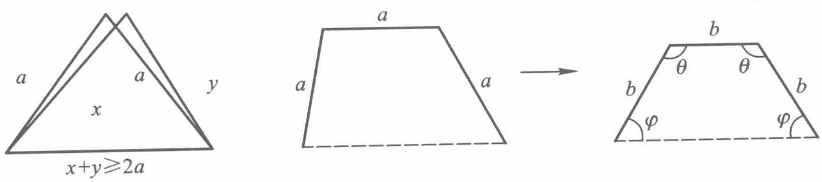

Fig. 1.2.

minimizer must be regular by establishing the fact that among the quadrilaterals of area 1 with the same base, the trapezoid of the same top two angles has the smallest circumlength.

The Steinitz's solution above is an example of variational principle on triangulated 1-manifolds.

The main mathematical goal of our study in this book is to consider the 2-dimensional analogous of the Steinitz's solution. We will consider triangulated surfaces and construct the associated energy functions. Among the main results obtained in this work is the complete list of all possible energy functions of localized nature for 2-dimensional variational principles on triangulated surfaces.

1.2 Polyhedral Metrics and Polyhedral Surfaces

Take a finite set of points in the Euclidean 3-space $\mathbb{E}^3$ and the convex hull of these points, we obtain a convex polytope whose vertices are among the given finite set. If the points are generic, then the convex polytope has triangle faces. In this case, the boundary surface is a polyhedral surface. It has two properties. Firstly the surface is triangulated, secondly, the induced metric on the surface is locally flat except at the vertices. Recall that a triangulation of a surface is defined as follows. Take a finite collection of disjoint triangles and identify all pairs of edges by homeomorphisms. The quotient space is a surface with a triangulation whose cells are the quotients of triangles, edges and vertices in the disjoint union.

Definition 1.1. *An Euclidean polyhedral surface is a triple (S,T,d) where S is a closed surface, T is a triangulation of S and d is a metric on S whose restriction to each triangle is isometric to an Euclidean triangle. We will call the metric d a polyhedral metric. The discrete curvature k_0 of (S,T,d) is a function which assigns each vertex 2π less the sum of inner angles at the vertex, i.e.,*

$$k_0(v) = 2\pi - \sum_{i=1}^{m} \alpha_i,$$

where $\alpha_1,\ldots,\alpha_m$ are the inner angles (of triangles) at the vertex v.

We will also call above an $\mathbb{E}^2$ polyhedral surface. If we use the spherical (or hyperbolic) triangles instead of Euclidean triangles in Definition 1.1, the result is called a spherical (or $\mathbb{S}^2$) polyhedral surface (resp. $\mathbb{H}^2$ polyhedral surface). Spherical and hyperbolic polyhedral surfaces will be covered extensively in this book. It is due to the fact the $\mathbb{S}^2$ and $\mathbb{H}^2$ polyhedral surfaces constitute generic phenomena in polyhedral geometry so that Euclidean ones are the limiting cases. The discrete curvature k_0 is defined by the same formula for $\mathbb{S}^2$ and $\mathbb{H}^2$ polyhedral surfaces.

For a triangulated surface (S,T), we use $E = E(T)$ and $V = V(T)$ to denote respectively the sets of all edges and vertices of the triangulation. A polyhedral metric d on (S,T) is determined by specifying the lengths of edges, i.e., by the edge length function $l_d : E \to \mathbb{R}_{>0}$. In the sequel, we will identify a polyhedral metric d with its edge length function. In particular, we say two polyhedral metrics d, d', are equivalent if they have the same edge length function, i.e., $l_d = l_{d'}$.

From the definition, it is clear that the basic unit of discrete curvature is the inner angle. Furthermore, the metric-curvature relation is given by the cosine law.

Just like the smooth case, one of the main problem of study in polyhedral surface is to understand the relationship between the metric and curvature. Naturally we should study the cosine law carefully.

1.3 A Brief History on Geometry of Polyhedral Surface

The study of polyhedral geometry has a long history. The Greeks were able to find the five regular polytopes in $\mathbb{E}^3$ long time ago. Modern developments started with Euler's famous theorem concerning the number of vertices, edges and faces of convex polytopes in $\mathbb{E}^3$. Additional ideas and results came form Legendre, Cauchy, Steiner, Sylvester, Cayley, Möbius, Schlafli, Tait, Minkovski, Steinitz, and others.

We just mention a few remarkable theorems concerning rigidity and existence of the convex polyhedral surfaces here.

Theorem 1.2 (Cauchy, 1813). *Suppose P,Q are two compact convex polytopes in $\mathbb{E}^3$ so that there exists an isometry $f : \partial P \to \partial Q$ between their boundary surfaces preserving vertices, edges and facess. Then P and Q differ by a rigid motion of $\mathbb{E}^3$.*

Cauchy's original proof contained a gap which was fixed by Steinitz. A 3-connected graph is a graph with the property that if one removes any two vertices, the remaining subgraph is still connected.

Theorem 1.3 (Steinitz, 1917). ([Gru]) *For any 3-connected graph G in the 2-sphere $\mathbb{S}^2$, there exists a compact convex polytope P in $\mathbb{E}^3$ whose edge graph is isomorphism to G by a homeomorphism from ∂P to $\mathbb{S}^2$.*

A very important problem is to understand the space of all compact convex polytopes with combinatorics G. This is called the realization space. Steinitz proved that the space is connected. Conjecturally the space should be a cell.

Theorem 1.4 (Aleksandrov, 1942). *If (S^2, T, d) is a Euclidean polyhedral surface with curvature $k_0 \geqslant 0$, then there exists a compact convex polytope in $\mathbb{E}^3$ whose boundary is isometric to (S^2, d).*

Furthermore, the convex polytope in Aleksandrov's theorem is unique up to isometry of $\mathbb{E}^3$. Note, the uniqueness does not follow directly from Cauchy's rigidity theorem which assumes that boundary isometry preserves combinators. The most general rigidity theorem was established by Aleksandrov and Pogorelov.

Theorem 1.5 (Aleksandrov and Pogorelov, 1948). *If P and Q are two compact convex bodies in $\mathbb{E}^3$ so that their boundaries are isometric, they differ by a rigid motion of $\mathbb{E}^3$.*

Combining Aleksandrov realization theorem and the rigidity theorem, one sees that for a polyhedral surface (S^2, T, d) of non-negation discrete curvature $k_0 \geqslant 0$, there exists a natural cell decomposition T_1 of S^2 coming from the convex polytope realization. On the other hand, for the singular flat metric (S^2, d), there exists a natural Delaunay cell-decomposition T_2 of S^2. It is known that T_1 and T_2 may be different. It is conjecture that T_1 and T_2 differ by a bounded number of flips which is linear in the number of vertices in T_1 (or T_2) (see Fig. 1.3). To be more precise, let N be the number of vertices in T_1. By adding diagonals to non-triangular faces of T_1 and T_2 (without introducing extra vertices), we obtain two triangulations $\widetilde{T_1}$ and $\widetilde{T_2}$. Conjecturally, $\widetilde{T_1}$ and $\widetilde{T_2}$ differ by a finite number of flip operations so that the number of flips is linear in N.

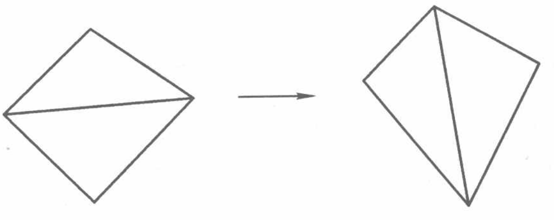

Fig. 1.3.

1.4 Recent Works on Polyhedral Surfaces

The recent insurgent of interests on polyhedral surfaces is mainly due to the work of William Thurston on geometrization of 3-manifolds. Recall that a polyhedral metric (S, T, d) is called a circle packing metric if there exists a positive function $r : V \to \mathbb{R}_{>0}$ so that the edge length l_d of an edge uv with end points u and v is given by

$$l_d(u, v) = r(u) + r(v).$$

For instance a triangle $\Delta v_1 v_2 v_3$ in T is realized by an Euclidean triangle of edge lengths $r_1 + r_2, r_2 + r_3, r_3 + r_1$. Geometrically, the vertices v_1, v_2, v_3 are the centers of three pairwise tangent circles of radiu r_1, r_2 and r_3 (see Fig. 1.4).

Theorem 1.6 (Thurston[Thu], Andreev[An]). *Suppose (S, T) is a closed triangulated surface. Then*

(a) *An Euclidean circle packing metric on (S, T) is determined up to scaling by its discrete curvature k_0.*

(b) *A hyperbolic circle packing metric on (S, T) is determined by its discrete curvature k_0.*

Furthermore, the space of all discrete curvatures $\{k_0\}$ of $\mathbb{E}^2$ (or $\mathbb{H}^2$) circle packing metric on (S, T) is a convex polytope.

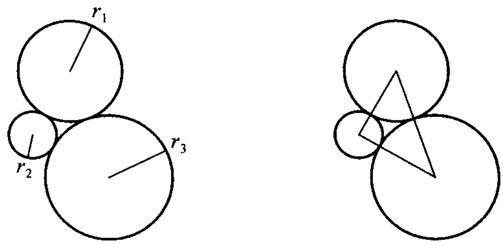

Fig. 1.4.

Definition 1.7 (Leibon[Le]). *Suppose (S,T,d) is a K^2 polyhedral surface where $K^2 = \mathbb{E}^2$ or $\mathbb{S}^2$ or $\mathbb{H}^2$. Then the ψ_0-curvature of it is the map $\psi_0 : \{edges\ in\ T\} \to \mathbb{R}$ so that $\psi_0(e) = \frac{1}{2}(b+b'+c+c'-a-a')$ where a,a' are the angles facing the edge e and b,b',c,c' are the angles adjacent to e.*

Note that for Euclidean polyhedral surfaces, $b+c = \pi - a$ and $b'+c' = \pi - a - a'$. Thus, $\psi_0(e) = \pi - a - a'$.

Definition 1.8. *A polyhedral surface (S,T,d) is called Delaunay if $\psi_0(e) \geqslant 0$ for all edges e in the triangulation. In the case that the discrete curvature $k_0(v) \equiv 0$ at all vertices, i.e, the polyhedral surface (S,T,d) is the geometric triangulation of a constant curvature surface, the Delaunay condition is the some as the classical no-vertex-circle condition. Namely, $\psi_0(e) \geqslant 0$ for all e if and only if the circum-circle of each triangle contains no vertex in its interior (in the universal cover).*

Two remarkable rigidity theorems for Euclidean and hyperbolic polyhedral surfaces are:

Theorem 1.9 (Rivin, 1995). *Any Euclidean polyhedral metric on a closed triangulated surface (S,T) is determined up to scaling by its ϕ_0-curvature $\phi_0 : E \to \mathbb{R}$ so that $\phi_0(e) = \pi - a - a'$ where a,a' are the angles facing e. Furthermore, the set of all ϕ_0-curvatures $\{\phi_0\}$ of all Delaunay Euclidean polyhedral metrics on (S,T) is a convex polytope.*

Note by the remarks above, we have $\phi_0(e) = \psi_0(e)$ in the Euclidean case. Thus Delaunay means $\phi_0 \geqslant 0$.

Theorem 1.10 (Leibon, 2002). *A hyperbolic polyhedral metric on a triangulated closed surface (S,T) is determined by its ψ_0-curvature. Furthermore, the set of all ψ_0-curvatures $\{\psi_0\}$ of Delaunay hyperbolic metrics on (S,T) is a convex polytope.*

1.5 Some of Our Results

Following [Lu1], we introduce three families of curvatures. The relationships between the polyhedral metrics and these curvatures like quantities are the main focus of the study in this book.

Suppose (S,T) is a closed triangulated surface so that T is the triangulation, E and V are the sets of all edges and vertices respectively. Let $\mathbb{E}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$ be the Euclidean, the spherical and the hyperbolic 2-dimensional geometries recpectily.

Definition 1.11. *Let $h \in \mathbb{R}$. Given a K^2 polyhedral metric l on (S,T) where $K^2 = \mathbb{E}^2$, or $\mathbb{S}^2$ or $\mathbb{H}^2$, the ϕ_h-curvature of the polyhedral metric l is the function $\phi_h : E \to \mathbb{R}$ sending an edge e to:*

$$\phi_h(e) = \int_a^{\pi/2} \sin^h(t)dt + \int_{a'}^{\pi/2} \sin^h(t)dt \tag{1.1}$$

where a, a' are the inner angles facing the edge e. See Fig. 1.5.

The ψ_h-curvature of the metric l is the function $\psi_h : E \to \mathbb{R}$ sending an edge e to

$$\psi_h(e) = \int_0^{\frac{b+c-a}{2}} \cos^h(t)dt + \int_0^{\frac{b'+c'-a'}{2}} \cos^h(t)dt \tag{1.2}$$

where b, b', c, c' are the inner angles adjacent to the edge e and a, a' are the angles facing the edge e. See Fig. 1.5.

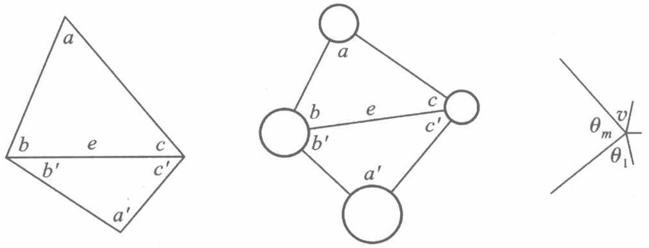

Fig. 1.5.

The h-th discrete curvature k_h of the polyhedral metric l on (S,T) is the function $k_h : V \to \mathbb{R}$ sending a vertex v to

$$k_h(v) = \frac{4-m}{2}\pi + \sum_{i=1}^{m} \int_{\theta_i}^{\pi/2} \tan^h(t/2)dt \tag{1.3}$$

where $\theta_1, \cdots, \theta_m$ are all inner angles at the vertex v. See Fig. 1.5.

We remark that ϕ_0- and ψ_0-curvatures were first introduced by I. Rivin [Ri] and G. Leibon [Le] respectively.

We have

Theorem 1.12 ([Lu1]). *Let $h \in \mathbb{R}$ and (S,T) be a closed triangulated surface.*

(1) *A Euclidean circle packing metric on (S,T) is determined up to isometry and scaling by its k_h-th discrete curvature.*
(2) *A hyperbolic circle packing metric on (S,T) is determined up to isometry by its k_h-th discrete curvature.*
(3) *If $h \leqslant -1$, a Euclidean polyhedral metric on (S,T) is determined up to isometry and scaling by its ϕ_h-curvature.*
(4) *If $h \leqslant -1$ or $h \geqslant 0$, a spherical polyhedral metric on (S,T) is determined up to isometry by its ϕ_h-curvature.*
(5) *If $h \leqslant -1$ or $h \geqslant 0$, a hyperbolic polyhedral surface is determined up to isometry by its ψ_h-curvature.*

For any $h \in \mathbb{R}$, there are local rigidity theorems in cases (3), (4), (5). We believe that above theorem holds for all $h \in \mathbb{R}$.

The counterpart of Theorem 1.12(5) for hyperbolic metrics with totally geodesic boundary on an ideal triangulated compact surface is the following. Recall that an *ideal triangulated compact surface* with boundary (S,T) is obtained by removing a small open regular neighborhood of the vertices of a triangulation of a closed surface. The *edges* of an ideal triangulation T correspond bijectively to the edges of the triangulation of the closed surface. Given a hyperbolic metric l with geodesic boundary on an ideal triangulated surface (S,T), there is a unique geometric ideal triangulation T^* isotopic to T so that all edges are geodesics orthogonal to the boundary. The edges in T^* decompose the surface into hyperbolic right-angled hexagons. The *ψ_h-curvature* of the hyperbolic metric l is defined to be the map $\psi_h : \{$ all edges in $T\} \to \mathbb{R}$ sending each edge e to

$$\psi_h(e) = \int_0^{\frac{b+c-a}{2}} \cosh^h(t)dt + \int_0^{\frac{b'+c'-a'}{2}} \cosh^h(t)dt \tag{1.4}$$

where a, a' are lengths of arcs in the boundary (in the ideal triangulation T^*) facing the edge and b, b', c, c', are the lengths of arcs in the boundary adjacent to the edge so that a, b, c lie in a hexagon. See Fig. 1.5.

Theorem 1.13 ([Lu1]). *A hyperbolic metric with totally geodesic boundary on an ideal triangulated compact surface is determined up to isometry by its ψ_h-curvature. Furthermore, if $h \geqslant 0$, then the set of all ψ_h-curvatures on a fixed ideal triangulated surface is an explicit open convex polytope P_h in a Euclidean space so that $P_h = P_0$.*

The case when $h < 0$ has been recently established by Ren Guo [Gu1]. He proved

Theorem 1.14 ([Gu1]). *Under the same assumption as in theorem 1.13, if $h < 0$, the set of all ψ_h-edge invariants on a fixed ideal triangulated surface is an explicit bounded open convex polytope P_h in a Euclidean space. Furthermore, if $h < h'$, then $P_h \subset P_{h'}$.*

The interesting part of Theorem 1.13 is that the images of the Teichmüller space in these coordinates (for $h \geqslant 0$) are all the same. Whether these coordinates are related to quantum Teichmüller theory is an interesting topic. See [FC, Ka, BL, Te] for more information. Combining Theorem 1.13 with the work of Ushijima [Us] and Kojima [Ko], one obtains for each $h \geqslant 0$ a cell decomposition of the Teichmüller space invariant under the action of the mapping class group.

The space of all polyhedral surfaces will also be studied in this book. In particular, we will give a Marden-Rodin's proof of Thurston's theorem that the space of all discrete curvatures $\{k_0\}$ of hyperbolic circle packing metrics on (S,T) is a convex polytope.

1.6 The Method of Proofs and Related Works

The variational principles were used to the proofs of above theorems. The use of variational principle on triangulated surfaces recently appeared in the seminal work of Colin de Verdiere [CV1] in 1991. In this work, Colin de Verdiere gave a new proof of Thurston-Andreev's circle packing theorem using a variational principle. The most striking part of the proof in [CV1] is the discovery of an action functional for the geometric triangles. This action functional of Colin de Verdiere can be considered as a 2-dimensional analogous of the Schlaefli formula. Recall that for a tetrahedron in $\mathbb{H}^3$ or $\mathbb{S}^3$, if l_e and θ_e are the length and dihedral angle at the edge e, then Schlaefli formula says