Serge Lang

UNDERGRADUATE ANALYSIS

Second Edition

高等数学分析 第2版

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Serge Lang

Undergraduate Analysis

Second Edition

With 91 Illustrations

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Foreword to the First Edition

The present volume is a text designed for a first course in analysis. Although it is logically self-contained, it presupposes the mathematical maturity acquired by students who will ordinarily have had two years of calculus. When used in this context, most of the first part can be omitted, or reviewed extremely rapidly, or left to the students to read by themselves. The course can proceed immediately into Part Two after covering Chapters 0 and I. However, the techniques of Part One are precisely those which are not emphasized in elementary calculus courses, since they are regarded as too sophisticated. The context of a third-year course is the first time that they are given proper emphasis, and thus it is important that Part One be thoroughly mastered. Emphasis has shifted from computational aspects of calculus to theoretical aspects: proofs for theorems concerning continuous functions; sketching curves like x^2e^{-x}, $x \log x$, $x^{1/x}$ which are usually regarded as too difficult for the more elementary courses; and other similar matters.

Roughly speaking, the course centers around those properties which have to do with uniform convergence, uniform limits, and uniformity in general, whether in the context of differentiation or integration. It is natural to introduce the sup norm and convergence with respect to the sup norm as one of the most basic notions. One of the fundamental purposes of the course is to teach the reader fundamental estimating techniques involving the triangle inequality, especially as it applies to limits of sequences of functions. On the one hand, this requires a basic discussion of open and closed sets in metric spaces (and I place special emphasis on normed vector spaces, without any loss of generality), compact sets, continuous functions on compact sets, etc. On the other hand, it is also necessary to include the classical techniques of determining convergence for series, Fourier series in particular. A number of convergence theorems are subsumed under the general technique of Dirac sequences, applying as well to the Landau proof of the Weierstrass approximation theorem as to the proof of uniform convergence of the Cesaro sum of a Fourier series to a continuous function; or to the construction by means of the Poisson kernel of a harmonic function on the disc having given boundary value on the circle. Thus concrete classical examples are emphasized.

The theory of functions or mappings on $\mathbf{R}^n$ is split into two parts. One chapter deals with those properties of functions (real valued) which can essentially be reduced to one variable techniques by inducing the function on a curve. This includes the derivation of the tangent plane of a surface, the study of the gradient, potential functions, curve integrals, and Taylor's formula in several variables. These topics require only a minimum of linear algebra, specifically only n-tuples in $\mathbf{R}^n$ and the basic facts about the scalar product. The next chapters deal with maps of $\mathbf{R}^n$ into $\mathbf{R}^m$ and thus require somewhat more linear algebra, but only the basic facts about matrices and determinants. Although I recall briefly some of these facts, it is now reasonably standard that third-year students have had a term of linear algebra and are at ease with matrices. Systematic expositions are easily found elsewhere, for instance in my *Introduction to Linear Algebra*.

Only the formal aspect of Stokes' theorem is treated, on simplices. The computational aspects in dimension 2 or 3 should have been covered in another course, for instance as in my book *Calculus of Several Variables*; while the more theoretical aspects on manifolds deserve a monograph to themselves and inclusion in this book would have unbalanced the book, which already includes more material than can be covered in one year. The emphasis here is on analysis (rather than geometry) and the basic estimates of analysis. The inclusion of extra material provides alternatives depending on the degree of maturity of the students and the taste of the instructor. For instance, I preferred to provide a complete and thorough treatment of the existence and uniqueness theorem for differential equations, and the dependence on initial conditions, rather than slant the book toward more geometric topics.

The book has been so written that it can also be used as a text for an honors course addressed to first- and second-year students in universities who had calculus in high school, and it can then be used for both years. The first part (calculus at a more theoretical level) should be treated thoroughly in this case. In addition, the course can reasonably include Chapters VI, VII, the first three sections of Chapter VIII, the treatment of the integral given in Chapter X and Chapter XV on partial derivatives. In addition, some linear algebra should be included.

Traditional courses in "advanced calculus" were too computational, and the curriculum did not separate the "calculus" part from the "analysis" part, as it does mostly today. I hope that this *Undergraduate Analysis* will meet the need of those who want to learn the basic techniques of analysis for the first time. My *Real and Functional Analysis* may then be used as a continuation at a more advanced level, into Lebesgue integration and functional analysis, requiring precisely the background of this undergraduate course.

New Haven
Spring 1983

SERGE LANG

Foreword to the Second Edition

The main addition is a new chapter on locally integrable vector fields, giving a criterion for such vector fields to be globally integrable in terms of the winding number. The theorem will of course reappear in a subsequent course on complex analysis, as the global version of Cauchy's theorem in the context of complex differentiable functions. However, it seems valuable and efficient to carry out this globalization already in the undergraduate real analysis course, so that students not only learn a genuinely real theorem, but are then well prepared for the complex analysis course. The "genuinely real theorem" also involves an independent theorem about circuits in the plane, which provides a good introduction to other considerations involving the topology of the plane and homotopy. However, the sections on homotopy will probably be omitted for lack of time. They may be used for supplementary reading.

Aside from the new chapter, I have rewritten many sections, I have expanded others, for instance: there is a new section on the heat kernel in the context of Dirac families (giving also a good example of improper integrals); there is a new section on the completion of a normed vector space; and I have included a proof of the fundamental lemma of (Lebesgue) integration, showing how an L^1-Cauchy sequence converges pointwise almost everywhere. Such a proof, which is quite short, illustrates concepts in the present book, and also provides a nice introduction to future courses which begin with Lebesgue integration.

I have also added more exercises. I emphasize that the exercises are an integral part of the development of the course. Some things proved later are earlier assigned as exercises to give students a chance to think about something before it is dealt with formally in the course. Furthermore, some exercises work out some items to prepare for their use later. For instance, bump functions can be constructed as an application of the exponential function in an early chapter, but they come up later in certain theoretical and practical contexts, for various purposes of approximations. The bump functions provide an aspect of calculus merging with analysis in a way which is usually not covered in the introductory calculus courses.

I personally became aware of the Bruhat–Tits situation with the semi-parallelogram law only in 1996, and realized it was really a topic in basic undergraduate analysis. So I have given the basic results in this direction as exercises when complete metric spaces are first introduced in Chapter VI.

The book has more material than can be covered completely in one year. The new chapter may provide good reading material for special projects outside class, or it may be included at the cost of not covering other material. For instance, the chapter on differential equations may be covered by a separate course on that subject. Much depends on how extensively the first five chapters need to be reviewed or actually covered. In my experiences, a lot.

I am much indebted to Allen Altman and Akira Komoto for a long list of corrections. I am also indebted to Rami Shakarchi for preparing an answer book, and also for several corrections.

New Haven, 1996 SERGE LANG

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PART ONE

Review of Calculus

CHAPTER 0

Sets and Mappings

In this chapter, we have put together a number of definitions concerning the basic terminology of mathematics. The reader could reasonably start reading Chapter I immediately, and refer to the present chapter only when he comes across a word which he does not understand. Most concepts will in fact be familiar to most readers.

We shall use some examples which logically belong with later topics in the book, but which most readers will have already encountered. Such examples are needed to make the text intelligible.

0, §1. SETS

A collection of objects is called a **set**. A member of this collection is also called an **element** of the set. If a is an element of a set S, we also say that a **lies** in S, and write $a \in S$. To denote the fact that S consists of elements $a, b, \ldots$ we often use the notation $S = \{a, b, \ldots\}$. We assume that the reader is acquainted with the set of positive integers, denoted by $\mathbf{Z}^+$, and consisting of the numbers $1, 2, \ldots$. The set consisting of all positive integers and the number 0 is called the set of **natural numbers**. It is denoted by $\mathbf{N}$.

A set is often determined by describing the properties which an object must satisfy in order to be in the set. For instance, we may define a set S by saying that it is the set of all real numbers $\geqq 1$. Sometimes, when defining a set by certain conditions on its elements, it may happen that there is no element satisfying these conditions. Then we say that the set is **empty**. Example: The set of all real numbers x which are both > 1 and < 0 is empty because there is no such number.

If S and S' are sets, and if every element of S' is an element of S, then we say that S' is a **subset** of S. Thus the set of all even positive integers $\{2, 4, 6, \ldots\}$ is a subset of the set of positive integers. To say that S' is a subset of S is to say that S' is part of S. Observe that our definition of a subset does not exclude the possibility that $S' = S$. If S' is a subset of S, but $S' \neq S$, then we shall say that S' is a **proper** subset of S. Thus the set of even integers is a proper subset of the set of natural numbers. To denote the fact that S' is a subset of S, we write $S' \subset S$, or $S \supset S'$; we also say that S' is **contained** in S. If $S' \subset S$ and $S \subset S'$ then $S = S'$.

If S_1, S_2 are sets, then the **intersection** of S_1 and S_2, denoted by $S_1 \cap S_2$, is the set of elements which lie in both S_1 and S_2. For instance, if S_1 is the set of natural numbers $\geqq 3$, and S_2 is the set of natural numbers $\leqq 3$, then $S_1 \cap S_2 = \{3\}$ is the set consisting of the number 3 alone.

The **union** of S_1 and S_2, denoted by $S_1 \cup S_2$, is the set of elements which lie in S_1 or S_2. For example, if S_1 is the set of all odd numbers $\{1, 3, 5, 7, \ldots\}$ and S_2 consists of all even numbers $\{2, 4, 6, \ldots\}$, then $S_1 \cup S_2$ is the set of positive integers.

If S' is a subset of a set S, then by the **complement** of S' in S we shall mean the set of all elements $x \in S$ such that x does not lie in S' (written $x \notin S'$). In the example of the preceding paragraph, the complement of S_1 in $\mathbf{Z}^+$ is the set S_2, and conversely.

Finally, if S, T are sets, we denote by $S \times T$ the set of all pairs (x, y) with $x \in S$ and $y \in T$. Note that if S or T is empty, then $S \times T$ is also empty. Similarly, if $S_1, \ldots, S_n$ are sets, we denote by $S_1 \times \cdots \times S_n$, or

$$\prod_{i=1}^{n} S_i$$

the set of all n-tuples $(x_1, \ldots, x_n)$ with $x_i \in S_i$.

0, §2. MAPPINGS

Let S, T be sets. A **mapping** or map, from S to T is an association which to every element of S associates an element of T. Instead of saying that f is a mapping of S into T, we shall often write the symbols $f: S \to T$.

If $f: S \to T$ is a mapping, and x is an element of S, then we denote by $f(x)$ the element of T associated to x by f. We call $f(x)$ the **value** of f at x, or also the **image** of x under f. The set of all elements $f(x)$, for all $x \in S$, is called the image of f. If S' is a subset of S, then the set of elements $f(x)$ for all $x \in S'$, is called the image of S' and is denoted by $f(S')$.

If f is as above, we often write $x \mapsto f(x)$ to denote the association of $f(x)$ to x. We thus distinguish two types of arrows, namely $\to$ and $\mapsto$.