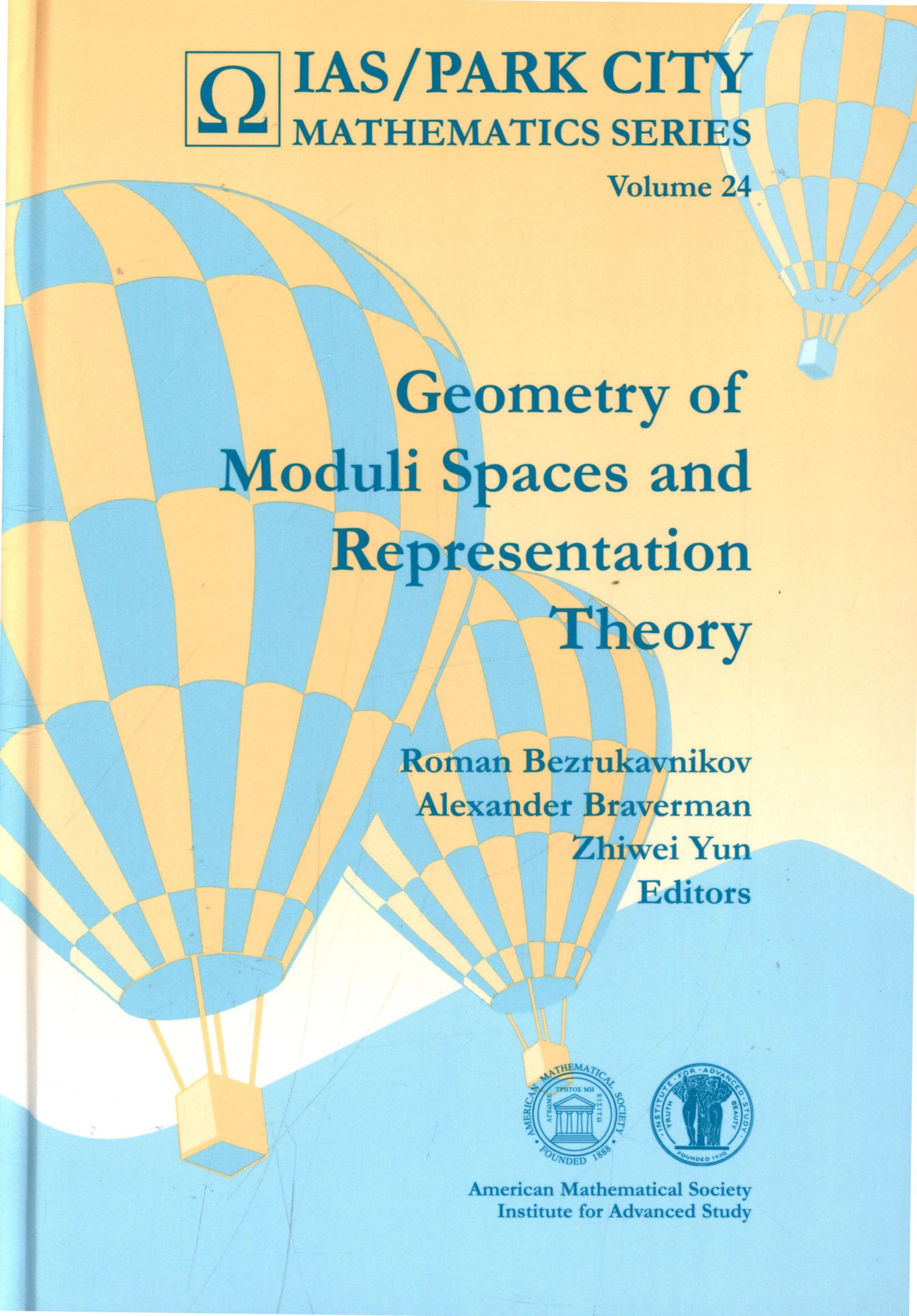

IAS/PARK CITY
MATHEMATICS SERIES

Volume 24

Geometry of Moduli Spaces and Representation Theory

Roman Bezrukavnikov
Alexander Braverman
Zhiwei Yun
Editors

American Mathematical Society
Institute for Advanced Study

This book is based on lectures given at the Graduate Summer School of the 2015 Park City Mathematics Institute program "Geometry of moduli spaces and representation theory", and is devoted to several interrelated topics in algebraic geometry, topology of algebraic varieties, and representation theory.

Geometric representation theory is a young but fast developing research area at the intersection of these subjects. An early profound achievement was the famous conjecture by Kazhdan–Lusztig about characters of highest weight modules over a complex semi-simple Lie algebra, and its subsequent proof by Beilinson-Bernstein and Brylinski-Kashiwara. Two remarkable features of this proof have inspired much of subsequent development: intricate algebraic data turned out to be encoded in topological invariants of singular geometric spaces, while proving this fact required deep general theorems from algebraic geometry.

Another focus of the program was enumerative algebraic geometry. Recent progress showed the role of Lie theoretic structures in problems such as calculation of quantum cohomology, K-theory, etc. Although the motivation and technical background of these constructions is quite different from that of geometric Langlands duality, both theories deal with topological invariants of moduli spaces of maps from a target of complex dimension one. Thus they are at least heuristically related, while several recent works indicate possible strong technical connections.

The main goal of this collection of notes is to provide young researchers and experts alike with an introduction to these areas of active research and promote interaction between the two related directions.

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Roman Bezrukavnikov, Alexander Braverman, and Zhiwei Yun, Volume Editors.

The IAS/Park City Mathematics Institute runs mathematics education programs that bring together high school mathematics teachers, researchers in mathematics and mathematics education, undergraduate mathematics faculty, graduate students, and undergraduates to participate in distinct but overlapping programs of research and education. This volume contains the lecture notes from the Graduate Summer School program.

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Geometry of Moduli Spaces and Representation Theory

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Preface

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the Regional Geometry Institute initiative of the National Science Foundation. In mid-1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school students, undergraduate faculty, K-12 teachers, and international teachers and education researchers. The Teacher Leadership Program also includes weekend workshops and other activities during the academic year.

One of PCMI's main goals is to make all of the participants aware of the full range of activities that occur in research, mathematics training and mathematics education: the intention is to involve professional mathematicians in education and to bring current concepts in mathematics to the attention of educators. To that end, late afternoons during the summer institute are devoted to seminars and discussions of common interest to all participants, meant to encourage interaction among the various groups. Many deal with current issues in education: others treat mathematical topics at a level which encourages broad participation.

Each year the Research Program and Graduate Summer School focuses on a different mathematical area, chosen to represent some major thread of current mathematical interest. Activities in the Undergraduate Summer School and Undergraduate Faculty Program are also linked to this topic, the better to encourage interaction between participants at all levels. Lecture notes from the Graduate Summer School are published each year in this series. The prior volumes are:

- Volume 1: *Geometry and Quantum Field Theory* (1991)
- Volume 2: *Nonlinear Partial Differential Equations in Differential Geometry* (1992)
- Volume 3: *Complex Algebraic Geometry* (1993)
- Volume 4: *Gauge Theory and the Topology of Four-Manifolds* (1994)
- Volume 5: *Hyperbolic Equations and Frequency Interactions* (1995)
- Volume 6: *Probability Theory and Applications* (1996)
- Volume 7: *Symplectic Geometry and Topology* (1997)
- Volume 8: *Representation Theory of Lie Groups* (1998)
- Volume 9: *Arithmetic Algebraic Geometry* (1999)

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- Volume 10: *Computational Complexity Theory* (2000)
- Volume 11: *Quantum Field Theory, Supersymmetry, and Enumerative Geometry* (2001)
- Volume 12: *Automorphic Forms and their Applications* (2002)
- Volume 13: *Geometric Combinatorics* (2004)
- Volume 14: *Mathematical Biology* (2005)
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- Volume 18: *Arithmetic of L-functions* (2009)
- Volume 19: *Mathematics in Image Processing* (2010)
- Volume 20: *Moduli Spaces of Riemann Surfaces* (2011)
- Volume 21: *Geometric Group Theory* (2012)
- Volume 22: *Geometric Analysis* (2013)
- Volume 23: *Mathematics and Materials* (2014)

The American Mathematical Society publishes material from the Undergraduate Summer School in their Student Mathematical Library and from the Teacher Leadership Program in the series IAS/PCMI—The Teacher Program.

After more than 25 years, PCMI retains its intellectual vitality and continues to draw a remarkable group of participants each year from across the entire spectrum of mathematics, from Fields Medalists to elementary school teachers.

Rafe Mazzeo
PCMI Director
March 2017

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Introduction

Roman Bezrukavnikov, Alexander Braverman, and Zhiwei Yun

The 2015 Park City Mathematics Institute program on "Geometry of moduli spaces and representation theory" was devoted to a combination of interrelated topics in algebraic geometry, topology of algebraic varieties and representation theory.

Geometric representation theory is a young but fast developing research area at the intersection of the those subjects. An early profound achievement was the formulation, in the late 70's, of Kazhdan and Lusztig's famous conjecture on characters of highest weight modules over a complex semi-simple Lie algebra, and its subsequent proof by Beilinson–Bernstein and Brylinski–Kashiwara. Two remarkable features of this proof have inspired much of subsequent development: intricate algebraic data turned out to be encoded in topological invariants of singular geometric spaces, while proving this fact required deep general theorems from algebraic geometry. The topological invariants in question have to do with *intersection cohomology* of Schubert varieties, while the key algebro-geometric result used in the proof is a generalization of Weil's conjecture by Beilinson, Bernstein and Deligne involving perverse sheaves.

The geometric spaces appearing in the Kazhdan–Lusztig conjectures are closed subvarieties in the flag variety, a homogeneous space which is a basic ingredient in the theory of algebraic groups. A later major direction in geometric representation theory, shaped by contributions of Lusztig, Nakajima and others, develops a similar relation between representation theory and moduli spaces of linear algebra data (quiver varieties).

More intricate geometric objects have entered the subject with the emergence of the geometric Langlands program. This direction, pioneered by Beilinson and Drinfeld in the 90's, is partly inspired by Langlands' conjectural nonabelian reciprocity laws from number theory. In the last decade, Kapustin and Witten have discovered its close connection to S-duality in quantum field theory. While employing some of the techniques of Kazhdan-Lusztig theory, geometric Langlands duality deals with more sophisticated geometric spaces, such as the moduli space (or stack) of principal bundles on a complete algebraic curve and its local counterpart, the affine Grassmannian, also known as the loop Grassmannian. A large part of the PCMI program was devoted to introducing this circle of ideas.

Another focus of the program was on some aspects of enumerative algebraic geometry. Recent progress in that area has been increasingly bringing to light

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the role of Lie theoretic structures in problems such as calculation of (equivariant) quantum cohomology, *K*-theory etc. Although the motivation and technical background of these constructions is quite different from that of geometric Langlands duality, both theories deal with topological invariants of moduli spaces of maps from a target of complex dimension one. Thus they are at least heuristically related, while several recent works indicate possible strong technical connections.

The goal of the program was to provide an introduction to these areas of active research and promote interaction between the two related directions. Our hope is that this will help to write a new chapter in the glorious history of the interaction between representation theory and algebraic geometry. Just as *D*-modules, perverse sheaves and the generalizations of Weil's conjecture have become standard tools in studying many algebraic questions in representation theory, we hope that keys to resolving other outstanding questions may lie in the recent techniques of enumerative algebraic geometry

The program included minicourses by Alexander Braverman, Mark de Cataldo, Victor Ginzburg, Davesh Maulik, Hiraku Nakajima, Xinwen Zhu, Zhiwei Yun, and Clay Scholars Ngô Bảo Châu and Andrei Okounkov. This volume contains contributions by Mark de Cataldo, Hiraku Nakajima, Ngô Bảo Châu, Andrei Okounkov, Xinwen Zhu and Zhiwei Yun.

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Perverse sheaves and the topology of algebraic varieties

Mark Andrea de Cataldo

Dedicato a Mikki

Contents

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Introduction

Goal of the lectures. The goal of these lectures is to introduce the novice to the use of perverse sheaves in complex algebraic geometry and to what is perhaps the deepest known fact relating the homological invariants of the source and target of a proper map of complex algebraic varieties, namely the decomposition theorem.

Notation. A variety is a complex algebraic variety, which we do not assume to be irreducible, nor reduced. We work with cohomology with $\mathbb{Q}$-coefficients as $\mathbb{Z}$-coefficients do not fit well in our story. As we rarely focus on a single cohomological degree, for the most part we consider the total, graded cohomology groups, which we denote by $H^*(X, \mathbb{Q})$.

Bibliographical references. The main reference is the survey [19] and the extensive bibliography contained in it, most of which is not reproduced here. This allowed me to try to minimize the continuous distractions related to the peeling apart of the various versions of the results and of the attributions. The reader may also consult the discussions in [18] that did not make it into the very different final version [19].

Style of the lectures and of the lecture notes. I hope to deliver my lectures in a rather informal style. I plan to introduce some main ideas, followed by what I believe to be a striking application, often with an idea of proof. The lecture notes are not intended to replace in any way the existing literature on the subject, they are a mere amplification of what I can possibly touch upon during the five one-hour lectures. As it is usual when meeting a new concept, the theorems and the applications are very important, but I also believe that working with examples, no matter how lowly they may seem, can be truly illuminating and useful in building one's own local and global picture. Because of the time factor, I cannot possibly fit many of these examples in the flow of the lectures. This is why there are plenty of exercises, which are not just about examples, but at times deal head-on with actual important theorems. I could have laid-out several more exercises (you can look at my lecture notes [22], or at my little book [9] for more exercises), but I tried to choose ones that would complement well the lectures; too much of anything is not a good thing anyway.

What is missing from these lectures? A lot! Two related topics come to mind: vanishing/nearby cycles and constructions of perverse sheaves; see the survey [19] for a quick introduction to both. To compound this infamy, there is no discussion of the equivariant picture [3].

An afterthought. The 2015 PCMI is now over. Even though I have been away from Mikki, Caterina, Amelie (Amie!) and Dylan for three weeks, my PCMI experience has been wonderful. If you love math, then you should consider participating in future PCMIs. Now, let us get to Lecture 1.

Lecture 1: The decomposition theorem

Summary of Lecture 1. *Deligne's theorem on the degeneration of the Leray spectral sequence for smooth projective maps; this is the 1968 prototype of the 1982 decomposition theorem. Application, via the use of the theory of mixed Hodge structures, to the global invariant cycle theorem, a remarkable topological property enjoyed by families of projective manifolds and compactifications of their total spaces. The main theorem of these lectures, the decomposition theorem, stated in cohomology. Application to a proof of the local invariant cycle theorem, another remarkable topological property concerning degenerations of families of projective manifolds. Deligne's theorem, including semi-simplicity of the direct image sheaves, in the derived category. The decomposition theorem: the direct image complex splits in the derived category into a direct sum of shifted and twisted intersection complexes supported on the target of a proper map.*

1.1. Deligne's theorem in cohomology

Let us start with a warm-up: the Künneth formula and a question.

Let Y, F be varieties. Then

$$H^*(Y \times F, \mathbb{Q}) = \bigoplus_{q \geqslant 0} H^{*-q}(Y, \mathbb{Q}) \otimes H^q(F, \mathbb{Q}). \tag{1.1.1}$$

Note that the restriction map $H^*(Y \times F, \mathbb{Q}) \to H^*(F, \mathbb{Q})$ is surjective. This surjectivity fails in the context of (differentiable) fiber bundles: take the Hopf fibration $h : S^3 \to S^2$ (cf. Exercise 1.7.2), for example. It is a remarkable fact that, in the context of algebraic geometry, one has indeed this surjectivity property, and more. Let us start discussing this phenomenon by asking the following

Question 1.1.2. *Let* $f : X \to Y$ *be a family of projective manifolds. What can we say about the restriction maps* $H^*(X, \mathbb{Q}) \to H^*(f^{-1}(y), \mathbb{Q})$*? Let* $\overline{X}$ *be a projective manifold completing* X *(i.e.* X *is open and Zariski-dense in* $\overline{X}$*). What can we say about the restriction maps* $H^*(\overline{X}, \mathbb{Q}) \to H^*(f^{-1}(y), \mathbb{Q})$*?*

Answer: The answers are given, respectively, by (1.2.1) and by the global invariant cycle Theorem 1.2.2. Both rely on Deligne's Theorem, which we review next.

The decomposition theorem has an important precursor in Deligne's theorem, which can be viewed as the decomposition theorem in the absence of singularities of the domain, of the target *and* of the map. We start by stating the cohomological version of his theorem.

Theorem 1.1.3. (**Blanchard-Deligne 1968 theorem in cohomology** [24]) *For any smooth projective map*[1] $f : X \to Y$ *of algebraic manifolds, there is an isomorphism*

$$H^*(X,\mathbb{Q}) \cong \bigoplus_{q\geqslant 0} H^{*-q}(Y, R^q f_* \mathbb{Q}_X), \tag{1.1.4}$$

where $R^q f_* \mathbb{Q}_X$ *denotes the* q*-th direct image sheaf of the sheaf* $\mathbb{Q}_X$ *via the morphism* f*; see §1.2. More precisely, the Leray spectral sequence (see §1.7) of the map* f *is* E_2*-degenerate.*

Proof. Exercise 1.7.3 guides you through Deligne's classical trick (the Deligne-Lefschetz criterion) of using the hard Lefschetz theorem on the fibers to force the triviality of the differentials of the Leray spectral sequence. □

Compare (1.1.1) and (1.1.4): both present cohomological shifts; both express the cohomology of the l.h.s. via cohomology groups on Y; in the former case, we have cohomology with constant coefficients; in the latter, and this is an important distinction, we have cohomology with locally constant coefficients.

Deligne's theorem is central in the study of the topology of algebraic varieties. Let us discuss one striking application of this result: the global invariant cycle theorem.

1.2. The global invariant cycle theorem

Let $f : X \to Y$ be a smooth and projective map of algebraic manifolds, let $j : X \to \overline{X}$ be an open immersion into a projective manifold and let $y \in Y$. What are the images of $H^*(X,\mathbb{Q})$ and $H^*(\overline{X},\mathbb{Q})$ via the restriction maps into $H^*(f^{-1}(y),\mathbb{Q})$? The answer is the global invariant cycle Theorem 1.2.2 below.

The direct image sheaf $\mathcal{R}^q := R^q f_* \mathbb{Q}_X$ on Y is the sheaf associated with the pre-sheaf

$$U \mapsto H^q(f^{-1}(U),\mathbb{Q}).$$

In view of Ehresmann's lemma, the proper[2] submersion f is a C^∞ fiber bundle. The sheaf $\mathcal{R}^q$ is then locally constant with stalk

$$\mathcal{R}^q_y = H^q(f^{-1}(y),\mathbb{Q}).$$

The fundamental group $\pi_1(Y,y)$ acts via linear transformations on $\mathcal{R}^q_y$: pick a loop $\gamma(t)$ at y and use a trivialization of the bundle along the loop to move vectors in $\mathcal{R}^q_y$ along $\mathcal{R}^q_{\gamma(t)}$, back to $\mathcal{R}^q_y$ (monodromy action for the locally constant sheaf $\mathcal{R}^q$).

---

[1]Smooth: submersion; projective: factors as $X \to Y \times \mathbb{P} \to Y$ (closed embedding, projection).
[2]Proper := the pre-image of compact is compact; it is the "relative" version of compactness.

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