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MATRIX-TENSOR METHODS IN CONTINUUM MECHANICS

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PREFACE

This book is based on a preprint edition, *An Introduction to Matrix-Tensor Methods in Theoretical and Applied Mechanics,* which was issued on a more or less interim basis and has been used by the author in at least six of his graduate courses. The present book represents a complete rewriting and bringing up to date of the earlier work, in the light of classroom experience. The purposes of the text are:

1. To introduce the engineer to the very important (and increasingly important) discipline in applied mathematics—tensor methods. Because the author's classroom experience has convinced him that the engineer can follow tensor theory most easily when it is presented in matrix form, this has been the method used in the text.
2. To show the fundamental unity of the different fields in continuum mechanics—with the unifying material formed by the matrix-tensor theory. Too often the student loses sight of the real connections between fields that we have artificially decompartmentalized. A truer understanding of the important and basic segment of engineering—mechanics of continua—can be obtained, the author feels, when the various portions of this field are presented as part of the complete fabric.
3. To present to the engineer modern engineering problems. For this reason, mathematical arguments have been kept to a minimum or avoided entirely where they would tend to add little to the physical understanding of the phenomenon being discussed. However, it should also be emphasized that the book is not to be thought of as nonmathematical. It requires of the student an understanding of differential and integral calculus, vector analysis, complex variable theory, mathematical analysis, and related topics usually considered to be the equipment of the graduate engineer or scientist. The fundamentals of matrix and tensor theory are covered in a form sufficient for the purposes of the text—and for additional advanced use as well.

In the first chapter the fundamentals of matrix algebra and calculus are presented, as well as a brief review of vector analysis and the introductory complex variable theory. This coverage, together with the current undergraduate mathematical training of engineers, should be sufficient preparation.

Chapter 2 presents the elements of tensor theory. The fundamental nature of the tensor is emphasized: the requirement that it behave in a certain manner under a transformation (rotation) of axes about the origin. The connection between the tensor and the matrix is brought out, and the groundwork in matrix-tensor analysis is laid.

Curvilinear coordinates, one of the most useful and important topics in applied mathematics to the engineer, is discussed in Chapter 3. The entire development, presented in matrix-tensor form, leads to expressions which permit one to put all of the equations of mathematical physics in any orthogonal curvilinear form whatever.

The remaining chapters indicate the applications of the theory to continuum mechanics—to fluids and to solids. Chapters 4 and 5 give the theory and some applications in the mathematical theory of elasticity. The essential tensors are derived, and their position in the theory is described in detail. Chapter 6 presents a discussion of matrix-tensor methods as they occur in structural engineering. Chapter 7 presents the application of matrix-tensor methods to plate and shell theory; Chapter 8 considers viscous flow phenomena, Chapter 9, plasticity. In all cases, the arguments and theory are presented from the matrix-tensor point of view, and the similarities (as well as essential differences) between the various fields are constantly brought out.

Chapter 10 presents a subject that is based squarely upon the matrix-tensor theory and that crosses all the fields considered in the text (and others). A form of dimensional analysis is described that is based upon tensoral invariance arguments, enabling one to give, without derivation, the qualitative form of many of the equations of mathematical physics and hence engineering.

A list of references to the standard works in the various fields considered, and to other special reports and books mentioned, is supplied. At the end of each chapter is a problems section.

In the author's graduate courses, he found it possible to complete essentially the entire book in a single three-hour-a-week semester. A graduate course in engineering mathematics was a prerequisite for this course. In his senior elective course the author was able to complete Chapters 1 through 4 in a three-hour-a-week semester. As a senior course, it should be possible to present the entire text in two semesters.

The author is indebted to Professor Francis Murnaghan whose inspiring lectures at Johns Hopkins University first introduced him to applied matrix-tensor methods. Professor Murnaghan's textbooks have been referred to liberally for basic source material. Several of the treatments

presented are those given by Dr. Murnaghan in his lectures. However, in the interests of engineering simplification, the author has taken some liberties in the form of presentation. If there are errors in this material as given here, the fault lies with the author.

S. F. Borg

CONTENTS

Chapter 1

MATHEMATICAL PRELIMINARIES

1-1 Introduction. In this chapter a brief treatment of matrix algebra is presented. In addition a discussion is given, in abbreviated form, of vector analysis and complex variable theory. The presentation of the topics in this chapter is utilitarian in form and, insofar as the vector analysis and complex variable portions are concerned, it is more in the nature of a review and refresher of the introductory phases of these subjects. A knowledge of the material presented in this and the next chapter will give an adequate mathematical background for the later portions of the text.

1-2 Definition of a Matrix. A rectangular array of m rows and n columns of numbers or other quantities is called a *matrix*. We designate this matrix with a capital letter, as A, and show it in its expanded form as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \tag{1-1}$$

In the above expression, a_{ij} represents an *element* of the matrix. Note particularly that the subscripts of the elements carry a position significance. That is, the first subscript represents the row position of the element and the second subscript represents the column position.

A matrix is *not* a determinant.[1] As a reminder of this, the enclosing

[1] More precisely:

1. A determinant is a *quantity* associated with a *square* array of n^2 elements. Thus,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is also given by the quantity $a_{11}a_{22} - a_{12}a_{21}$.

2. A matrix need not be square. It is simply a set of $m \times n$ elements in an ordered array. Thus

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

does not imply any particular operation need be performed on the elements a_{ij}.

Footnote continued on page 2.

bars are shown curved as against the ordinary usage of straight bars for the determinant.

The number of rows in a matrix need not necessarily be the same as the number of columns. If the number of rows does equal the number of columns then the matrix is a *square matrix*.

A matrix which consists of elements in a single row is sometimes called a *row matrix*. If the elements are in a single column it is sometimes called a *column matrix*. No particular distinction, in general, need be made between the one and the other.

The elements of a matrix may or may not have any physical significance. For example, the elements may be pure numbers, as

$$\begin{pmatrix} 6 & -3.2 & 7\tfrac{1}{2} \end{pmatrix} \tag{1-2}$$

or the elements may be components of a velocity vector, as

$$V = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{1-3}$$

Indeed, they could even be colors, as

(red blue green)

or animals, as

$$\begin{pmatrix} \text{cat} \\ \text{dog} \\ \text{hare} \end{pmatrix} \tag{1-4}$$

or they could be mixtures of any or all of the above.

No significance must be attached to the use of a row for the numbers and colors and a column form for the velocity and animals in the above matrices.

The elements of a matrix may also be complex quantities, chemical symbols, equations, or, in fact, any quantity whatever.

The *zero, or null, matrix*, 0, has all elements equal to zero. Thus, we have

$$0 = (0 \ 0 \ 0) \tag{1-5}$$

or

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1-6}$$

Footnote continued from page 1.

3. We may, however, *define* the determinant of a square matrix. This is, by definition, the quantity obtained by treating the elements of the matrix as elements of a determinant—the position of the elements being the same in both the matrix and the determinant.

We shall also in the following sections indicate various operations (arithmetic, algebraic, and other) that may, by definition, be performed on and by matrices.

or any similar arrangement. Note: the zero matrix may be either a row or column or square matrix, or a general matrix of rectangular form.

The *unit matrix* E_n is an n-by-n-square matrix whose diagonal elements (top left to bottom right) equal unity and whose off-diagonal elements equal zero. That is,

$$\text{in} \quad E_n, \quad \begin{cases} a_{ij} = 1 & \text{if} \quad i = j \\ a_{ij} = 0 & \text{if} \quad i \neq j \end{cases} \tag{1-7}$$

and, as an example,

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1-8}$$

A square matrix is *symmetrical* if

$$a_{ij} = a_{ji} \tag{1-9}$$

An example of a symmetrical matrix is the following:

$$\begin{pmatrix} x & e^t & z^2 \\ e^t & yz & 3t \\ z^2 & 3t & 1 \end{pmatrix} \tag{1-10}$$

A square matrix is *antisymmetric* or *skew-symmetric* if

$$a_{ij} = -a_{ji} \tag{1-11}$$

An example of a skew-symmetric matrix is

$$\begin{pmatrix} 0 & -3t \\ 3t & 0 \end{pmatrix} \tag{1-12}$$

Note that in a skew-symmetric matrix the main diagonal (upper left to lower right) elements must be zero, for only then will $a_{ij} = -a_{ji}$ be true for these elements.

The *transpose* of a matrix A is shown as $A^\star$ and is obtained by interchanging the rows and columns of A. Thus, if

$$A = \begin{pmatrix} z & xe^t & 2-y \\ 4 & 3xy & 0 \end{pmatrix} \tag{1-13}$$

then

$$A^\star = \begin{pmatrix} z & 4 \\ xe^t & 3xy \\ 2-y & 0 \end{pmatrix} \tag{1-14}$$

The foregoing represents the basic definitions or nomenclature in matrix theory.

1-3 Matrix Arithmetic, Algebra, and Calculus. Up to this point we have defined, in some detail, exactly what a matrix is and we have discussed some special matrices. If matrices are to be useful in engineering or physical applications, then they must behave in certain set ways when subjected to particular conditions. In our work in engineering and science we are primarily concerned with *quantitative* relations, and therefore we shall be most interested in the behavior of matrices in arithmetical and related mathematical operations. Matrices will be of use to us if, and only if, the theory of matrices can be developed along logical mathematical lines.

The simplest mathematical operations are those of arithmetic—equality, addition, subtraction, multiplication, and division. We discuss these first.

Two matrices A and B are *equal* only if each has the same number of rows and the same number of columns and if corresponding elements are equal. Thus, given

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \tag{1-15}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \tag{1-16}$$

then if

$$a_{ij} = b_{ij} \tag{1-17}$$

it follows

$$A = B \tag{1-18}$$

Thus, the simple algebraic equations

$$\left.\begin{aligned} a &= p + 2u \\ b &= q + 7v \\ c &= r + 1.6w \\ d &= s + 17x \end{aligned}\right\} \tag{1-19}$$

may be given in matrix form as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p + 2u & q + 7v \\ r + 1.6w & s + 17x \end{pmatrix} \tag{1-20}$$

We define the *sum* of two matrices A and B only if A and B have the same number of rows and of columns. The sum $A + B$ is then a matrix C of the same number of rows as A (and B) and the same number of columns as A (and B) and with

$$c_{ij} = a_{ij} + b_{ij} \tag{1-21}$$

For example, the algebraic equations

$$\left.\begin{aligned} c_{11} &= a_{11} + b_{11} \\ c_{12} &= a_{12} + b_{12} \\ c_{21} &= a_{21} + b_{21} \\ c_{22} &= a_{22} + b_{22} \end{aligned}\right\} \tag{1-22}$$

are equivalent[2] to

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{1-23}$$

It may be shown (the student should verify this and the following statement for typical matrices) that the sum of two matrices is *commutative*. That is,

$$A + B = B + A \tag{1-24}$$

Also, it may be shown that the addition of matrices is *associative*. That is,

$$(A + B) + C = A + (B + C) \tag{1-25}$$

The *difference* of two matrices A and B is defined similarly. Thus,

$$C = A - B \tag{1-26}$$

with

$$c_{ij} = a_{ij} - b_{ij} \tag{1-27}$$

We may define multiplication of a matrix A by a scalar k as follows: the elements of kA are given by ka_{ij}, so that, for example, if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1-28}$$

then

$$kA = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix} \tag{1-29}$$

[2] Alternatively, this may be expressed in the following essentially equivalent form:

$$\begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix}$$

in which the positional significance of the subscripts has been waived.

Note that this is consistent with the usual notation

$$kA = \underbrace{A+A+\cdots+A}_{k\text{ times}} \tag{1-30}$$

An important property of square matrices which follows directly from the law of addition and subtraction is the following:

Any square matrix may be given as the sum of a symmetrical and antisymmetrical matrix. For, if A is a square matrix, then obviously

$$A = \frac{A+A^\star}{2}+\frac{A-A^\star}{2} \tag{1-31}$$

The first term on the right is a symmetrical matrix and the second term is an antisymmetrical matrix. This may be verified for a 2×2 matrix as follows:

If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1-32}$$

then

$$A^\star = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \tag{1-33}$$

so that

$$\frac{A+A^\star}{2} = \begin{pmatrix} a_{11} & \dfrac{a_{12}+a_{21}}{2} \\ \dfrac{a_{21}+a_{12}}{2} & a_{22} \end{pmatrix} \tag{1-34}$$

and

$$\frac{A-A^\star}{2} = \begin{pmatrix} 0 & \dfrac{a_{12}-a_{21}}{2} \\ \dfrac{a_{21}-a_{12}}{2} & 0 \end{pmatrix} \tag{1-35}$$

A very important operation in matrix arithmetic is the *product* of two matrices. The previous operations are not too different from the more familiar ones of elementary arithmetic. The product operation, however, is quite different.

The *product* of two matrices is obtained as follows: given two matrices A and B such that the number of rows in B equals the number of columns in A, then the product AB is given by C, in which the element c_{ij} is obtained by multiplying each element of the i^{th} row of A by the corresponding element of the j^{th} column of B and adding. For example,

$$\left.\begin{aligned} C &= AB \\ \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} \\ \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{pmatrix} \\ \text{or}^3 \quad c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{aligned}\right\} \quad (1\text{-}36)$$

Note, in the above product, C is a 2×1 matrix. In general, if

$$C = AB \quad (1\text{-}37)$$

and if k is the number of rows in A and l is the number of columns in B, then the matrix C will have k rows and l columns.

It will be obvious from the above example that, in general,

$$AB \neq BA \quad (1\text{-}38)$$

that is, the position of a matrix in a matrix multiplication is *not* immaterial.

The student should also note that

$$CE = C \quad (1\text{-}39)$$

where C is any matrix, E is a unit matrix of same number of rows as C has columns. For example,

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix} \quad (1\text{-}40)$$

Thus, E plays the same role in matrix multiplication that unity plays in algebraic multiplication.

[3] Equation 1-36 indicates the usefulness of the given definition of matrix product. The first line of Equation 1-36 is a compact expression for the two linear equations shown in the last two lines of Equation 1-36. In general, systems of linear equations can be shown very compactly by utilizing the definition of matrix product.

As another example of matrix multiplication, the student should satisfy himself that the set of algebraic equations

$$\left.\begin{aligned} a &= 2ex + 3gy \\ b &= 2ev + 3gw \\ c &= -tx + s^2y \\ d &= -tv + s^2w \end{aligned}\right\} \tag{1-41}$$

is equivalent to the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2e & 3g \\ -t & s^2 \end{pmatrix}\begin{pmatrix} x & v \\ y & w \end{pmatrix} \tag{1-42}$$

It may be shown (the student should verify this and the following statement using simple matrices) that the product of matrices is *associative*. That is,

$$(AB)C = A(BC) \tag{1-43}$$

Also, the product of matrices is *distributive*. That is,

$$A(B+C) = AB+AC \tag{1-44}$$

The following expression is the statement of the very important *transpose product rule*:

$$(AC)^\star = C^\star A^\star \tag{1-45}$$

The student should verify this for a simple case.

Division of matrices is a non-unique process and therefore must remain undefined and not part of the algebra of matrices.[4] To illustrate what is meant by this, consider the product

$$AB = C \tag{1-46}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1-47}$$

$$B = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \tag{1-48}$$

[4] Although division, as noted herein, is an undefined operation, division corresponding to the relation $AA^{-1} = E_n$ or $A^{-1} = E_n/A$ in which A^{-1} is the "inverse of A", is defined as shown on p. 10. When it exists, A^{-1} is analogous to the reciprocal of a number A in arithmetic.