A Practical Course in Differential Equations and Mathematical Modelling

Classical and New Methods
Nonlinear Mathematical Models
Symmetry and Invariance Principles

微分方程与数学物理问题

经典方法和现代新方法
非线性数学物理问题
对称性和不变性理论

Nail H. Ibragimov

Higher Education Press

A Practical Course in Differential Equations and Mathematical Modelling

Classical and New Methods
Nonlinear Mathematical Models
Symmetry and Invariance Principles

微分方程与数学物理问题

Higher Education Press

World Scientific

NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI · CHENNAI

Nail H. Ibragimov
Department of Mathematics and Science
Blekinge Institute of Technology
S-371 79 Karlskrona, Sweden

图书在版编目（CIP）数据

微分方程和数学物理问题 = A Practical Course in Differential Equations and Mathematical Modelling: 英文/(瑞典)伊布拉基莫夫(Ibragimov,N.H.)著. 北京：高等教育出版社，2009.8
ISBN 978-7-04-027603-9

Ⅰ.微… Ⅱ.伊… Ⅲ.①微分方程-英文②数学物理方法-英文 Ⅳ.O175 O411.1

中国版本图书馆CIP数据核字(2009)第132064号

策划编辑 王丽萍 **责任编辑** 王丽萍 **封面设计** 于　涛
策划校对 王　莉 **责任印制** 陈伟光

出版发行	高等教育出版社	购书热线	010-58581118
社　　址	北京市西城区德外大街4号	免费咨询	400-810-0598
邮政编码	100120	网　　址	http://www.hep.edu.cn
总　　机	010-58581000		http://www.hep.com.cn
		网上订购	http://www.landraco.com
经　　销	蓝色畅想图书发行有限公司		http://www.landraco.com.cn
印　　刷	涿州市星河印刷有限公司	畅想教育	http://www.widedu.com
开　　本	787×1092　1/16	版　　次	2009年8月第1版
印　　张	22.75	印　　次	2009年8月第1次印刷
字　　数	255 000	定　　价	68.00元

物料号　27603-00

Preface

Modern mathematics has over 300 years of history. From the very beginning, it was focused on differential equations as a major tool for mathematical modelling. Most of mathematical models in physics, engineering sciences, biomathematics, etc. lead to nonlinear differential equations.

Today's engineering and science students and researchers routinely confront problems in mathematical modelling involving solution techniques for differential equations. Sometimes these solutions can be obtained analytically by numerous traditional ad hoc methods appropriate for integrating particular types of equations. More often, however, the solutions cannot be obtained by these methods, in spite of the fact that, e.g. over 400 types of integrable second-order ordinary differential equations were accumulated due to ad hoc approaches and summarized in voluminous catalogues.

On the other hand, the fundamental natural laws and technological problems formulated in terms of differential equations can be successfully treated and solved by Lie group methods. For example, Lie group analysis reduces the classical 400 types of equations to 4 types only! Development of group analysis furnished ample evidence that the theory provides a universal tool for tackling considerable numbers of differential equations even when other means of integration fail. In fact, group analysis is the only universal and effective method for solving nonlinear differential equations analytically. The old integration methods rely essentially on linearity as well as on constant coefficients. Group analysis deals equally easily with *linear and nonlinear* equations, as well as with constant and variable coefficients. For example, from the traditional point of view, the linear equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_{n-1} \frac{\mathrm{d}y}{\mathrm{d}x} + a_n\, y = 0$$

with constant coefficients $a_1, \ldots, a_n$ is different from the equation

$$\bar{x}^n \frac{\mathrm{d}^n \bar{y}}{\mathrm{d}\bar{x}^n} + a_1\, \bar{x}^{n-1} \frac{\mathrm{d}^{n-1} \bar{y}}{\mathrm{d}\bar{x}^{n-1}} + \cdots + a_{n-1}\, \bar{x} \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} + a_n\, \bar{y} = 0$$

known as *Euler's equation.* From the group standpoint, however, these equations are merely two different representations of one and the same equation with two known commuting symmetries, namely,

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = y\frac{\partial}{\partial y} \quad \text{and} \quad \overline{X}_1 = \bar{x}\frac{\partial}{\partial \bar{x}}, \quad \overline{X}_2 = \bar{y}\frac{\partial}{\partial \bar{y}}$$

for the first and second equation, respectively. These symmetries span two similar Lie algebras and readily lead to the transformation $x = \ln |\bar{x}|$ converting Euler's equation to the equation with constant coefficients.

I believe that Lie groups are interesting first of all due to their utilization for solving differential equations. It was a mistake to isolate them from this natural application and treat as a branch of abstract mathematics. *"To isolate mathematics from the practical demands of the sciences is to invite the sterility of a cow shut away from the bulls"* (P.L. Chebyshev, 1821~1894).

Today group analysis is becoming part of curricula in differential equations and nonlinear mathematical modelling and attracts more and more students. For example, the course in *Partial Differential Equations* at Moscow Institute of Physics and Technology attracted more than 100 students when I used Lie group methods, instead of 10 students that we had in the traditional course. The same happened when I delivered similar lectures for science students in South Africa and Sweden.

The present text is based on these lectures and reflects, to a certain extent, my own taste and experience. Primarily, it has been designed for the course in differential equations delivered at the Blekinge Institute of Technology for engineering, mathematics and science students. Then the text has been revised, enlarged and is used now in the following courses:

Differential equations: The course covers both ordinary and partial differential equations; it combines basic classical methods, mainly for linear equations, with new methods for solving nonlinear equations analytically; designed for beginners; students learn how to find symmetries of differential equations by solving determining equations.

Analytical methods in mathematical modelling: The emphasis in this course is on nonlinear mathematical models in physics, biology and engineering sciences; the course covers such topics as nonlinear superposition, symmetry and conservation laws, group invariant solutions.

Group analysis of differential equations: The course introduces students of mathematics and engineering to those areas of the theory of transformations groups and Lie algebras which are most important in practical applications; during the course, students develop analytic skills in modern methods for solving nonlinear ordinary and partial differential equations.

Distributions and invariance principle in initial value problems: An easy to follow introduction to basic concepts of the distribution theory with emphasis on useful tools; Lie's infinitesimal technique is extended to the space of distributions and used, together with an invariance principle, for calculating fundamental solutions and solving initial value problems for equations with constant and variable coefficients.

In my presentation, I have striven to make the group analysis of differential equations more accessible for engineering and science students. Therefore, the emphasis in this book is on applications of known symmetries rather than on their computation. In order to formulate the essence of my experience in solving various types of differential equations, I rephrase the famous French aphorism *cherchez la femme* as follows:

If you cannot solve a nonlinear differential equation, cherchez le groupe.

My sincere thanks are due to my colleague Claes Jogréus for his lasting help. My wife Raisa read the manuscript at various stages of completion of the second edition, corrected misprints and contributed numerous valuable criticisms, for which I make grateful acknowledgement. It is also a pleasure to thank my daughters Sania and Alia for several helpful comments.

Karlskrona, 3 March 2009 Nail H. Ibragimov

Contents

Chapter 1

Selected topics from analysis

This preparatory chapter is designed to meet the needs of beginners and provides a background in elementary mathematics and mathematical analysis which is necessary in the succeeding parts of the book.

Additional reading: E. Goursat [10].

1.1 Elementary mathematics

1.1.1 Numbers, variables and elementary functions

Real numbers appear in our practical activities (e.g., while measuring distances, weights, etc.) as approximate decimal numbers. For example, the distance to the moon at perigee is S km, where the number S is approximately equal to 356630. A more accurate estimation of the distance is 356629 km and 744 m. Hence,

$$S \approx 356629.744 \equiv 356629 + \frac{744}{1000} = 356629 + \frac{7}{10} + \frac{4}{100} + \frac{4}{1000}.$$

If one will continue further, one will get even better approximations and obtain a representation of the number S as an infinite decimal. Thus, we use the following definition.

Definition 1.1.1. *Real numbers* are identified with infinite decimals

$$a = a_0.a_1a_2 \ldots a_n \ldots, \tag{1.1.1}$$

where a_0 is an integer, and $a_1, a_2, \ldots, a_n \ldots$ are digits, i.e., they can assume any of ten Arabic number symbols, 0 through 9. Eq. (1.1.1) means that

$$a = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n} + \cdots. \tag{1.1.2}$$

Remark 1.1.1. If (1.1.1) is a periodical decimal, and only in this case, a is a *rational* number, i.e., $a = p/q$, where p and q are integers, $q \neq 0$. The real numbers determined by non-periodical infinite decimals are termed *irrational* numbers. The numbers 0. (9)= 0. 9999... and 1 are identified.

Example 1.1.1. Famous examples of irrational numbers are:

$$\begin{aligned}
\sqrt{2} &= 1.4142136\ldots &&\approx 1.41\\
\pi &= 3.1415926535\ldots &&\approx 3.14\\
\mathrm{e} &= 2.718281828459045\ldots &&\approx 2.72\\
\gamma &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right) &&\approx 0.58
\end{aligned}$$

where γ is known as *Euler's number.*

Remark 1.1.2. It is a historical accident that we represent real numbers in the decimal system. If Babylonian culture would last much longer we would probably use the Babylonian sexagesimal system and employ, instead of (1.1.2), the representation

$$a = a_0 + \frac{a_1}{60} + \frac{a_2}{60^2} + \cdots + \frac{a_n}{60^n} + \cdots. \tag{1.1.3}$$

Definition 1.1.2. A *variable* x is a quantity to which any numerical value can be assigned. A quantity with a fixed value is called a *constant.* One should distinguish *arbitrary* constants from *absolute* constants. An arbitrary constant retains any given value throughout the investigation, while an absolute constant retains the same value in all problems.

Example 1.1.2. In the equation of a circle, $x^2+y^2 = R^2$, x and y are variables representing the coordinates of a point moving along the circle, while the radius R is an arbitrary constant. On the other hand, the formula $C = 2\pi R$ for the circumference of the circle contains, along with the arbitrary constant R, two absolute constants, 2 and $\pi \approx 3.14$.

Theorem 1.1.1. Any real number a is a limit of a sequence of rational numbers $r_n = p_n/q_n$, where p_n and $q_n \neq 0$ are integers:

$$a = \lim_{n\to\infty} r_n. \tag{1.1.4}$$

Proof. Let the real number a be given by Eq. (1.1.1). We take for r_n the finite sums of the corresponding infinite series (1.1.2):

$$r_1 = a_0 + \frac{a_1}{10},\ r_2 = a_0 + \frac{a_1}{10} + \frac{a_2}{100},\ \ldots,\ r_n = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n}.$$

They provide a sequence of rational numbers $\{r_n\}$ satisfying Eq. (1.1.4).

The following definition is based on Theorem 1.1.1.

Definition 1.1.3. The exponential function $y = a^x$, where $a > 0$ is any real number, is defined by the following equations:

$$a^0 = 1,\ a^1 = a,\ a^n = \underbrace{a \cdots a}_{n},\quad n = 2, 3, \dots, \qquad (\text{here } x = n);$$

$$a^{\frac{1}{n}} \equiv \sqrt[n]{a} = b \Leftrightarrow b^n = a;\quad a^{\frac{p}{q}} = \sqrt[q]{a^p}, \qquad (\text{here } x = p/q);$$

$$a^x = \lim_{n\to\infty} a^{x_n} \equiv \lim_{n\to\infty} \sqrt[q_n]{a^{p_n}},\quad (\text{here } x = \lim_{n\to\infty} x_n,\ x_n = p_n/q_n)\cdot$$

Basic laws of exponents:

$$a^{-x} = \frac{1}{a^x},\quad a^x\, a^y = a^{x+y},\quad (ab)^x = a^x\, b^x,\quad (a^x)^y = a^{xy}.$$

Example 1.1.3. Consider the number $10^{\sqrt{2}} = 25.954\ldots$. We have $\sqrt{2} = \lim_{n\to\infty} x_n$, where $x_0 = 1,\ x_1 = 1.4,\ x_2 = 1.41, \dots$. Accordingly, $10^{\sqrt{2}} = \lim_{n\to\infty} y_n$, where $y_0 = 10^{x_0} = 10,\ y_1 = 10^{x_1} \approx 25.12,\ y_2 = 10^{x_2} \approx 25.70, \dots$.

In solving differential equations, one often encounters the exponential function

$$y = \mathrm{e}^x. \tag{1.1.5}$$

Here e is a real number determined by one of the most important limits in mathematical analysis:

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n. \tag{1.1.6}$$

Its value, accurate to fifteen decimal places, is given in Example 1.1.1.

Function (1.1.5) is a representative of so-called *elementary functions* defined as follows.

Definition 1.1.4. The *basic elementary functions* are:

$$y = C,\quad C = \text{const.};$$
$$y = x^\alpha,\quad \text{where } x > 0,\ \alpha \text{ is a real number};$$
$$y = a^x,\quad \text{where } a > 0,\ a \neq 1;$$
$$y = \log_a x,\quad \text{where } a > 0,\ a \neq 1;\quad x > 0;$$
$$y = \sin x,\ y = \cos x,\ y = \operatorname{tg} x\ (\equiv \tan x),\ y = \operatorname{ctg} x;$$
$$y = \arcsin x,\ y = \arccos x,\ y = \operatorname{arctg} x,\ y = \operatorname{arcctg} x.$$

A function $y = f(x)$ is called an *elementary function* if it is obtained from the basic elementary functions by a finite number of operations involving *addition, subtraction, multiplication, division,* and *superposition.*

Remark 1.1.3. The logarithm $\log_a x$ with $a = \mathrm{e}$ is called the *natural logarithm* and denoted by $\ln x$.

Remark 1.1.4. The basic trigonometric functions can be obtained from one of them, e.g., from $\sin x$, in combination with other basic elementary functions. Indeed,

$$\cos x = \sqrt{1-\sin^2 x}, \quad \tan x = \frac{\sin x}{\sqrt{1-\sin^2 x}}, \quad \cot x = \frac{\sqrt{1-\sin^2 x}}{\sin x}.$$

Similar relations exist between inverse trigonometric functions as well, e.g.

$$\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}}\cdot \tag{1.1.7}$$

Example 1.1.4. The following hyperbolic functions provide examples of non-basic elementary functions:

$$\sinh\, x = \frac{e^x - e^{-x}}{2}, \quad \cosh\, x = \frac{e^x + e^{-x}}{2}, \quad \tanh\, x = \frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot \tag{1.1.8}$$

Elementary functions of many variables are obtained in a similar way.

Example 1.1.5. The following function $\psi(t, x, z)$ is an elementary function of three variables, t, x, z :

$$\psi = -\frac{1}{4}\ln\left|M + \frac{1}{t}\left(2\sin^2 x + l_1 e^{-z}\sin x + l_2 e^{-2z}\right)\right| \tag{1.1.9}$$

involving three arbitrary constants, l_1, l_2 and M.

Example 1.1.6. The following functions that often occur in applications are given by integrals and are not elementary:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt \quad \text{(the integral sine)}, \tag{1.1.10}$$

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt \quad \text{(the integral cosine)}, \tag{1.1.11}$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \text{(the error function)}, \tag{1.1.12}$$

$$\mathrm{Ei}(x) = -\int_{-\infty}^x \frac{e^t}{t}\,dt, \quad \mathrm{li}(x) = \int_0^x \frac{dt}{\ln t} \equiv \mathrm{Ei}(\ln\, x), \tag{1.1.13}$$

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt \quad \text{(the Gamma function)}. \tag{1.1.14}$$

The Gamma function plays an important part in analysis and differential equations. It has interesting general properties, e.g.

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{1.1.15}$$

and the remarkable numerical values (see, e.g. [34]):

$$\Gamma(1) = 1,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \Gamma\left(\frac{n}{2}\right) = \frac{2\pi^{n/2}}{\omega_n},\ \Gamma(n+1) = n!, \tag{1.1.16}$$

where ω_n is the surface area of the unit sphere in n dimensions.

1.1.2 Quadratic and cubic equations

Problems of elementary mathematics can often be solved by the method of transformations. Let us begin with elementary algebra.

Recall that the roots $x = x_1$ and $x = x_2$ of the general quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0, \tag{1.1.17}$$

are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \cdot \tag{1.1.18}$$

The expression

$$\Delta = b^2 - 4ac \tag{1.1.19}$$

is known as the *discriminant* of the quadratic equation (1.1.17). It is manifest from (1.1.18) that the vanishing of discriminant (1.1.19),

$$\Delta = b^2 - 4ac = 0, \tag{1.1.20}$$

is the condition for Eq. (1.1.17) to have two equal roots, $x_1 = x_2$.

In accordance with tradition, students learn from school to derive solution (1.1.18) by completing the square. Indeed, this method is simple but it is not suitable for tackling the general cubic as well as equations of higher degrees.

The idea of transformation of equations, unlike the method of completing the square appropriate only for the quadratic equation, furnishes a general method appropriate for solution of the quadratic equation as well as for a simplification of equations of higher degrees. The simplest transformation of equations is provided by a linear transformation of the variable x :

$$y = x + \varepsilon. \tag{1.1.21}$$

It converts any equation of degree n into an equation of the same degree. In particular, the quadratic equation (1.1.17) after the substitution $x = y - \varepsilon$ becomes $ay^2 + (b - 2a\varepsilon)y + a\varepsilon^2 - b\varepsilon + c = 0$. Hence, transformation (1.1.21) converts (1.1.17) into a new quadratic equation,

$$\bar{a}\,y^2 + \bar{b}\,y + \bar{c} = 0,$$

where

$$\bar{a} = a, \quad \bar{b} = b - 2a\varepsilon, \quad \bar{c} = c + a\varepsilon^2 - b\varepsilon. \tag{1.1.22}$$

Defining ε from $b - 2a\varepsilon = 0$, one obtains $\bar{b} = 0$ and $\bar{c} = c - b^2/(4a)$. Hence, the transformation

$$y = x + \frac{b}{2a} \tag{1.1.23}$$

converts (1.1.17) into the equation

$$ay^2 - \frac{b^2 - 4ac}{4a} = 0.$$