国外数学名著系列

（影印版）27

Masamichi Takesaki

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# 《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

# Preface to the Encyclopaedia Subseries on Operator Algebras and Non-Commutative Geometry

The theory of von Neumann algebras was initiated in a series of papers by Murray and von Neumann in the 1930's and 1940's. A von Neumann algebra is a self-adjoint unital subalgebra $M$ of the algebra of bounded operators of a Hilbert space which is closed in the weak operator topology. According to von Neumann's bicommutant theorem, $M$ is closed in the weak operator topology if and only if it is equal to the commutant of its commutant. A *factor* is a von Neumann algebra with trivial centre and the work of Murray and von Neumann contained a reduction of all von Neumann algebras to factors and a classification of factors into types I, II and III.

$C^*$-algebras are self-adjoint operator algebras on Hilbert space which are closed in the norm topology. Their study was begun in the work of Gelfand and Naimark who showed that such algebras can be characterized abstractly as involutive Banach algebras, satisfying an algebraic relation connecting the norm and the involution. They also obtained the fundamental result that a commutative unital $C^*$-algebra is isomorphic to the algebra of complex valued continuous functions on a compact space – its spectrum.

Since then the subject of operator algebras has evolved into a huge mathematical endeavour interacting with almost every branch of mathematics and several areas of theoretical physics.

Up into the sixties much of the work on $C^*$-algebras was centered around representation theory and the study of $C^*$-algebras of type I (these algebras are characterized by the fact that they have a well behaved representation theory). Finite dimensional $C^*$-algebras are easily seen to be just direct sums of matrix algebras. However, by taking algebras which are closures in norm of finite dimensional algebras one obtains already a rich class of $C^*$-algebras – the so-called AF-algebras – which are not of type I. The idea of taking the closure of an inductive limit of finite-dimensional algebras had already appeared in the work of Murray-von Neumann who used it to construct a fundamental example of a factor of type II – the "hyperfinite" (nowadays also called approximately finite dimensional) factor.

One key to an understanding of the class of AF-algebras turned out to be $K$-theory. The techniques of $K$-theory, along with its dual, *Ext*-theory, also found immediate applications in the study of many new examples of $C^*$-algebras that arose in the end of the seventies. These examples include for instance "the noncommutative tori" or other crossed products of abelian $C^*$-algebras by groups of homeomorphisms and abstract $C^*$-algebras generated by isometries with certain relations, now known as the algebras $\mathcal{O}_n$. At the same time, examples of algebras were increasingly studied that codify data from differential geometry or from topological dynamical systems.

On the other hand, a little earlier in the seventies, the theory of von Neumann algebras underwent a vigorous growth after the discovery of a natural infinite family of pairwise nonisomorphic factors of type III and the advent of Tomita-Takesaki theory. This development culminated in Connes' great classification theorems for approximately finite dimensional ("injective") von Neumann algebras.

Perhaps the most significant area in which operator algebras have been used is mathematical physics, especially in quantum statistical mechanics and in the foundations of quantum field theory. Von Neumann explicitly mentioned quantum theory as one of his motivations for developing the theory of rings of operators and his foresight was confirmed in the algebraic quantum field theory proposed by Haag and Kastler. In this theory a von Neumann algebra is associated with each region of space-time, obeying certain axioms. The inductive limit of these von Neumann algebras is a $C^*$-algebra which contains a lot of information on the quantum field theory in question. This point of view was particularly successful in the analysis of superselection sectors.

In 1980 the subject of operator algebras was entirely covered in a single big three weeks meeting in Kingston Ontario. This meeting served as a review of the classification theorems for von Neumann algebras and the success of $K$-theory as a tool in $C^*$-algebras. But the meeting also contained a preview of what was to be an explosive growth in the field. The study of the von Neumann algebra of a foliation was being developed in the far more precise $C^*$-framework which would lead to index theorems for foliations incorporating techniques and ideas from many branches of mathematics hitherto unconnected with operator algebras.

Many of the new developments began in the decade following the Kingston meeting. On the $C^*$-side was Kasparov's $KK$-theory – the bivariant form of $K$-theory for which operator algebraic methods are absolutely essential. Cyclic cohomology was discovered through an analysis of the fine structure of extensions of $C^*$-algebras These ideas and many others were integrated into Connes' vast *Noncommutative Geometry* program. In cyclic theory and in connection with many other aspects of noncommutative geometry, the need for going beyond the class of $C^*$-algebras became apparent. Thanks to recent progress, both on the cyclic homology side as well as on the $K$-theory side, there is now a well developed bivariant $K$-theory and cyclic theory for a natural class of topological algebras as well as a bivariant character taking $K$-theory to cyclic theory. The 1990's also saw huge progress in the classification theory of nuclear $C^*$-algebras in terms of $K$-theoretic invariants, based on new insight into the structure of exact $C^*$-algebras.

On the von Neumann algebra side, the study of subfactors began in 1982 with the definition of the *index* of a subfactor in terms of the Murray-von Neumann theory and a result showing that the index was surprisingly restricted in its possible values. A rich theory was developed refining and clarifying the index. Surprising connections with knot theory, statistical mechanics and quantum field theory have been found. The superselection theory mentioned above turned out to have fascinating links to subfactor theory. The subfactors themselves were constructed in the representation theory of loop groups.

Beginning in the early 1990's Voiculescu initiated the theory of free probability and showed how to understand the free group von Neumann algebras in terms of random matrices, leading to the extraordinary result that the von Neumann algebra $M$ of the free group on infinitely many generators has full fundamental group, i.e. $pMp$ is isomorphic to $M$ for every non-zero projection $p \in M$. The subsequent introduction of free entropy led to the solution of more old problems in von Neumann algebras such as the lack of a Cartan subalgebra in the free group von Neumann algebras.

Many of the topics mentioned in the (obviously incomplete) list above have become large industries in their own right. So it is clear that a conference like the one in Kingston is no longer possible. Nevertheless the subject does retain a certain unity and sense of identity so we felt it appropriate and useful to create a series of encylopaedia volumes documenting the fundamentals of the theory and defining the current state of the subject.

In particular, our series will include volumes treating the essential technical results of $C^*$-algebra theory and von Neumann algebra theory including sections on noncommutative dynamical systems, entropy and derivations. It will include an account of $K$-theory and bivariant $K$-theory with applications and in particular the index theorem for foliations. Another volume will be devoted to cyclic homology and bivariant $K$-theory for topological algebras with applications to index theorems. On the von Neumann algebra side, we plan volumes on the structure of subfactors and on free probability and free entropy. Another volume shall be dedicated to the connections between operator algebras and quantum field theory.

October 2001

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# Introduction

*Mathematics* for infinite dimensional objects is becoming more and more important today both in theory and application. *Rings of operators*, renamed *von Neumann algebras* by J. Dixmier, were first introduced by J. von Neumann fifty years ago, 1929, in [254] with his grand aim of giving a sound foundation to mathematical sciences of infinite nature. J. von Neumann and his collaborator F. J. Murray laid down the foundation for this new field of mathematics, *operator algebras*, in a series of papers, [240], [241], [242], [257] and [259], during the period of the 1930s and early in the 1940s. In the introduction to this series of investigations, they stated *Their solution* (to the problems of understanding rings of operators)[1] *seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalize the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.* Since then there has appeared a large volume of literature, and a great deal of progress has been achieved by many mathematicians. The motivations of Murray and von Neumann seem to have been fully verified. Many important results and powerful techniques were added to the theory. Various related fields of mathematics have emerged, and a number of topics in this subject have branched out to independent fields.

[1] Added by the author.

The main characteristic of this subject can be stated as a complex of analysis and algebra: the results are phrased in algebraic terms, while the techniques are highly analytic. Sometimes, one might run into problems directly related to the foundation of mathematics such as the continuum hypothesis. One might be amazed to realize the possibility of such an elaborated algebraic structure in this wild area involving high degrees of infinity.

The theory of operator algebras is concerned with self-adjoint algebras of bounded linear operators on a Hilbert space closed under the norm topology, $C^*$-algebras, or the weak operator topology, von Neumann algebras. $C^*$-algebras are characterized as a special class of Banach algebras by means of a simple system of axioms. A concrete realization of a $C^*$-algebra as an algebra of operators on a Hilbert space is regarded as a representation of the algebra. Thus, the study of $C^*$-algebras consists of two parts: one is concerned with the intrinsic structure of algebras and the other deals with the representations of a $C^*$-algebra. Needless to say, these two parts are closely related, and indeed the algebraic structure of a $C^*$-algebra is studied through various representations of the algebra. Thus, this division of the theory stays at a formal level. Nevertheless, the separation of problems has positive effects: for instance, a systematic usage of inequivalent representations of a $C^*$-algebra provides flexible techniques even if it is given as a concrete algebra of operators on a specially chosen Hilbert space. Indeed, this freedom in choosing an appropriate representation is one of the main merits of the axiomatic approach to operator algebras.

Being infinite dimensional, our problems require careful investigation of approximation process; thus the study of topological structures is inevitable. For this reason, the topological, analytical aspect of operator algebras receives more of our attention than the algebraic aspect in this first volume. After establishing the basic foundation in Chapter I, the Banach space duality for operator algebras will be studied throughout the text. The reader will find a strong similarity between our theory and measure theory on locally compact spaces. In fact, the study of abelian $C^*$-algebras will be reduced to that of locally compact spaces, and a substantial part of our theory is called noncommutative integration theory.

Each chapter begins with an introduction to its basic facts. Sections and paragraphs with * sign are somewhat technical; the reader who wants to get rather a quick grasp of the theory may postpone these parts. The sign ** indicates the end of the technical paragraph. Comments and historic background are placed at the end of each chapter and some sections as notes. Complements to a section or a chapter and some results of special interest are stated as exercises with † sign and references.

In the succeeding volume, the author will discuss further, among other topics, noncommutative integration theory, the so-called Tomita–Takesaki theory, automorphism groups of operator algebras, crossed products, infinite tensor products, the structure of von Neumann algebras of type III, approximately finite dimensional von Neumann algebras, and the existence of a continuum of nonisomorphic factors.

The author would like to express here his sincere gratitude to Professors H. A. Dye, R. V. Kadison, D. Kastler, M. Nakamura, Y. Misonou and J. Tomiyama from whom he received scientific as well as moral support at several stages of the work. A major part of the preparation was done at the University of Aix-Marseille-Luminy, ZiF, the University of Bielefeld, while the author was on leave from the University of California, Los Angeles. He acknowledges gratefully a generous support extended to him, for a part of the preparation, from the Guggenheim Foundation. The author is very grateful to Mrs. L. Beerman for typing the manuscript skillfully with great patience.

# Contents

# Theory of Operator Algebras I

# Chapter I
# Fundamentals of Banach Algebras and $C^*$-Algebras

## 0. Introduction

In this this first chapter, we lay the foundation for later discussion, giving elementary results in Banach algebras and $C^*$-algebras. The first three sections are devoted to the general Banach algebras. The most important results in these sections are Theorem 2.5, Corollary 2.6, and Theorem 3.11, which are really fundamental in the theory of Banach algebras. Discussion of $C^*$-algebras starts from Section 4. As an object of the theory of operator algebras, a $C^*$-algebra is a uniformly closed self-adjoint algebra $A$ of bounded linear operators on a Hilbert space $\mathfrak{H}$. The major task of the theory of operator algebras is to find descriptions of the structure of $\{A,\mathfrak{H}\}$. This problem splits into two problems:

(a) Find descriptions of the algebraic structure of $A$ alone;
(b) Given an algebra $A$, find all possible pairs $\{B,\mathfrak{K}\}$ such that $B$ is isomorphic to $A$ as an abstract algebra.

The first approach to problem (a) is to characterize a uniformly closed self-adjoint algebra of bounded linear operators on a Hilbert space as an abstract algebra, i.e., without using a Hilbert space. A solution to this question is given by postulates (i)–(vi) in Section 1, for a $C^*$-algebra, and is proved in Theorem 9.18. Problem (b) leads us to the representation theory of $C^*$-algebras. Namely, an action of a $C^*$-algebra $A$ is viewed as a representation on a Hilbert space, and problem (b) is translated in this terminology as follows:

(b′) Find descriptions of all representations of a given $C^*$-algebra.

The obvious question after the postulates were once laid down is the existence of representations, which is answered, as mentioned, by Theorem 9.18. It turns out (Theorem 9.14) that there is a strong link between positive linear functionals and representations. Section 9 is the highlight of the chapter. A characterization of extreme points of the unit ball of a $C^*$-algebra is given in Section 10, which will be used in Chapter III to show that a $W^*$-algebra is unital. Section 11 is devoted to a sketch of finite dimensional $C^*$-algebras and their representations.

## 1. Banach Algebras

Let **R** and **C** denote always the real number field and the complex number field, respectively.

**Definition 1.1.** Let $A$ be a Banach space over **C**. If $A$ is an algebra over **C** in which the multiplication satisfies the inequality

$$\|xy\| \leq \|x\|\,\|y\|,$$

then $A$ is called a *Banach algebra*.

The inequality

$$\|x_1y_1 - x_2y_2\| \leq \|x_1\|\,\|y_1 - y_2\| + \|x_1 - x_2\|\,\|y_2\|$$

shows that the product $xy$ is a continuous function of two variables $x$ and $y$.

If $E$ is a Banach space over **C**, then the set $\mathscr{L}(E)$ of all bounded operators on $E$ is a Banach algebra with the natural algebraic operations and norm.

**Definition 1.2.** If a Banach algebra $A$ admits a map: $x \mapsto x^* \in A$ with the following properties:

(i) $(x^*)^* = x$;

(ii) $(x + y)^* = x^* + y^*$;

(iii) $(\alpha x)^* = \bar{\alpha}x^*$;

(iv) $(xy)^* = y^*x^*$;

(v) $\|x^*\| = \|x\|$;

for every $x, y \in A$ and $\alpha \in \mathbf{C}$, then $A$ is called an *involutive Banach algebra* and the map: $x \mapsto x^*$ the *involution* (or *$*$-operation*) of $A$. If the involution of $A$ satisfies the following additional condition:

(vi) $\|x^*x\| = \|x^*\|\,\|x\|$, $x \in A$;

then $A$ is called a *$C^*$-algebra*.

Let $\Omega$ be a locally compact space. The set $C_\infty(\Omega)$ of all continuous functions on $\Omega$ vanishing at infinity is a $C^*$-algebra with the following structure:

$$(\lambda x + \mu y)(\omega) = \lambda x(\omega) + \mu y(\omega);$$
$$(xy)(\omega) = x(\omega)y(\omega);$$
$$x^*(\omega) = \overline{x(\omega)};$$
$$\|x\| = \sup\{|x(\omega)|:\omega \in \Omega\};$$

for every $x,y \in C_\infty(\Omega)$, $\lambda,\mu \in \mathbf{C}$ and $\omega \in \Omega$. The $C^*$-algebra $C_\infty(\Omega)$ is abelian. The algebra $C_\infty(\Omega)$ has an identity if and only if $\Omega$ is compact. In this case, $C_\infty(\Omega)$ is denoted simply by $C(\Omega)$.

If $\mathfrak{H}$ is a Hilbert space, then the Banach algebra $\mathscr{L}(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$ is a $C^*$-algebra with the involution: $x \mapsto x^*$ defined as the adjoint operator $x^*$ of $x$. If the dimension of $\mathfrak{H}$ is greater than one, then $\mathscr{L}(\mathfrak{H})$ is not abelian.

**Proposition 1.3.** *If $A$ is a Banach algebra with an identity* 1, *then there exists a norm $\|\cdot\|_0$ on $A$ such that*: (i) *the new norm $\|\cdot\|_0$ is equivalent to the original norm $\|\cdot\|$*; (ii) *$(A,\|\cdot\|_0)$ is a Banach algebra*; (iii) $\|1\|_0 = 1$.

PROOF. For each $x \in A$, let $L_x$ denote the operator: $y \in A \mapsto xy \in A$. The map: $x \mapsto L_x$ is then injective because $L_x 1 = x$. Put $\|x\|_0 = \|L_x\|$, $x \in A$. By the inequality $\|xy\| \le \|x\|\|y\|$, we have $\|x\|_0 \le \|x\|$. On the other hand, we have

$$\|x\|_0 = \|L_x\| = \sup\{\|xy\|:\|y\| \le 1\} \ge \|x\|/\|1\|.$$

Hence the norm $\|\cdot\|_0$ is equivalent to the original norm. Assertions (ii) and (iii) are almost automatic now. Q.E.D.

By this proposition, we assume always that the norm of the identity is one if it exists. A Banach algebra with an identity is said to be *unital*.

*Remark 1.4.* If $A$ is a unital involutive Banach algebra, then we have $1^* = 1$. Furthermore, if $A$ is a unital $C^*$-algebra, then the condition $\|1\| = 1$ follows automatically from postulate (vi).

If a given involutive Banach algebra $A$ is not unital, then we can imbed $A$ into a unital involutive Banach algebra $A_1$ as an ideal in the following way: We take the direct sum $A \oplus \mathbf{C}$ as a linear space $A_1$, in which we define a Banach algebra structure by

$$(x,\lambda)(y,\mu) = (xy + \mu x + \lambda y,\lambda\mu);$$
$$(x,\lambda)^* = (x^*,\bar{\lambda});$$
$$\|(x,\lambda)\| = \|x\| + |\lambda|;$$