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彭龙 著

有限元概率算法及其高精度分析

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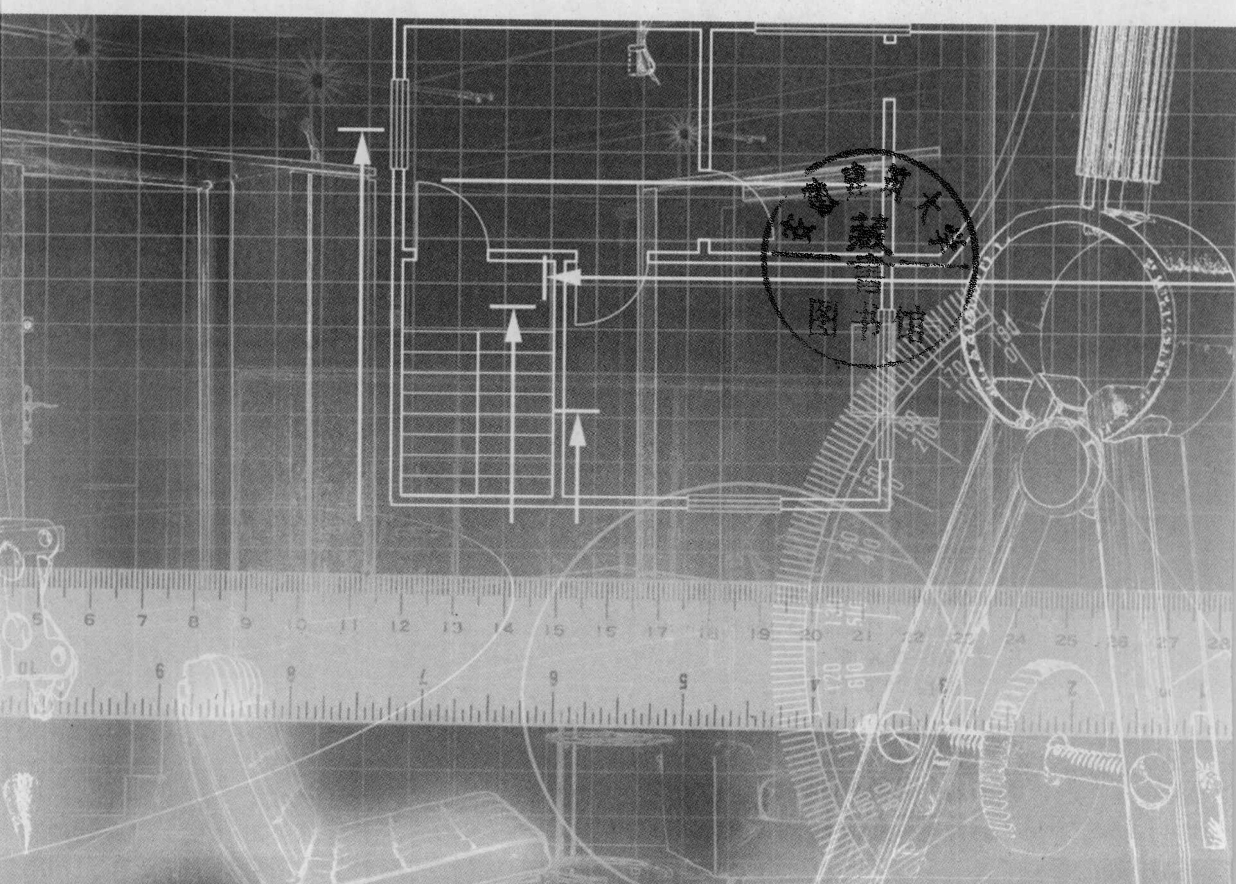

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近年来，有限元概率算法在国内外引起了不同的反响，有的高度评价这种新的算法，有的基本上否定这种算法。本书进一步讨论这一方法，得到了一些新的结果。

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作者简介

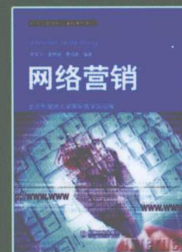
彭龙 男，1964年出生，经济学教授，博士生导师，享受国务院政府特殊津贴。1997年7月在中国科学院系统科学研究所获得博士学位，2001年7月至2003年7月于北京大学光华管理学院从事博士后研究。

现任北京外国语大学国际商学院院长、国际金融与商务研究所所长，中国金融学会金融工程专业委员会常务委员，中国国际贸易学会国际商务英语研究委员会副主任，北京邮电大学经济管理学院教授、博士生导师，对外经济贸易大学国际经贸学院兼职教授，中国人民大学商学院兼职教授，以及数家中国香港及国内上市公司独立董事。

曾任东南大学应用数学系（原数学力学系）副教授、副系主任，上海丰华（集团）股份有限公司（股票代码：600615）董事长，阳光股份有限公司（股票代码：000608）独立董事。

曾在美国Columbia University做高级研究学者，英国University of Loughborough、Lancaster University做高级访问学者，美国University of Wyoming做访问学者。

曾主持国家自然科学基金两项，出版专著及教材多部，并在国际国内核心期刊发表学术论文20余篇。研究方向：金融工程、证券投资、衍生证券定价理论。



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前　　言

有限元概率算法是应用蒙特卡罗方法去求偏微分方程有限元逼近的近似解。使用这种方法可以在不形成总刚矩阵的情况下直接求出有限元解在某个点或少数几个点处的近似值，不但能节省计算机内存单元且程序易于实现。

近年来，有限元概率算法在国内外引起了不同的反响，有的高度评价这种新的算法，有的基本上否定这种算法。本书进一步讨论这一方法，得到了一些新的结果。

本文在第 5 章将以二维 Laplace 方程第一边值问题为例构造两种提高精度的方法，应用多重网格法的思想，让质点在两个不同尺度的网格上以一定的概率交替的变网格游动。这样得到的随机变量的数学期望 $E(\zeta)$ 与 u 的二次元逼近一致。本文的第 7 章证明了有限元概率算法的与维数无关性。本文第 8 章给出一种快速有限元概率算法，我们 P'_k 引入型有限元空间 $V_h \in S_0^h$(P_k 型有限元空间)，其维数 $DimV_h = n-1$ 等于 P_1 型有限元空间的维数，只需用相当于分片线性有限元结点构成 P'_k 的型有限元空间就可达到 P_k 型元的超收敛精度，它使存贮量减少，计算速度提高。如果直接在有限元空间上建立概率模型，那么质点游动的结点有 $kn+1$ 个，工作量是很大的，而我们在有限元空间 V_h 上建立概率模型，质点游动的结点只有 $n+1$ 个，工作量就小多了，这是一种快速概率算法。

Foreward

The finite element probability computing method is the method which uses the Monte Carlo method to get the approximate solution of finite element approximation of the partial differential equation. Using this method, the approximate value on or a few nodes of the finite element solution can be directly calculated without forming the total stiffness matrix. Thus, not only the computer's inner memory unit can be saved, but also the procedure is easy to be achieved.

In this thesis, we would like to further discuss the method. The first section introduces Markov chain and probability transfer matrices. Using these matrices, we establish the basic framework of the probability computing methods. In the second chapter the finite element probability computing method is introduced. Based on the above results, two kinds of methods with high accuracy (the probability multigrid method and the boundary thickening method) are presented in the fourth chapter. In the sixth chapter, the rectangular element probability computing method is presented, and a proof of its convergence is given. The numerical experiment shows that is has a better approximation than that for the triangular element probability computing method. The seventh chapter proves that the computing amounts are the same for 1-d solution or for 2-d solution or 3-d solution at a note. In the eighth chapter, the fast computing scheme of the finite element method for the two point boundary problem is presented. The ninth chapter gives the example analysis for finite element probability computing methods.

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Chapter 1

Introduction

To use mathematical models on a computer one needs numerical methods. Only in the very simplest cases is it possible to find exact analytical solutions of the equations in the model, and in general one has to rely on numerical techniques for finding approximate solutions. The finite element method is a general technique for numerical solution of differential and integral equations in science and engineering. The method was introduced by engineers in the late 50's and early 60's for the numerical solution of partial differential equations in structural engineering (elasticity equations, plate equations, etc). At this point the method was thought of as a generalization of earlier methods in structural engineering for beams, frames and plates, where the structure was subdivided into small parts, so-called finite elements, with know simple behaviour. When the mathematical study of the finite element method started in the mid 60's it soon became clear that in fact the method is a general technique for numerical solution of partial differential equations with roots in the variational methods in mathematics introduced in the beginning of the century. During the 60's and 70's the method was developed, by engineers, mathematicians and numerical analysts, into a general method for numerical solution of partial differential equations equations and integral equations with applications in many areas of science and engineering. Today, finite element methods are used extensively for problems in structural engineering, strength of materials, fluid mechanics, nuclear engineering, electro-magnetism, wave-propagat-ion, scattering, heat conduction, convection-diffusion processes, integrated circuits, petroleum engineering, reaction-diffusion processes and many other areas.

We often encounter some concentrated load problems in the aspect of engineering and physics. As for the solution to these problems, it is unnecessary for us to know the approximate value on all nodes of the finite element solution, we only need to calculate it on one or a few nodes. Concerning this kind of problems, we will waste too much time on the calculation of some unnecessary results if we apply the finite element method directly.

The finite element probability computing method (see [1], [2]) is the method which uses the Monte Carlo method to get the approximate solution of finite element approximation of the partial differential equation. Using this method, the approximate value on one or a few nodes of the finite element solution can be directly calculated without forming the total stiffness matrix. Thus, not only the computers inner memory unit can be saved, but also the procedure is easy to be achieved.

Recently, the finite element probability computing methods arouse different responses in the world. Some people think this is a good method in computing, others don't think so. In this thesis, we would like to further discuss the method.

This thesis is divided roughly into seven sections. The first section introduces Markov chain and probability transfer matrices. Using these matrices, we will establish the basic framework of the probability methods. In the second section, the finite element probability computing method bolds, if the dividing step length h is very long. Therefore finding a method with high accuracy is significant. Based on the above results, two kinds of methods with high accuracy (the probability multi-grain method and the boundary thickening method) are presented in the third chapter. In the fourth chapter, the rectangular element probability computing methods is presented, and a proof of its convergence is given. The numerical experiment shows that it has a better approximation than that for the triangular element probability computing method. Probability computing methods have the advantages and the disadvantages.

Their mostly valuable character is the dimensional independence in solving a PDE (at one or a few notes). The fifth section proves that the computing amounts are the same for 1-d solution or for 2-d solution of 3-d solution at the sixth chapter. In the seventh chapter, the fast computing scheme of the finite element method for the two point boundary problem is presented. The eighth chapter gives the example analyses for finite element probability computing methods.

Chapter 2

Brief Introduction of Markov Chain and Nonnegative Matrices

Let $E = \{P_1, P_2, \cdots, P_m\}$ (or $E = \{1, 2, \cdots, m\}$ for short) denote the finite point set in Euclid space R_d ($d = 1, 2, 3, \cdots$). Assume that a particle walks on E as random which walks once a unit time. Let ζ_0, ζ_1, ζ_2, $\cdots$ denote the location of particle on the movement $t = 0$, 1, 2, $\cdots$, respectively. Sometimes, E is called state space. ζ_n ($n = 0, 1, 2, \cdots$) is a random process regarding $(E, \rightarrow)$ as state space, where $\rightarrow$ is the set of all subsets $P(\zeta_0 = i_0, \zeta_1 = i_1, \cdots \zeta_s = i_s)$. We may define conditional probability:

$$P(\zeta_{n+1} = i_{n+1} \mid \zeta_0 = i_0, \cdots \zeta_n = i_n)$$
$$= P(\zeta_{n+1} = i_{n+1}, \cdots, \zeta_0 = i_0) /$$
$$P(\zeta_n = i_n, \cdots, \zeta_0 = i_0),$$

$\{\zeta_n, n \geqslant 0\}$ is called a homogeneous Markov chain (or

Markov chain for short) if it satisfies following conditions:

a) (Markov property)

$$P(\xi_{n+1}=i_{n+1}|\xi_0=i_0,\cdots\xi_n=i_n)$$
$$=P(\xi_{n+1}=i_{n+1}|\xi_n=i_n),$$

b) (Homogeneity)

$$P(x_{n+1}=j|x_n=i)=P(x_{m+1}=j|x_m=i).$$

Let $p_{ij}=P(\xi_{n+1}=j|\xi_n=i)$ which is called transfer probability from $i\in E$ to $j\in E$. The matrix $P=(p_{ij})_{m\times n}$ is a transfer probability matrix, which satisfies the following conditions:

(i) $p_{ij}\geqslant 0$,

(ii) $\sum_{j=1}^{m} p_{ij} = 1.$

The k-th product $P^{(k)}=(p_{ij}(k))$ of P is called k-th transfer matrix, which satisfies $P^{(m+k)}=P^{(m)}\cdot P^{(k)}$ obviously. In addition,

c) $p_{ij}\geqslant 0$,

d) $\sum_{j} p_{ij} = 1$,

e) Chapman-kolme property:

$$p_{ij}(n+k) = \sum_{r} p_{ir}(n)p_{rj}(k).$$

Example 1. Random walk with absorbing barriers. Assume that the state space of random walk ξ_n, $n\geqslant 0$ is the set $e=\{0,1,2,\cdots,b\}$ in $[0,b]$, the points 1, b are the absorbing barriers, i. e. the particle will stop forever if it is at the point 0 or b:

$$p_{00}=1,\qquad p_{bb}=1$$