Advanced Mathematics for Engineers & Scientists

MURRAY R. SPIEGEL

INCLUDING 950 SOLVED PROBLEMS

SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

ADVANCED MATHEMATICS

for

Engineers and Scientists

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BY

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Preface

In recent years the number of topics in mathematics required of engineers and scientists has greatly increased. This is to be expected since mathematics plays a vital role as a language in the formulation and solution of problems involving science and engineering and as these problems become more complex it is natural that the mathematical methods needed for their solution should increase in number and complexity.

It is the purpose of this book to provide important advanced mathematical concepts and methods needed by engineers and scientists as well as mathematicians who are interested in the applications of their field. The book has been designed as a supplement to all current standard textbooks or as a textbook for a formal course in the mathematical methods of engineering and science.

Each chapter begins with a clear statement of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems and derivations of basic results are included among the solved problems. The large number of supplementary problems serve as a review and possible extension of the material of each chapter.

Topics covered include ordinary differential equations, Laplace transforms, vector analysis, Fourier series, Fourier integrals, gamma, beta and other special functions, Bessel functions, Legendre and other orthogonal functions, partial differential equations, complex variables and conformal mapping, matrices and calculus of variations. The first chapter which provides a review of fundamental concepts of algebra, trigonometry, analytic geometry and calculus may either be read at the beginning or referred to as needed depending on the background of the student.

Considerably more material has been included here than can be covered in most courses. This has been done to make the book more flexible, to provide a more useful book of reference and to stimulate further interest in the topics.

I wish to take this opportunity to thank Daniel Schaum, Nicola Monti and Hank Hayden for their splendid cooperation.

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Chapter 1

Review of Fundamental Concepts

REAL NUMBERS

At the very foundations of mathematics is the concept of a *set* or *collection* of objects and, in particular, sets of *numbers* on which we base our quantitative work in science and engineering. The student is already familiar with the following important sets of numbers.

1. **Natural Numbers** $1, 2, 3, 4, \ldots$ or *positive integers* used in counting.
2. **Integers** $0, \pm 1, \pm 2, \pm 3, \ldots$. These numbers arose in order to provide meaning to *subtraction* [inverse of *addition*] of any two natural numbers. Thus $2-6=-4$, $8-8=0$, etc.
3. **Rational Numbers** such as $2/3$, $-10/7$, etc. arose in order to provide meaning to *division* [inverse of *multiplication*] or *quotient* of any two integers with the exception that division by zero is not defined.
4. **Irrational Numbers** such as $\sqrt{2}$, π, etc. are numbers which cannot be expressed as the quotient of two integers.

Note that the set of natural numbers is a *subset*, i.e. a part, of the set of integers which in turn is a subset of the set of rational numbers.

The set of numbers which are either rational or irrational is called the set of *real numbers* [to distinguish them from *imaginary* or *complex numbers* on page 11] and is composed of *positive* and *negative numbers* and *zero*. The real numbers can be represented as *points* on a line as indicated in Fig. 1-1. For this reason we often use *point* and *number* interchangeably.

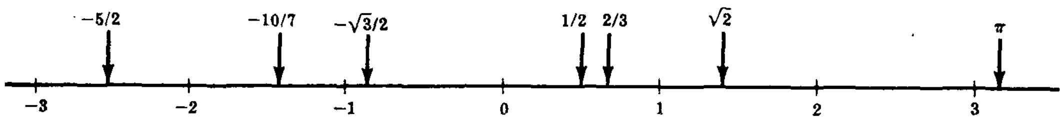

Fig. 1-1

The student is also familiar with the concept of *inequality*. Thus we say that the real number a is *greater than* or *less than* b [symbolized by $a > b$ or $a < b$] if $a - b$ is a positive or negative number respectively. For any real numbers a and b we must have $a > b$, $a = b$ or $a < b$.

RULES OF ALGEBRA

If a, b, c are any real numbers, the following rules of algebra hold.

1.	$a + b = b + a$	Commutative law for addition
2.	$a + (b + c) = (a + b) + c$	Associative law for addition
3.	$ab = ba$	Commutative law for multiplication
4.	$a(bc) = (ab)c$	Associative law for multiplication
5.	$a(b + c) = ab + ac$	Distributive law

It is from these rules [if we accept them as *axioms* or *postulates*] that we can prove the usual rules of signs, as for example $(-5)(3) = -15$, $(-2)(-3) = 6$, etc.

The student is also familiar with the usual rules of *exponents*:

$$a^m \cdot a^n = a^{m+n}, \quad a^m/a^n = a^{m-n},\ a \neq 0, \quad (a^m)^n = a^{mn} \tag{1}$$

FUNCTIONS

Another important concept is that of *function*. The student will recall that a function f is a rule which assigns to each object x, also called *member* or *element*, of a set A an element y of a set B. To indicate this correspondence we write $y = f(x)$ where $f(x)$ is called the *value* of the function at x.

Example 1. If $f(x) = x^2 - 3x + 2$, then $f(2) = 2^2 - 3(2) + 2 = 0$.

The student is also familiar with the process of "graphing functions" by obtaining number pairs (x, y) and considering these as points plotted on an xy coordinate system. In general $y = f(x)$ is represented graphically by a *curve*. Because y is usually determined from x, we sometimes call x the *independent variable* and y the *dependent variable*.

SPECIAL TYPES OF FUNCTIONS

1. **Polynomials** $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$. If $a_0 \neq 0$, n is called the *degree* of the polynomial. The polynomial equation $f(x) = 0$ has exactly n *roots* provided we count repetitions. For example $x^3 - 3x^2 + 3x - 1 = 0$ can be written $(x-1)^3 = 0$ so that the 3 roots are 1, 1, 1. Note that here we have used the *binomial theorem*

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots + x^n \tag{2}$$

where the *binomial coefficients* are given by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{3}$$

and where *factorial* n, i.e. $n!$, $= n(n-1)(n-2)\cdots 1$ while $0! = 1$ by definition.

2. **Exponential Functions** $f(x) = a^x$. These functions obey the rules *(1)*.

An important special case occurs where $a = e = 2.7182818\cdots$.

3. **Logarithmic Functions** $f(x) = \log_a x$. These functions are *inverses* of the exponential functions, i.e. if $a^x = y$ then $x = \log_a y$ where a is called the *base* of the logarithm. Interchanging x and y gives $y = \log_a x$. If $a = e$, which is often called the *natural base* of logarithms, we denote $\log_e x$ by $\ln x$, called *the natural logarithm* of x. The fundamental rules satisfied by natural logarithms [or logarithms to any base] are

$$\ln(mn) = \ln m + \ln n, \quad \ln\frac{m}{n} = \ln m - \ln n, \quad \ln m^p = p \ln m \tag{4}$$

4. **Trigonometric Functions** $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$.

Some fundamental relationships among these functions are as follows.

(a) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, $\quad \cos x = \sin\left(\frac{\pi}{2} - x\right)$, $\quad \tan x = \dfrac{\sin x}{\cos x}$,

$\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$, $\quad \sec x = \dfrac{1}{\cos x}$, $\quad \csc x = \dfrac{1}{\sin x}$

(*b*) $\sin^2 x + \cos^2 x = 1, \quad \sec^2 x - \tan^2 x = 1, \quad \csc^2 x - \cot^2 x = 1$

(*c*) $\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$

(*d*) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

(*e*) $A \cos x + B \sin x = \sqrt{A^2 + B^2} \sin(x + \alpha)$ where $\tan \alpha = A/B$

The trigonometric functions are *periodic*. For example $\sin x$ and $\cos x$, shown in Fig. 1-2 and 1-3 respectively, have period 2π.

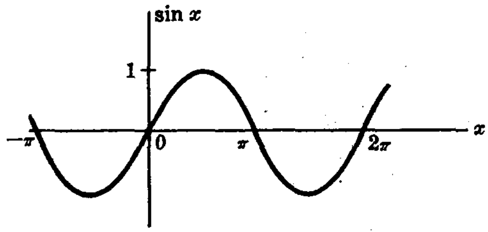

Fig. 1-2

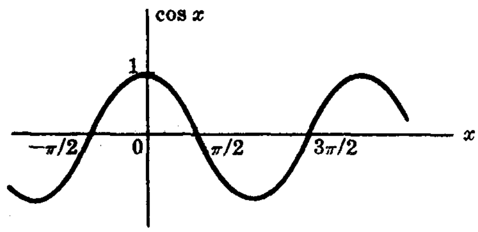

Fig. 1-3

5. Inverse Trigonometric Functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$. These are *inverses* of the trigonometric functions. For example if $\sin x = y$ then $x = \sin^{-1} y$, or on interchanging x and y, $y = \sin^{-1} x$.

6. Hyperbolic Functions. These are defined in terms of exponential functions as follows.

(*a*) $\sinh x = \dfrac{e^x - e^{-x}}{2}, \qquad \cosh x = \dfrac{e^x + e^{-x}}{2},$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Some fundamental identities analogous to those for trigonometric functions are

(*b*) $\cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x + \tanh^2 x = 1, \quad \coth^2 x - \operatorname{csch}^2 x = 1$

(*c*) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

The inverse hyperbolic functions, given by $\sinh^{-1} x$, $\cosh^{-1} x$, etc. can be expressed in terms of logarithms [see Problem 1.9, for example].

LIMITS

The function $f(x)$ is said to have the *limit* l as x approaches a, abbreviated $\lim_{x \to a} f(x) = l$, if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

Note that $|p|$, i.e. the *absolute value* of p, is equal to p if $p > 0$, $-p$ if $p < 0$ and 0 if $p = 0$.

Example 2. $\lim_{x \to 1} (x^2 - 4x + 8) = 5, \quad \lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4, \quad \lim_{x \to 0} \dfrac{\sin x}{x} = 1$

If $\lim_{x\to a} f_1(x) = l_1$, $\lim_{x\to a} f_2(x) = l_2$ then we have the following *theorems on limits.*

(a) $\lim_{x\to a} [f_1(x) \pm f_2(x)] = \lim_{x\to a} f_1(x) \pm \lim_{x\to a} f_2(x) = l_1 \pm l_2$

(b) $\lim_{x\to a} [f_1(x) f_2(x)] = \left[\lim_{x\to a} f_1(x)\right]\left[\lim_{x\to a} f_2(x)\right] = l_1 l_2$

(c) $\lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)} = \frac{l_1}{l_2}$ if $l_2 \neq 0$

CONTINUITY

The function $f(x)$ is said to be *continuous* at a if $\lim_{x\to a} f(x) = f(a)$.

Example 3. $f(x) = x^2 - 4x + 8$ is continuous at $x = 1$. However, if $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 6 & x = 2 \end{cases}$ then $f(x)$ is not continuous [or is *discontinuous*] at $x = 2$ and $x = 2$ is called a *discontinuity* of $f(x)$.

If $f(x)$ is continuous at each point of an interval such as $x_1 \leqq x \leqq x_2$ or $x_1 < x \leqq x_2$, etc., it is said to be *continuous in the interval.*

If $f_1(x)$ and $f_2(x)$ are continuous in an interval then $f_1(x) \pm f_2(x)$, $f_1(x) f_2(x)$ and $f_1(x)/f_2(x)$ where $f_2(x) \neq 0$ are also continuous in the interval.

DERIVATIVES

The *derivative* of $y = f(x)$ at a point x is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \tag{5}$$

where $h = \Delta x$, $\Delta y = f(x+h) - f(x) = f(x+\Delta x) - f(x)$ provided the limit exists.

The *differential* of $y = f(x)$ is defined by

$$dy = f'(x)\,dx \quad \text{where } dx = \Delta x \tag{6}$$

The process of finding derivatives is called *differentiation.* By taking derivatives of $y' = dy/dx = f'(x)$ we can find second, third and higher order derivatives, denoted by $y'' = d^2y/dx^2 = f''(x)$, $y''' = d^3y/dx^3 = f'''(x)$, etc.

Geometrically the derivative of a function $f(x)$ at a point represents the *slope of the tangent line* drawn to the curve $y = f(x)$ at the point.

If a function has a derivative at a point, then it is continuous at the point. However, the converse is not necessarily true.

DIFFERENTIATION FORMULAS

In the following u, v represent functions of x while a, c, p represent constants. We assume of course that the derivatives of u and v exist, i.e. u and v are *differentiable.*

1. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
2. $\frac{d}{dx}(cu) = c\frac{du}{dx}$
3. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$
5. $\frac{d}{dx} u^p = pu^{p-1}\frac{du}{dx}$
6. $\frac{d}{dx}(a^u) = a^u \ln a$

7. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$

8. $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$

9. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$

10. $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$

11. $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$

12. $\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$

13. $\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$

14. $\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}$

15. $\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

16. $\frac{d}{dx} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

17. $\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$

18. $\frac{d}{dx} \cot^{-1} u = \frac{-1}{1+u^2} \frac{du}{dx}$

19. $\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}$

20. $\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}$

In the special case where $u = x$, the above formulas are simplified since in such case $du/dx = 1$.

INTEGRALS

If $dy/dx = f(x)$, then we call y an *indefinite integral* or *anti-derivative* of $f(x)$ and denote it by

$$\int f(x)\,dx \tag{7}$$

Since the derivative of a constant is zero, all indefinite integrals of $f(x)$ can differ only by a constant.

The *definite integral* of $f(x)$ between $x = a$ and $x = b$ is defined as

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)] \tag{8}$$

provided this limit exists. Geometrically if $f(x) \geqq 0$, this represents the area under the curve $y = f(x)$ bounded by the x axis and the ordinates at $x = a$ and $x = b$. The integral will exist if $f(x)$ is continuous in $a \leqq x \leqq b$.

Definite and indefinite integrals are related by the following theorem.

Theorem 1-1 **[Fundamental Theorem of Calculus].** If $f(x) = \frac{d}{dx} g(x)$, then

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx} g(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$$

Example 4. $\int_1^2 x^2\,dx = \int_1^2 \frac{d}{dx}\left(\frac{x^3}{3}\right) dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$

The process of finding integrals is called *integration*.

INTEGRATION FORMULAS

In the following u, v represent functions of x while a, b, c, p represent constants. In all cases we omit the constant of integration, which nevertheless is implied.

1. $\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$

2. $\int cu\,dx = c \int u\,dx$

3. $\int u\left(\frac{dv}{dx}\right) dx = uv - \int v\left(\frac{du}{dx}\right) dx$ or $\int u\,dv = uv - \int v\,du$

This is called *integration by parts*.

4. $\int F[u(x)]\,dx = \int F(w)\frac{dw}{w'}$ where $w = u(x)$ and $w' = dw/dx$ expressed as a function of w. This is called *integration by substitution* or *transformation.*

5. $\int u^p\,du = \frac{u^{p+1}}{p+1},\ p \neq -1$

6. $\int u^{-1}\,du = \int \frac{du}{u} = \ln u$

7. $\int a^u\,du = \frac{a^u}{\ln a},\ a \neq 0, 1$

8. $\int e^u\,du = e^u$

9. $\int \sin u\,du = -\cos u$

10. $\int \cos u\,du = \sin u$

11. $\int \tan u\,du = -\ln \cos u$

12. $\int \cot u\,du = \ln \sin u$

13. $\int \sec u\,du = \ln(\sec u + \tan u)$

14. $\int \csc u\,du = \ln(\csc u - \cot u)$

15. $\int e^{au}\sin bu\,du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$

16. $\int e^{au}\cos bu\,du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$

17. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$

18. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

19. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2})$

20. $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2})$

21. $\int \sinh u\,du = \cosh u$

22. $\int \cosh u\,du = \sinh u$

SEQUENCES AND SERIES

A *sequence,* indicated by $u_1, u_2, \ldots$ or briefly by $\langle u_n \rangle$, is a function defined on the set of natural numbers. The sequence is said to have the *limit* l or to *converge* to l, if given any $\epsilon > 0$ there exists a number $N > 0$ such that $|u_n - l| < \epsilon$ for all $n > N$, and in such case we write $\lim_{n\to\infty} u_n = l$. If the sequence does not converge we say that it *diverges.*

Consider the sequence $u_1,\ u_1 + u_2,\ u_1 + u_2 + u_3, \ldots$ or $S_1, S_2, S_3, \ldots$ where $S_n = u_1 + u_2 + \cdots + u_n$. We call $\langle S_n \rangle$ the sequence of *partial sums* of the sequence $\langle u_n \rangle$. The symbol

$$u_1 + u_2 + u_3 + \cdots \quad \text{or} \quad \sum_{n=1}^{\infty} u_n \quad \text{or briefly} \quad \sum u_n \tag{9}$$

is defined as synonymous with $\langle S_n \rangle$ and is called an *infinite series.* This series will converge or diverge according as $\langle S_n \rangle$ converges or diverges. If it converges to S we call S the *sum* of the series.

The following are some important theorems concerning infinite series.

Theorem 1-2. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 1-3. If $\Sigma |u_n|$ converges and $|v_n| \leq |u_n|$, then $\Sigma |v_n|$ converges.

Theorem 1-4. If $\Sigma |u_n|$ converges, then Σu_n converges.

In such case we say that Σu_n *converges absolutely* or is *absolutely convergent.* A property of such series is that the terms can be rearranged without affecting the sum.

Theorem 1-5. If $\Sigma |u_n|$ diverges and $|v_n| \geq |u_n|$, then $\Sigma |v_n|$ and Σv_n both diverge.

Theorem 1-6. The series $\Sigma|u_n|$, where $|u_n| = f(n) \geqq 0$, converges or diverges according as $\int_1^\infty f(x)\,dx = \lim_{M\to\infty} \int_1^M f(x)\,dx$ exists or does not exist.

This theorem is often called the *integral test*.

Theorem 1-7. The series $\Sigma|u_n|$ diverges if $\lim_{n\to\infty} |u_n| \neq 0$. However, if $\lim_{n\to\infty} |u_n| = 0$ the series may or may not converge [see Problem 1.31].

Theorem 1-8. Suppose that $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = r$. Then the series Σu_n converges (absolutely) if $r < 1$ and diverges if $r > 1$. If $r = 1$, no conclusion can be drawn.

This theorem is often referred to as the *ratio test*.

The above ideas can be extended to the case where the u_n are functions of x denoted by $u_n(x)$. In such case the sequences or series will converge or diverge according to the particular values of x. The set of values of x for which a sequence or series converges is called the *region of convergence*, denoted by $\mathcal{R}$.

Example 5. The series $1 + x + x^2 + x^3 + \cdots$ has a region of convergence $\mathcal{R}$ [in this case an interval] given by $-1 < x < 1$ if we restrict ourselves to real values of x.

UNIFORM CONVERGENCE

We can say that the series $u_1(x) + u_2(x) + \cdots$ converges to the sum $S(x)$ in a region $\mathcal{R}$, if given $\epsilon > 0$ there exists a number N, which in general depends on both ϵ and x, such that $|S(x) - S_n(x)| < \epsilon$ whenever $n > N$ where $S_n(x) = u_1(x) + \cdots + u_n(x)$. If we can find N depending only on ϵ and not on x, we say that the series converges *uniformly* to $S(x)$ in $\mathcal{R}$. Uniformly convergent series have many important advantages as indicated in the following theorems.

Theorem 1-9. If $u_n(x)$, $n = 1, 2, 3, \ldots$ are continuous in $a \leqq x \leqq b$ and $\Sigma u_n(x)$ is uniformly convergent to $S(x)$ in $a \leqq x \leqq b$, then $S(x)$ is continuous in $a \leqq x \leqq b$.

Theorem 1-10. If $\Sigma u_n(x)$ converges uniformly to $S(x)$ in $a \leqq x \leqq b$ and $u_n(x)$, $n = 1, 2, 3, \ldots$ are integrable in $a \leqq x \leqq b$, then

$$\int_a^b S(x)\,dx = \int_a^b \{u_1(x) + u_2(x) + \cdots\}\,dx = \int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots$$

Theorem 1-11. If $u_n(x)$, $n = 1, 2, 3, \ldots$ are continuous and have continuous derivatives in $a \leqq x \leqq b$ and if $\Sigma u_n(x)$ converges to $S(x)$ while $\Sigma u_n'(x)$ is uniformly convergent in $a \leqq x \leqq b$, then

$$S'(x) = \frac{d}{dx}\{u_1(x) + u_2(x) + \cdots\} = u_1'(x) + u_2'(x) + \cdots$$

An important test for uniform convergence, often called the *Weierstrass M test*, is given by the following.

Theorem 1-12. If there is a set of positive constants M_n, $n = 1, 2, 3, \ldots$ such that $|u_n(x)| \leqq M_n$ in $\mathcal{R}$ and ΣM_n converges, then $\Sigma u_n(x)$ is uniformly convergent [and also absolutely convergent] in $\mathcal{R}$.

TAYLOR SERIES

The *Taylor series* for $f(x)$ about $x = a$ is defined as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n \qquad (10)$$

where
$$R_n = \frac{f^{(n)}(x_0)(x-a)^n}{n!}, \qquad x_0 \text{ between } a \text{ and } x \qquad (11)$$

is called the *remainder* and where it is supposed that $f(x)$ has derivatives of order n at least. The case where $n = 1$ is often called the *law of the mean* or *mean-value theorem* and can be written as

$$\frac{f(x) - f(a)}{x - a} = f'(x_0) \qquad x_0 \text{ between } a \text{ and } x \qquad (12)$$

The infinite series corresponding to (*10*), also called the *formal Taylor series* for $f(x)$, will converge in some interval if $\lim_{n\to\infty} R_n = 0$ in this interval. Some important Taylor series together with their intervals of convergence are as follows.

1. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots \qquad -\infty < x < \infty$

2. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$

3. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$

4. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots \qquad -1 < x \leqq 1$

5. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots \qquad -1 \leqq x \leqq 1$

A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ is often called a *power series*. Such power series are uniformly convergent in any interval which lies entirely within the interval of convergence [see Problem 1.120].

FUNCTIONS OF TWO OR MORE VARIABLES

The concept of function of one variable given on page 2 can be extended to functions of two or more variables. Thus for example $z = f(x,y)$ defines a function f which assigns to the number pair (x,y) the number z.

Example 6. If $f(x,y) = x^2 + 3xy + 2y^2$, then $f(-1,2) = (-1)^2 + 3(-1)(2) + 2(2)^2 = 3$.

The student is familiar with graphing $z = f(x,y)$ in a 3-dimensional xyz coordinate system to obtain a *surface*. We sometimes call x and y *independent* variables and z a *dependent variable*. Occasionally we write $z = z(x,y)$ rather than $z = f(x,y)$, using the symbol z in two different senses. However, no confusion should result.

The ideas of limits and continuity for functions of two or more variables pattern closely those for one variable.

PARTIAL DERIVATIVES

The *partial derivatives* of $f(x,y)$ with respect to x and y are defined by

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \qquad \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k} \qquad (13)$$

if these limits exist. We often write $h = \Delta x$, $k = \Delta y$. Note that $\partial f/\partial x$ is simply the ordinary derivative of f with respect to x *keeping y constant*, while $\partial f/\partial y$ is the ordinary derivative of f with respect to y *keeping x constant*. Thus the usual differentiation formulas on pages 4 and 5 apply.

Example 7. If $f(x,y) = 3x^2 - 4xy + 2y^2$ then $\dfrac{\partial f}{\partial x} = 6x - 4y$, $\dfrac{\partial f}{\partial y} = -4x + 4y$.

Higher derivatives are defined similarly. For example, we have the second order derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \tag{14}$$

The derivatives in *(13)* are sometimes denoted by f_x and f_y. In such case $f_x(a,b)$, $f_y(a,b)$ denote these partial derivatives evaluated at (a,b). Similarly the derivatives in *(14)* are denoted by $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ respectively. The second and third results in *(14)* will be the same if f has continuous partial derivatives of second order at least.

The *differential* of $f(x,y)$ is defined as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{15}$$

where $h = \Delta x = dx$, $k = \Delta y = dy$.

Generalizations of these results are easily made.

TAYLOR SERIES FOR FUNCTIONS OF TWO OR MORE VARIABLES

The ideas involved in Taylor series for functions of one variable can be generalized. For example, the Taylor series for $f(x,y)$ about $x = a$, $y = b$ can be written

$$\begin{aligned} f(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2!}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2] + \cdots \end{aligned} \tag{16}$$

LINEAR EQUATIONS AND DETERMINANTS

Consider the system of linear equations

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \tag{17}$$

These represent two lines in the xy plane, and in general will meet in a point whose coordinates (x,y) are found by solving *(17)* simultaneously. We find

$$x = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \qquad y = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2} \tag{18}$$

It is convenient to write these in *determinant form* as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{19}$$

where we define a *determinant of the second order* or *order 2* to be

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{20}$$