

*Advanced
Mathematics
for Engineers & Scientists*

MURRAY R. SPIEGEL

INCLUDING 950 SOLVED PROBLEMS

SCHAUM'S OUTLINE OF
THEORY AND PROBLEMS
OF
ADVANCED MATHEMATICS
for
Engineers and Scientists

•

BY
MURRAY R. SPIEGEL, Ph.D.
Professor of Mathematics
Rensselaer Polytechnic Institute

•

SCHAUM'S OUTLINE SERIES

McGRAW-HILL BOOK COMPANY

*New York, St. Louis, San Francisco, Düsseldorf, Johannesburg, Kuala Lumpur, London, Mexico,
Montreal, New Delhi, Panama, Rio de Janeiro, Singapore, Sydney, and Toronto*

Copyright © 1971 by McGraw-Hill, Inc. All Rights Reserved. Printed in the United States of America. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher.

07-060216-6

1 2 3 4 5 6 7 8 9 0 SH SH 7 5 4 3 2 1

Preface

In recent years the number of topics in mathematics required of engineers and scientists has greatly increased. This is to be expected since mathematics plays a vital role as a language in the formulation and solution of problems involving science and engineering and as these problems become more complex it is natural that the mathematical methods needed for their solution should increase in number and complexity.

It is the purpose of this book to provide important advanced mathematical concepts and methods needed by engineers and scientists as well as mathematicians who are interested in the applications of their field. The book has been designed as a supplement to all current standard textbooks or as a textbook for a formal course in the mathematical methods of engineering and science.

Each chapter begins with a clear statement of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems and derivations of basic results are included among the solved problems. The large number of supplementary problems serve as a review and possible extension of the material of each chapter.

Topics covered include ordinary differential equations, Laplace transforms, vector analysis, Fourier series, Fourier integrals, gamma, beta and other special functions, Bessel functions, Legendre and other orthogonal functions, partial differential equations, complex variables and conformal mapping, matrices and calculus of variations. The first chapter which provides a review of fundamental concepts of algebra, trigonometry, analytic geometry and calculus may either be read at the beginning or referred to as needed depending on the background of the student.

Considerably more material has been included here than can be covered in most courses. This has been done to make the book more flexible, to provide a more useful book of reference and to stimulate further interest in the topics.

I wish to take this opportunity to thank Daniel Schaum, Nicola Monti and Hank Hayden for their splendid cooperation.

M. R. SPIEGEL

Rensselaer Polytechnic Institute
January, 1971

SCHAUM'S OUTLINE SERIES

COLLEGE PHYSICS

including 625 SOLVED PROBLEMS
 Edited by CAREL W. van der MERWE, Ph.D.,
Professor of Physics, New York University

COLLEGE CHEMISTRY

including 385 SOLVED PROBLEMS
 Edited by JEROME L. ROSENBERG, Ph.D.,
Professor of Chemistry, University of Pittsburgh

GENETICS

including 500 SOLVED PROBLEMS
 By WILLIAM D. STANSFIELD, Ph.D.,
Dept. of Biological Sciences, Calif. State Polytech.

MATHEMATICAL HANDBOOK

including 2400 FORMULAS and 60 TABLES
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

First Yr. COLLEGE MATHEMATICS

including 1850 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

COLLEGE ALGEBRA

including 1940 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

TRIGONOMETRY

including 680 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

MATHEMATICS OF FINANCE

including 500 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

PROBABILITY

including 500 SOLVED PROBLEMS
 By SEYMOUR LIPSCHUTZ, Ph.D.,
Assoc. Prof. of Math., Temple University

STATISTICS

including 875 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

ANALYTIC GEOMETRY

including 345 SOLVED PROBLEMS
 By JOSEPH H. KINDLE, Ph.D.,
Professor of Mathematics, University of Cincinnati

DIFFERENTIAL GEOMETRY

including 500 SOLVED PROBLEMS
 By MARTIN LIPSCHUTZ, Ph.D.,
Professor of Mathematics, University of Bridgeport

CALCULUS

including 1175 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

DIFFERENTIAL EQUATIONS

including 560 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

SET THEORY and Related Topics

including 530 SOLVED PROBLEMS
 By SEYMOUR LIPSCHUTZ, Ph.D.,
Assoc. Prof. of Math., Temple University

FINITE MATHEMATICS

including 750 SOLVED PROBLEMS
 By SEYMOUR LIPSCHUTZ, Ph.D.,
Assoc. Prof. of Math., Temple University

MODERN ALGEBRA

including 425 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

LINEAR ALGEBRA

including 600 SOLVED PROBLEMS
 By SEYMOUR LIPSCHUTZ, Ph.D.,
Assoc. Prof. of Math., Temple University

MATRICES

including 340 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

PROJECTIVE GEOMETRY

including 200 SOLVED PROBLEMS
 By FRANK AYRES, Jr., Ph.D.,
Professor of Mathematics, Dickinson College

GENERAL TOPOLOGY

including 650 SOLVED PROBLEMS
 By SEYMOUR LIPSCHUTZ, Ph.D.,
Assoc. Prof. of Math., Temple University

GROUP THEORY

including 600 SOLVED PROBLEMS
 By B. BAUMSLAG, B. CHANDLER, Ph.D.,
Mathematics Dept., New York University

VECTOR ANALYSIS

including 480 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

ADVANCED CALCULUS

including 925 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

COMPLEX VARIABLES

including 640 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

LAPLACE TRANSFORMS

including 450 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

NUMERICAL ANALYSIS

including 775 SOLVED PROBLEMS
 By FRANCIS SCHEID, Ph.D.,
Professor of Mathematics, Boston University

DESCRIPTIVE GEOMETRY

including 175 SOLVED PROBLEMS
 By MINOR C. HAWK, Head of
Engineering Graphics Dept., Carnegie Inst. of Tech.

ENGINEERING MECHANICS

including 460 SOLVED PROBLEMS
 By W. G. McLEAN, B.S. in E.E., M.S.,
Professor of Mechanics, Lafayette College
 and E. W. NELSON, B.S. in M.E., M. Adm. E.,
Engineering Supervisor, Western Electric Co.

THEORETICAL MECHANICS

including 720 SOLVED PROBLEMS
 By MURRAY R. SPIEGEL, Ph.D.,
Professor of Math., Rensselaer Polytech. Inst.

LAGRANGIAN DYNAMICS

including 275 SOLVED PROBLEMS
 By D. A. WELLS, Ph.D.,
Professor of Physics, University of Cincinnati

STRENGTH OF MATERIALS

including 430 SOLVED PROBLEMS
 By WILLIAM A. NASH, Ph.D.,
Professor of Eng. Mechanics, University of Florida

FLUID MECHANICS and HYDRAULICS

including 475 SOLVED PROBLEMS
 By RANALD V. GILES, B.S., M.S. in C.E.,
Prof. of Civil Engineering, Drexel Inst. of Tech.

FLUID DYNAMICS

including 100 SOLVED PROBLEMS
 By WILLIAM F. HUGHES, Ph.D.,
Professor of Mech. Eng., Carnegie Inst. of Tech.
 and JOHN A. BRIGHTON, Ph.D.,
Asst. Prof. of Elec. Eng., Pennsylvania State U.

ELECTRIC CIRCUITS

including 350 SOLVED PROBLEMS
 By JOSEPH A. EDMINISTER, M.S.E.E.,
Assoc. Prof. of Elec. Eng., University of Akron

ELECTRONIC CIRCUITS

including 160 SOLVED PROBLEMS
 By EDWIN C. LOWENBERG, Ph.D.,
Professor of Elec. Eng., University of Nebraska

FEEDBACK & CONTROL SYSTEMS

including 680 SOLVED PROBLEMS
 By J. J. DiSTEFANO III, A. R. STUBBERUD,
 and I. J. WILLIAMS, Ph.D.,
Engineering Dept., University of Calif., at L.A.

TRANSMISSION LINES

including 165 SOLVED PROBLEMS
 By R. A. CHIPMAN, Ph.D.,
Professor of Electrical Eng., University of Toledo

REINFORCED CONCRETE DESIGN

including 200 SOLVED PROBLEMS
 By N. J. EVERARD, MSCE, Ph.D.,
Prof. of Eng. Mech. & Struc., Arlington State Col.
 and J. L. TANNER III, MSCE,
Technical Consultant, Texas Industries Inc.

MECHANICAL VIBRATIONS

including 225 SOLVED PROBLEMS
 By WILLIAM W. SETO, B.S. in M.E., M.S.,
Assoc. Prof. of Mech. Eng., San Jose State College

MACHINE DESIGN

including 320 SOLVED PROBLEMS
 By HALL, HOLOWENKO, LAUGHLIN
Professors of Mechanical Eng., Purdue University

BASIC ENGINEERING EQUATIONS

including 1400 BASIC EQUATIONS
 By W. F. HUGHES, E. W. GAYLORD, Ph.D.,
Professors of Mech. Eng., Carnegie Inst. of Tech.

ELEMENTARY ALGEBRA

including 2700 SOLVED PROBLEMS
 By BARNETT RICH, Ph.D.,
Head of Math. Dept., Brooklyn Tech. H.S.

PLANE GEOMETRY

including 850 SOLVED PROBLEMS
 By BARNETT RICH, Ph.D.,
Head of Math. Dept., Brooklyn Tech. H.S.

TEST ITEMS IN EDUCATION

including 3100 TEST ITEMS
 By G. J. MOULY, Ph.D., L. E. WALTON, Ph.D.,
Professors of Education, University of Miami

CONTENTS

	Page
Chapter 1 REVIEW OF FUNDAMENTAL CONCEPTS	1
Real numbers. Rules of algebra. Functions. Special types of functions. Limits. Continuity. Derivatives. Differentiation formulas. Integrals. Integration formulas. Sequences and series. Uniform convergence. Taylor series. Functions of two or more variables. Partial derivatives. Taylor series for functions of two or more variables. Linear equations and determinants. Maxima and minima. Method of Lagrange multipliers. Leibnitz's rule for differentiating an integral. Multiple integrals. Complex numbers.	
<hr/>	
Chapter 2 ORDINARY DIFFERENTIAL EQUATIONS	38
Definition of a differential equation. Order of a differential equation. Arbitrary constants. Solution of a differential equation. Differential equation of a family of curves. Special first order equations and solutions. Equations of higher order. Existence and uniqueness of solutions. Applications of differential equations. Some special applications. Mechanics. Electric circuits. Orthogonal trajectories. Deflection of beams. Miscellaneous problems. Numerical methods for solving differential equations.	
<hr/>	
Chapter 3 LINEAR DIFFERENTIAL EQUATIONS	71
General linear differential equation of order n . Existence and uniqueness theorem. Operator notation. Linear operators. Fundamental theorem on linear differential equations. Linear dependence and Wronskians. Solutions of linear equations with constant coefficients. Non-operator techniques. The complementary or homogeneous solution. The particular solution. Method of undetermined coefficients. Method of variation of parameters. Operator techniques. Method of reduction of order. Method of inverse operators. Linear equations with variable coefficients. Simultaneous differential equations. Applications.	
<hr/>	
Chapter 4 LAPLACE TRANSFORMS	98
Definition of a Laplace transform. Laplace transforms of some elementary functions. Sufficient conditions for existence of Laplace transforms. Inverse Laplace transforms. Laplace transforms of derivatives. The unit step function. Some special theorems on Laplace transforms. Partial fractions. Solutions of differential equations by Laplace transforms. Applications to physical problems. Laplace inversion formulas.	
<hr/>	
Chapter 5 VECTOR ANALYSIS	121
Vectors and scalars. Vector algebra. Laws of vector algebra. Unit vectors. Rectangular unit vectors. Components of a vector. Dot or scalar product. Cross or vector product. Triple products. Vector functions. Limits, continuity and derivatives of vector functions. Geometric interpretation of a vector derivative. Gradient, divergence and curl. Formulas involving ∇ . Orthogonal curvilinear coordinates. Jacobians. Gradient, divergence, curl and Laplacian in orthogonal curvilinear. Special curvilinear coordinates.	

CONTENTS

	Page
Chapter 6 MULTIPLE, LINE AND SURFACE INTEGRALS AND INTEGRAL THEOREMS	147
Double integrals. Iterated integrals. Triple integrals. Transformations of multiple integrals. Line integrals. Vector notation for line integrals. Evaluation of line integrals. Properties of line integrals. Simple closed curves. Simply and multiply-connected regions. Green's theorem in the plane. Conditions for a line integral to be independent of the path. Surface integrals. The divergence theorem. Stokes' theorem.	
<hr/>	
Chapter 7 FOURIER SERIES	182
Periodic functions. Fourier series. Dirichlet conditions. Odd and even functions. Half range Fourier sine or cosine series. Parseval's identity. Differentiation and integration of Fourier series. Complex notation for Fourier series. Complex notation for Fourier series. Orthogonal functions.	
<hr/>	
Chapter 8 FOURIER INTEGRALS	201
The Fourier integral. Equivalent forms of Fourier's integral theorem. Fourier transforms. Parseval's identities for Fourier integrals. The convolution theorem.	
<hr/>	
Chapter 9 GAMMA, BETA AND OTHER SPECIAL FUNCTIONS	210
The gamma function. Table of values and graph of the gamma function. Asymptotic formula for $\Gamma(n)$. Miscellaneous results involving the gamma function. The beta function. Dirichlet integrals. Other special functions. Error function. Exponential integral. Sine integral. Cosine integral. Fresnel sine integral. Fresnel cosine integral. Asymptotic series or expansions.	
<hr/>	
Chapter 10 BESSEL FUNCTIONS	224
Bessel's differential equation. Bessel functions of the first kind. Bessel functions of the second kind. Generating function for $J_n(x)$. Recurrence formulas. Functions related to Bessel functions. Hankel functions of first and second kinds. Modified Bessel functions. Ber, bei, ker, kei functions. Equations transformed into Bessel's equation. Asymptotic formulas for Bessel functions. Zeros of Bessel functions. Orthogonality of Bessel functions. Series of Bessel functions.	
<hr/>	
Chapter 11 LEGENDRE FUNCTIONS AND OTHER ORTHOGONAL FUNCTIONS	242
Legendre's differential equation. Legendre polynomials. Generating function for Legendre polynomials. Recurrence formulas. Legendre functions of the second kind. Orthogonality of Legendre polynomials. Series of Legendre polynomials. Associated Legendre functions. Other special functions. Hermite polynomials. Laguerre polynomials. Sturm-Liouville systems.	
<hr/>	
Chapter 12 PARTIAL DIFFERENTIAL EQUATIONS	258
Some definitions involving partial differential equations. Linear partial differential equations. Some important partial differential equations. Heat conduction equation. Vibrating string equation. Laplace's equation. Longitudinal vibrations of a beam. Transverse vibrations of a beam. Methods of solving boundary-value problems. General solutions. Separation of variables. Laplace transform methods.	

CONTENTS

	Page
Chapter 13 COMPLEX VARIABLES AND CONFORMAL MAPPING	286
Functions. Limits and continuity. Derivatives. Cauchy-Riemann equations. Integrals. Cauchy's theorem. Cauchy's integral formulas. Taylor's series. Singular points. Poles. Laurent's series. Residues. Residue theorem. Evaluation of definite integrals. Conformal mapping. Riemann's mapping theorem. Some general transformations. Mapping of a half plane on to a circle. The Schwarz-Christoffel transformation. Solutions of Laplace's equation by conformal mapping.	
<hr/>	
Chapter 14 COMPLEX INVERSION FORMULA FOR LAPLACE TRANSFORMS	324
The complex inversion formula. The Bromwich contour. Use of residue theorem in finding inverse Laplace transforms. A sufficient condition for the integral around Γ to approach zero. Modification of Bromwich contour in case of branch points. Case of infinitely many singularities. Applications to boundary-value problems.	
<hr/>	
Chapter 15 MATRICES	342
Definition of a matrix. Some special definitions and operations involving matrices. Determinants. Theorems on determinants. Inverse of a matrix. Orthogonal and unitary matrices. Orthogonal vectors. Systems of linear equations. Systems of n equations in n unknowns. Cramer's rule. Eigenvalues and eigenvectors. Theorems on eigenvalues and eigenvectors.	
<hr/>	
Chapter 16 CALCULUS OF VARIATIONS	375
Maximum or minimum of an integral. Euler's equation. Constraints. The variational notation. Generalizations. Hamilton's principle. Lagrange's equations. Sturm-Liouville systems and Rayleigh-Ritz methods. Operator interpretation of matrices.	
<hr/>	
INDEX	399

Chapter 1

Review of Fundamental Concepts

REAL NUMBERS

At the very foundations of mathematics is the concept of a *set* or *collection* of objects and, in particular, sets of *numbers* on which we base our quantitative work in science and engineering. The student is already familiar with the following important sets of numbers.

1. **Natural Numbers** 1, 2, 3, 4, ... or *positive integers* used in counting.
2. **Integers** 0, ± 1 , ± 2 , ± 3 , ... These numbers arose in order to provide meaning to *subtraction* [inverse of *addition*] of any two natural numbers. Thus $2 - 6 = -4$, $8 - 8 = 0$, etc.
3. **Rational Numbers** such as $2/3$, $-10/7$, etc. arose in order to provide meaning to *division* [inverse of *multiplication*] or *quotient* of any two integers with the exception that division by zero is not defined.
4. **Irrational Numbers** such as $\sqrt{2}$, π , etc. are numbers which cannot be expressed as the quotient of two integers.

Note that the set of natural numbers is a *subset*, i.e. a part, of the set of integers which in turn is a subset of the set of rational numbers.

The set of numbers which are either rational or irrational is called the set of *real numbers* [to distinguish them from *imaginary* or *complex numbers* on page 11] and is composed of *positive* and *negative numbers* and *zero*. The real numbers can be represented as *points* on a line as indicated in Fig. 1-1. For this reason we often use *point* and *number* interchangeably.

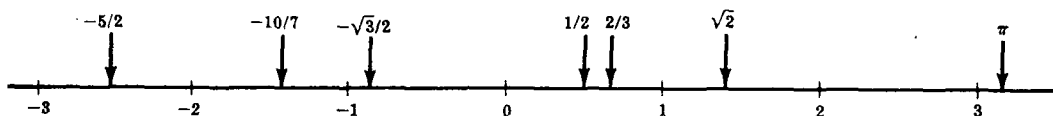


Fig. 1-1

The student is also familiar with the concept of *inequality*. Thus we say that the real number a is *greater than* or *less than* b [symbolized by $a > b$ or $a < b$] if $a - b$ is a positive or negative number respectively. For any real numbers a and b we must have $a > b$, $a = b$ or $a < b$.

RULES OF ALGEBRA

If a , b , c are any real numbers, the following rules of algebra hold.

1. $a + b = b + a$ Commutative law for addition
2. $a + (b + c) = (a + b) + c$ Associative law for addition
3. $ab = ba$ Commutative law for multiplication
4. $a(bc) = (ab)c$ Associative law for multiplication
5. $a(b + c) = ab + ac$ Distributive law

It is from these rules [if we accept them as *axioms* or *postulates*] that we can prove the usual rules of signs, as for example $(-5)(3) = -15$, $(-2)(-3) = 6$, etc.

The student is also familiar with the usual rules of *exponents*:

$$a^m \cdot a^n = a^{m+n}, \quad a^m/a^n = a^{m-n}, \quad a \neq 0, \quad (a^m)^n = a^{mn} \quad (1)$$

FUNCTIONS

Another important concept is that of *function*. The student will recall that a function f is a rule which assigns to each object x , also called *member* or *element*, of a set A an element y of a set B . To indicate this correspondence we write $y = f(x)$ where $f(x)$ is called the *value* of the function at x .

Example 1. If $f(x) = x^2 - 3x + 2$, then $f(2) = 2^2 - 3(2) + 2 = 0$.

The student is also familiar with the process of "graphing functions" by obtaining number pairs (x, y) and considering these as points plotted on an xy coordinate system. In general $y = f(x)$ is represented graphically by a *curve*. Because y is usually determined from x , we sometimes call x the *independent variable* and y the *dependent variable*.

SPECIAL TYPES OF FUNCTIONS

- Polynomials** $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$. If $a_0 \neq 0$, n is called the *degree* of the polynomial. The polynomial equation $f(x) = 0$ has exactly n roots provided we count repetitions. For example $x^3 - 3x^2 + 3x - 1 = 0$ can be written $(x-1)^3 = 0$ so that the 3 roots are 1, 1, 1. Note that here we have used the *binomial theorem*

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + x^n \quad (2)$$

where the *binomial coefficients* are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (3)$$

and where *factorial* n , i.e. $n!$, $= n(n-1)(n-2) \dots 1$ while $0! = 1$ by definition.

- Exponential Functions** $f(x) = a^x$. These functions obey the rules (1).

An important special case occurs where $a = e = 2.7182818 \dots$.

- Logarithmic Functions** $f(x) = \log_a x$. These functions are *inverses* of the exponential functions, i.e. if $a^x = y$ then $x = \log_a y$ where a is called the *base* of the logarithm. Interchanging x and y gives $y = \log_a x$. If $a = e$, which is often called the *natural base* of logarithms, we denote $\log_e x$ by $\ln x$, called *the natural logarithm* of x . The fundamental rules satisfied by natural logarithms [or logarithms to any base] are

$$\ln(mn) = \ln m + \ln n, \quad \ln \frac{m}{n} = \ln m - \ln n, \quad \ln m^p = p \ln m \quad (4)$$

- Trigonometric Functions** $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$.

Some fundamental relationships among these functions are as follows.

$$(a) \quad \sin x = \cos\left(\frac{\pi}{2} - x\right), \quad \cos x = \sin\left(\frac{\pi}{2} - x\right), \quad \tan x = \frac{\sin x}{\cos x},$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

$$(b) \sin^2 x + \cos^2 x = 1, \quad \sec^2 x - \tan^2 x = 1, \quad \csc^2 x - \cot^2 x = 1$$

$$(c) \sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$$

$$(d) \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$(e) A \cos x + B \sin x = \sqrt{A^2 + B^2} \sin(x + \alpha) \quad \text{where } \tan \alpha = A/B$$

The trigonometric functions are *periodic*. For example $\sin x$ and $\cos x$, shown in Fig. 1-2 and 1-3 respectively, have period 2π .

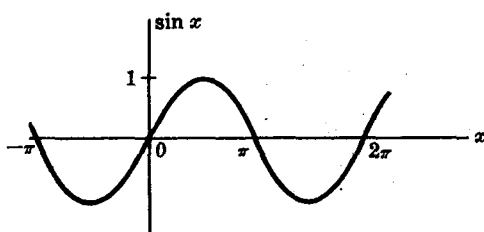


Fig. 1-2

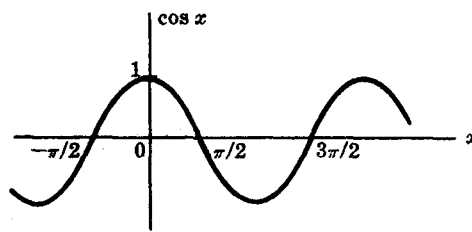


Fig. 1-3

5. **Inverse Trigonometric Functions** $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\csc^{-1}x$. These are *inverses* of the trigonometric functions. For example if $\sin x = y$ then $x = \sin^{-1}y$, or on interchanging x and y , $y = \sin^{-1}x$.

6. **Hyperbolic Functions.** These are defined in terms of exponential functions as follows.

$$(a) \sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Some fundamental identities analogous to those for trigonometric functions are

$$(b) \cosh^2 x - \sinh^2 x = 1, \quad \operatorname{sech}^2 x + \tanh^2 x = 1, \quad \coth^2 x - \operatorname{csch}^2 x = 1$$

$$(c) \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

The inverse hyperbolic functions, given by $\sinh^{-1}x$, $\cosh^{-1}x$, etc. can be expressed in terms of logarithms [see Problem 1.9, for example].

LIMITS

The function $f(x)$ is said to have the *limit* l as x approaches a , abbreviated $\lim_{x \rightarrow a} f(x) = l$, if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

Note that $|p|$, i.e. the *absolute value* of p , is equal to p if $p > 0$, $-p$ if $p < 0$ and 0 if $p = 0$.

$$\text{Example 2. } \lim_{x \rightarrow 1} (x^2 - 4x + 8) = 5, \quad \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4, \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

If $\lim_{x \rightarrow a} f_1(x) = l_1$, $\lim_{x \rightarrow a} f_2(x) = l_2$ then we have the following *theorems on limits*.

$$(a) \quad \lim_{x \rightarrow a} [f_1(x) \pm f_2(x)] = \lim_{x \rightarrow a} f_1(x) \pm \lim_{x \rightarrow a} f_2(x) = l_1 \pm l_2$$

$$(b) \quad \lim_{x \rightarrow a} [f_1(x) f_2(x)] = \left[\lim_{x \rightarrow a} f_1(x) \right] \left[\lim_{x \rightarrow a} f_2(x) \right] = l_1 l_2$$

$$(c) \quad \lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x \rightarrow a} f_1(x)}{\lim_{x \rightarrow a} f_2(x)} = \frac{l_1}{l_2} \quad \text{if } l_2 \neq 0$$

CONTINUITY

The function $f(x)$ is said to be *continuous* at a if $\lim_{x \rightarrow a} f(x) = f(a)$.

Example 3. $f(x) = x^2 - 4x + 8$ is continuous at $x = 1$. However, if $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x \neq 2 \\ 6 & x = 2 \end{cases}$ then $f(x)$ is not continuous [or is *discontinuous*] at $x = 2$ and $x = 2$ is called a *discontinuity* of $f(x)$.

If $f(x)$ is continuous at each point of an interval such as $x_1 \leq x \leq x_2$ or $x_1 < x \leq x_2$, etc., it is said to be *continuous in the interval*.

If $f_1(x)$ and $f_2(x)$ are continuous in an interval then $f_1(x) \pm f_2(x)$, $f_1(x) f_2(x)$ and $f_1(x)/f_2(x)$ where $f_2(x) \neq 0$ are also continuous in the interval.

DERIVATIVES

The *derivative* of $y = f(x)$ at a point x is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad (5)$$

where $h = \Delta x$, $\Delta y = f(x+h) - f(x) = f(x+\Delta x) - f(x)$ provided the limit exists.

The *differential* of $y = f(x)$ is defined by

$$dy = f'(x) dx \quad \text{where } dx = \Delta x \quad (6)$$

The process of finding derivatives is called *differentiation*. By taking derivatives of $y' = dy/dx = f'(x)$ we can find second, third and higher order derivatives, denoted by $y'' = d^2y/dx^2 = f''(x)$, $y''' = d^3y/dx^3 = f'''(x)$, etc.

Geometrically the derivative of a function $f(x)$ at a point represents the *slope of the tangent line* drawn to the curve $y = f(x)$ at the point.

If a function has a derivative at a point, then it is continuous at the point. However, the converse is not necessarily true.

DIFFERENTIATION FORMULAS

In the following u, v represent functions of x while a, c, p represent constants. We assume of course that the derivatives of u and v exist, i.e. u and v are *differentiable*.

1. $\frac{d}{dx} (u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
2. $\frac{d}{dx} (cu) = c \frac{du}{dx}$
3. $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$
4. $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$
5. $\frac{d}{dx} u^p = pu^{p-1} \frac{du}{dx}$
6. $\frac{d}{dx} (a^u) = a^u \ln a$

- | | |
|---|--|
| 7. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$ | 14. $\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}$ |
| 8. $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$ | 15. $\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$ |
| 9. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$ | 16. $\frac{d}{dx} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$ |
| 10. $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$ | 17. $\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$ |
| 11. $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$ | 18. $\frac{d}{dx} \cot^{-1} u = \frac{-1}{1+u^2} \frac{du}{dx}$ |
| 12. $\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$ | 19. $\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}$ |
| 13. $\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$ | 20. $\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}$ |

In the special case where $u = x$, the above formulas are simplified since in such case $du/dx = 1$.

INTEGRALS

If $dy/dx = f(x)$, then we call y an *indefinite integral* or *anti-derivative* of $f(x)$ and denote it by

$$\int f(x) dx \quad (7)$$

Since the derivative of a constant is zero, all indefinite integrals of $f(x)$ can differ only by a constant.

The *definite integral* of $f(x)$ between $x = a$ and $x = b$ is defined as

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)] \quad (8)$$

provided this limit exists. Geometrically if $f(x) \geq 0$, this represents the area under the curve $y = f(x)$ bounded by the x axis and the ordinates at $x = a$ and $x = b$. The integral will exist if $f(x)$ is continuous in $a \leq x \leq b$.

Definite and indefinite integrals are related by the following theorem.

Theorem 1-1 [Fundamental Theorem of Calculus]. If $f(x) = \frac{d}{dx} g(x)$, then

$$\int_a^b f(x) dx = \int_a^b \frac{d}{dx} g(x) dx = g(x) \Big|_a^b = g(b) - g(a)$$

Example 4. $\int_1^2 x^2 dx = \int_1^2 \frac{d}{dx} \left(\frac{x^3}{3} \right) dx = \frac{x^3}{3} \Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$

The process of finding integrals is called *integration*.

INTEGRATION FORMULAS

In the following u, v represent functions of x while a, b, c, p represent constants. In all cases we omit the constant of integration, which nevertheless is implied.

- $\int (u \pm v) dx = \int u dx \pm \int v dx$
- $\int cu dx = c \int u dx$
- $\int u \left(\frac{dv}{dx} \right) dx = uv - \int v \left(\frac{du}{dx} \right) dx$ or $\int u dv = uv - \int v du$

This is called *integration by parts*.

4. $\int F[u(x)] dx = \int F(w) \frac{dw}{w'}$ where $w = u(x)$ and $w' = dw/dx$ expressed as a function of w . This is called *integration by substitution or transformation*.

$$5. \int u^p du = \frac{u^{p+1}}{p+1}, \quad p \neq -1$$

$$14. \int \csc u du = \ln(\csc u - \cot u)$$

$$6. \int u^{-1} du = \int \frac{du}{u} = \ln u$$

$$15. \int e^{au} \sin bu du = \frac{e^{au}(a \sin bu - b \cos bu)}{a^2 + b^2}$$

$$7. \int a^u du = \frac{a^u}{\ln a}, \quad a \neq 0, 1$$

$$16. \int e^{au} \cos bu du = \frac{e^{au}(a \cos bu + b \sin bu)}{a^2 + b^2}$$

$$8. \int e^u du = e^u$$

$$17. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}$$

$$9. \int \sin u du = -\cos u$$

$$18. \int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}$$

$$10. \int \cos u du = \sin u$$

$$19. \int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2})$$

$$11. \int \tan u du = -\ln \cos u$$

$$20. \int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2})$$

$$12. \int \cot u du = \ln \sin u$$

$$21. \int \sinh u du = \cosh u$$

$$13. \int \sec u du = \ln(\sec u + \tan u)$$

$$22. \int \cosh u du = \sinh u$$

SEQUENCES AND SERIES

A *sequence*, indicated by u_1, u_2, \dots or briefly by $\langle u_n \rangle$, is a function defined on the set of natural numbers. The sequence is said to have the *limit* l or to *converge* to l , if given any $\epsilon > 0$ there exists a number $N > 0$ such that $|u_n - l| < \epsilon$ for all $n > N$, and in such case we write $\lim_{n \rightarrow \infty} u_n = l$. If the sequence does not converge we say that it *diverges*.

Consider the sequence $u_1, u_1 + u_2, u_1 + u_2 + u_3, \dots$ or S_1, S_2, S_3, \dots where $S_n = u_1 + u_2 + \dots + u_n$. We call $\langle S_n \rangle$ the sequence of *partial sums* of the sequence $\langle u_n \rangle$. The symbol

$$u_1 + u_2 + u_3 + \dots \quad \text{or} \quad \sum_{n=1}^{\infty} u_n \quad \text{or briefly} \quad \sum u_n \quad (9)$$

is defined as synonymous with $\langle S_n \rangle$ and is called an *infinite series*. This series will converge or diverge according as $\langle S_n \rangle$ converges or diverges. If it converges to S we call S the *sum* of the series.

The following are some important theorems concerning infinite series.

Theorem 1-2. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 1-3. If $\sum |u_n|$ converges and $|v_n| \leq |u_n|$, then $\sum |v_n|$ converges.

Theorem 1-4. If $\sum |u_n|$ converges, then $\sum u_n$ converges.

In such case we say that $\sum u_n$ *converges absolutely* or is *absolutely convergent*. A property of such series is that the terms can be rearranged without affecting the sum.

Theorem 1-5. If $\sum |u_n|$ diverges and $|v_n| \geq |u_n|$, then $\sum |v_n|$ and $\sum v_n$ both diverge.

Theorem 1-6. The series $\sum |u_n|$, where $|u_n| = f(n) \geq 0$, converges or diverges according as $\int_1^\infty f(x) dx = \lim_{M \rightarrow \infty} \int_1^M f(x) dx$ exists or does not exist.

This theorem is often called the *integral test*.

Theorem 1-7. The series $\sum |u_n|$ diverges if $\lim_{n \rightarrow \infty} |u_n| \neq 0$. However, if $\lim_{n \rightarrow \infty} |u_n| = 0$ the series may or may not converge [see Problem 1.31].

Theorem 1-8. Suppose that $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = r$. Then the series $\sum u_n$ converges (absolutely) if $r < 1$ and diverges if $r > 1$. If $r = 1$, no conclusion can be drawn.

This theorem is often referred to as the *ratio test*.

The above ideas can be extended to the case where the u_n are functions of x denoted by $u_n(x)$. In such case the sequences or series will converge or diverge according to the particular values of x . The set of values of x for which a sequence or series converges is called the *region of convergence*, denoted by \mathcal{R} .

Example 5. The series $1 + x + x^2 + x^3 + \cdots$ has a region of convergence \mathcal{P} [in this case an interval] given by $-1 < x < 1$ if we restrict ourselves to real values of x .

UNIFORM CONVERGENCE

We can say that the series $u_1(x) + u_2(x) + \cdots$ converges to the sum $S(x)$ in a region \mathcal{R} , if given $\epsilon > 0$ there exists a number N , which in general depends on both ϵ and x , such that $|S(x) - S_n(x)| < \epsilon$ whenever $n > N$ where $S_n(x) = u_1(x) + \cdots + u_n(x)$. If we can find N depending only on ϵ and not on x , we say that the series converges *uniformly* to $S(x)$ in \mathcal{R} . Uniformly convergent series have many important advantages as indicated in the following theorems.

Theorem 1-9. If $u_n(x)$, $n = 1, 2, 3, \dots$ are continuous in $a \leq x \leq b$ and $\sum u_n(x)$ is uniformly convergent to $S(x)$ in $a \leq x \leq b$, then $S(x)$ is continuous in $a \leq x \leq b$.

Theorem 1-10. If $\sum u_n(x)$ converges uniformly to $S(x)$ in $a \leq x \leq b$ and $u_n(x)$, $n = 1, 2, 3, \dots$ are integrable in $a \leq x \leq b$, then

$$\int_a^b S(x) dx = \int_a^b \{u_1(x) + u_2(x) + \cdots\} dx = \int_a^b u_1(x) dx + \int_a^b u_2(x) dx + \cdots$$

Theorem 1-11. If $u_n(x)$, $n = 1, 2, 3, \dots$ are continuous and have continuous derivatives in $a \leq x \leq b$ and if $\sum u_n(x)$ converges to $S(x)$ while $\sum u'_n(x)$ is uniformly convergent in $a \leq x \leq b$, then

$$S'(x) = \frac{d}{dx} \{u_1(x) + u_2(x) + \cdots\} = u'_1(x) + u'_2(x) + \cdots$$

An important test for uniform convergence, often called the *Weierstrass M test*, is given by the following.

Theorem 1-12. If there is a set of positive constants M_n , $n = 1, 2, 3, \dots$ such that $|u_n(x)| \leq M_n$ in \mathcal{R} and $\sum M_n$ converges, then $\sum u_n(x)$ is uniformly convergent [and also absolutely convergent] in \mathcal{R} .

TAYLOR SERIES

The *Taylor series* for $f(x)$ about $x = a$ is defined as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n \quad (10)$$

where
$$R_n = \frac{f^{(n)}(x_0)(x-a)^n}{n!}, \quad x_0 \text{ between } a \text{ and } x \quad (11)$$

is called the *remainder* and where it is supposed that $f(x)$ has derivatives of order n at least. The case where $n = 1$ is often called the *law of the mean* or *mean-value theorem* and can be written as

$$\frac{f(x) - f(a)}{x - a} = f'(x_0) \quad x_0 \text{ between } a \text{ and } x \quad (12)$$

The infinite series corresponding to (10), also called the *formal Taylor series* for $f(x)$, will converge in some interval if $\lim_{n \rightarrow \infty} R_n = 0$ in this interval. Some important Taylor series together with their intervals of convergence are as follows.

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad -\infty < x < \infty$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad -\infty < x < \infty$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad -\infty < x < \infty$
4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \leq 1$
5. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad -1 \leq x \leq 1$

A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ is often called a *power series*. Such power series are uniformly convergent in any interval which lies entirely within the interval of convergence [see Problem 1.120].

FUNCTIONS OF TWO OR MORE VARIABLES

The concept of function of one variable given on page 2 can be extended to functions of two or more variables. Thus for example $z = f(x, y)$ defines a function f which assigns to the number pair (x, y) the number z .

Example 6. If $f(x, y) = x^2 + 3xy + 2y^2$, then $f(-1, 2) = (-1)^2 + 3(-1)(2) + 2(2)^2 = 3$.

The student is familiar with graphing $z = f(x, y)$ in a 3-dimensional xyz coordinate system to obtain a *surface*. We sometimes call x and y *independent variables* and z a *dependent variable*. Occasionally we write $z = z(x, y)$ rather than $z = f(x, y)$, using the symbol z in two different senses. However, no confusion should result.

The ideas of limits and continuity for functions of two or more variables pattern closely those for one variable.

PARTIAL DERIVATIVES

The *partial derivatives* of $f(x, y)$ with respect to x and y are defined by

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k} \quad (13)$$

if these limits exist. We often write $h = \Delta x$, $k = \Delta y$. Note that $\partial f / \partial x$ is simply the ordinary derivative of f with respect to x keeping y constant, while $\partial f / \partial y$ is the ordinary derivative of f with respect to y keeping x constant. Thus the usual differentiation formulas on pages 4 and 5 apply.

Example 7. If $f(x, y) = 3x^2 - 4xy + 2y^2$ then $\frac{\partial f}{\partial x} = 6x - 4y$, $\frac{\partial f}{\partial y} = -4x + 4y$.

Higher derivatives are defined similarly. For example, we have the second order derivatives

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \quad (14)$$

The derivatives in (13) are sometimes denoted by f_x and f_y . In such case $f_x(a, b)$, $f_y(a, b)$ denote these partial derivatives evaluated at (a, b) . Similarly the derivatives in (14) are denoted by f_{xx} , f_{xy} , f_{yx} , f_{yy} respectively. The second and third results in (14) will be the same if f has continuous partial derivatives of second order at least.

The differential of $f(x, y)$ is defined as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad (15)$$

where $h = \Delta x = dx$, $k = \Delta y = dy$.

Generalizations of these results are easily made.

TAYLOR SERIES FOR FUNCTIONS OF TWO OR MORE VARIABLES

The ideas involved in Taylor series for functions of one variable can be generalized. For example, the Taylor series for $f(x, y)$ about $x = a$, $y = b$ can be written

$$\begin{aligned} f(x, y) = & f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ & + \frac{1}{2!} [f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2] + \dots \end{aligned} \quad (16)$$

LINEAR EQUATIONS AND DETERMINANTS

Consider the system of linear equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \quad (17)$$

These represent two lines in the xy plane, and in general will meet in a point whose coordinates (x, y) are found by solving (17) simultaneously. We find

$$x = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \quad y = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2} \quad (18)$$

It is convenient to write these in determinant form as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad (19)$$

where we define a determinant of the second order or order 2 to be

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (20)$$