Lecture Notes in Mathematics 1672

Satya Mandal

Projective Modules and Complete Intersections

Springer

Satya Mandal

Projective Modules and Complete Intersections

Springer

Author

Satya Mandal
Department of Mathematics
University of Kansas
Lawrence, Kansas 66045, USA
e-mail: mandal@math.ukans.edu

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Mandal, Satya:
Projective modules and complete intersections / Satya Mandal. - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ; Paris ; Santa Clara ; Singapore ; Tokyo : Springer, 1997
(Lecture notes in mathematics ; 1672)
ISBN 3-540-63564-5

Mathematics Subject Classification (1991): 13-XX, 13C10

ISSN 0075-8434
ISBN 3-540-63564-5 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1997
Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera-ready $\TeX$ output by the author
SPIN: 10553364 46/3142-543210 - Printed on acid-free paper

Preface

In these notes we give an account of the developments in research in *Projective Modules and Complete Intersections* since the proof of Serre's Conjecture due to Quillen and Suslin and the subsequent publication of T-Y Lam's book, *Serre's Conjecture.*

I expect these notes to be accessible to a wide range of readers, with or without a serious background in commutative algebra. These notes evolved out of class notes for a course on this topic that I taught several years ago at the University of Kansas to a group of students who had no prior serious exposure to commutative algebra. My students enjoyed the course. I would hope that the readers will find these notes enjoyable as well.

I need to thank a long list of people who helped me, directly or indirectly, to accomplish this goal. I thank Professor Amit Roy of the Tata Institute of Fundamental Research, Bombay, for the excellent training he gave me in my early career. I would also like to thank my friends Daniel Katz and Jeffrey Lang for their encouragement and for many helpful discussions. Thanks are also due to D. S. Nagaraj of the Institute of Mathematical Sciences, Madras, and to Ravi Rao and Raja Sridharan of the Tata Institute of Fundamental Research, Bombay, for many helpful discussions. I would like to thank my brothers, sister and our parents for their support. Thanks to my wife, Elsit, for her encouragement and patience and also to our little daughter Nila for being here with us on time for the occasion.

Last but not the least I thank Tim Buller and Aaron Johnson for helping me with the computer system.

This project was partially supported by grants (FY 94 and FY98) from the General Research Fund of the University of Kansas.

Contents

此为试读,需要完整PDF请访问: www.ertongbook.com

Chapter 0

Introduction

In these notes on *Projective Modules and Complete Intersections* we present an account of the developments in research on this subject since the proof of the Conjecture of Serre due to Quillen and Suslin.

After two preliminary chapters, we start with the proof of Serre's Conjecture and some associated results of Quillen and Suslin in Chapter 3.

Chapter 4 includes the Basic Element Theory of Eisenbud and Evans and the proofs of the Eisenbud-Evans Conjectures. Our treatment of the Basic element theory incorporates the idea of *generalized dimension functions* due to Plumstead.

In Chapter 5, we discuss the theory of matrices that we need in the later chapters. We tried to avoid the theory of elementary matrices in these notes. Instead, we talk about the *Isotopy Subgroup* of the General Linear Group in this chapter.

The theory of Complete Intersections is discussed in Chapter 6. Among the theorems in this chapter are

1. the theorem of Eisenbud and Evans on the number of set theoretic generators of ideals in polynomial rings,

2. the theorem of Ferrand-Szpiro,

3. the theorem of Boratynski on the number of set theoretic generators of ideals in polynomial rings over fields,

4. the theorem of Ferrand-Szpiro-Mohan Kumar on the local complete intersection curves in affine spaces,

5. the theorem of Cowsik-Nori on curves in affine spaces.

To prove the theorem of Cowsik-Nori, we also give a complete proof of the fact that a curve in an affine space over a perfect field is integral and birational to its projection to an affine two subspace, after a change of variables.

In Chapter 7, we discuss the theory of Projective modules over polynomial rings in several variables over noetherian commutative rings. The techniques used in this chapter are almost entirely due to Lindel. Among the theorems in this chapter are

1. Lindel's Theorem on Bass-Quillen Conjecture,
2. the theorem of Bhatwadekar-Roy on the existence of Unimodular elements in projective modules,
3. Lindel's Theorem on the transitivity of the action of the group of transvections on the set of unimodular elements of a projective module.

A large portion of these notes evolved out of class notes for a course on this topic that I taught some years ago at the University of Kansas. The students in this class did not have any previous serious exposure to commutative algebra. My approach while conducting this course was to

a) state and explain results and proofs that have a potential to excite the students;

b) skip those proofs that may become technical;

c) state, explain and use results from commutative algebra as and when needed.

With this approach, I was able to cover the materials in Chapter 1-6. Among the theorems that I stated in this course *without proof* were the theorems of Eisenbud-Evans (Theorem 4.1.1), Plumstead (Theorem 4.3.1 and 4.3.2) and Sathaye-Mohan Kumar(Theorem 4.3.3). On the other hand, I proved the theorem of Ferrand-Szpiro (Theorem 6.1.3). I finished the course with the proof of the theorem of Ferrand-Szpiro-Mohan Kumar(Theorem 6.2.5) that

a locally complete intersection ideal of height $n-1$ in a polynomial ring $k[X_1, \ldots, X_n]$ over a field k is set theoretically generated by $n-1$ polynomials,

and with the statement of the Cowsik-Nori theorem(Theorem 6.3.1) that

any ideal of pure height $n-1$ in a polynomial ring $k[X_1, \ldots, X_n]$ over a field k of positive characteristic is set theoretically generated by $n-1$ elements.

Chapter 1

Preliminaries

In this chapter we shall put together some notations, some terminologies and preliminaries from commutative algebra that we will be using throughout these notes.

1.1 Localization

Suppose R is a commutative ring. For a subset S of R, we say that S is a *multiplicative* subset of R if 1 is in S and for s and t in S, st is also in S.

For a multiplicative subset S of R and an R-module M, we have

$$M_S = \{m/t : m \in M \text{ and } t \in S\}.$$

For m, n in M and t, s in S, we have $m/t = n/s$ if $u(sm - tn) = 0$ for some u in S. M_S is called the *localization* of M at the multiplicative set S. The following are some facts about localization.

Fact 1.1.1 Suppose R is a commutative ring and S is a multiplicative subset of R. Let M be an R-module. Then the following are easy to see.

(a) R_S is a ring and the map $R \to R_S$ that sends r to $r/1$ is a ring homomorphism.

(b) M_S becomes an R_S-module under the natural operations

$$m/s + n/t = (tm + sn)/st \quad \text{and} \quad (a/u)(m/t) = am/ut$$

for m, n in M , a in R and s, t, u in S.

(c) The natural map $i : M \longrightarrow M_S$ that sends m to $m/1$ is an R-linear map.

(d) The natural map $i : M \longrightarrow M_S$ has the following universal property : Given an R_S-module N and an R-linear map $f : M \longrightarrow N$ there is a unique R_S-linear map $F : M_S \longrightarrow N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{i} & M_S \\ & f \searrow & \downarrow F \\ & & N \end{array}$$

commutes.

(e) For an element f in R, we write R_f for R_S and write M_f for M_S where $S = \{1, f, f^2, \ldots\}$. For a prime ideal $\wp$ of R, $R_\wp$ and $M_\wp$, respectively, denote R_S and M_S where $S = R - \wp$.

(f) We have $\quad M_S \approx M \otimes_R R_S \quad$ as R_S - modules.

(g) Let M, N be two R-modules and let $f : M \to N$ be an R-linear map. It follows from the universal property that there is an R_S-linear map $F : M_S \to N_S$ such that the following diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ M_S & \longrightarrow & N_S \end{array}$$

commutes.

Definition 1.1.1 A homomorphism $i : R \to A$ of commutative rings is called *flat* if for all short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules and R-linear maps, the induced sequence

$$0 \longrightarrow M' \otimes_R A \longrightarrow M \otimes_R A \longrightarrow M'' \otimes_R A \longrightarrow 0$$

is exact.

Proposition 1.1.1 For a commutative ring R and a multiplicative subset S of R, the natural map $i : R \longrightarrow R_S$ is flat.

Definition 1.1.2 Suppose R is a commutative ring and M is an R-module. We say that M is *finitely presented* if there is an exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-modules, for some nonnegative integer n and a finitely generated R-module K. Equivalently, M is finitely presented if there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-modules, where m and n are nonnegative integers.

Remark 1.1.1 We leave it as an exercise that *for a noetherian commutative ring R, an R-module M is finitely generated if and only if it is finitely presented.*

Notations 1.1.1 Let R be a commutative ring.

1. We denote the set of all prime ideals of R by $Spec(R)$.
2. The set of all maximal ideals of R will be denoted by $\max(R)$.
3. For an ideal I of R, $V(I)$ will denote the set of all prime ideals of R that contain I.
4. For an element f of R, $D(f)$ will denote the set of all the prime ideals of R that do not contain f.

Exercise 1.1.1 Suppose R is a ring and $S \subseteq T$ be two multiplicative subsets of R. Then the following diagram

$$\begin{array}{ccc} R & \longrightarrow & R_S \\ & \searrow & \downarrow \\ & & R_T \end{array}$$

of the natural maps commutes. Further if $\tilde{T}$ is the image of T in R_S then $(R_S)_{\tilde{T}}$ is naturally isomorphic to R_T. We say that *R_T is a further localization of R_S* to explain this phenomenon.

1.2 The Local-Global Principle

Lemma 1.2.1 Suppose R is a commutative ring and M is an R-module. Then the following are equivalent :

1. $M = 0$,
2. $M_\wp = 0$ for all $\wp$ in $Spec(R)$,
3. $M_m = 0$ for all maximal ideals m of R.

Proof. See the book of Kunz ([K1]), Chapter III.

Proposition 1.2.1 Suppose R is a commutative ring and M is an R-module. For two submodules M' and M'' of M, we have $M' = M''$ if and only if $M'_m = M''_m$ for all maximal ideals m of R.

Proof. See the book of Kunz ([K1]), Chapter IV.

此为试读，需要完整PDF请访问：www.ertongbook.com

Example 1.2.1 Suppose R is a Dedekind domain which is not a Principal ideal domain (PID). If I is an ideal of R that is not principal, then I is not isomorphic to R but $I_m \approx R_m$ for all maximal ideals m of R.

Corollary 1.2.1 Suppose I and J are two ideals of R. Then $I = J$ if and only if $I_m = J_m$ for all maximal ideals m of R containing $I \cap J$.

Proof. The corollary follows from Proposition 1.2.1 .

Corollary 1.2.2 Suppose that R is a commutative ring and M is an R-module. Let $\{m_i\}_{i\in I} = S$ be a subset of M. Then the set $\{m_i\}_{i\in I}$ generates M if and only if the image $\{m_i/1\}_{i\in I}$ of S in M_m generates M_m for all maximal ideals m of R.

Proof. Let N be the submodule of M generated by S. Now the proof is an immediate consequence of Proposition 1.2.1.

Lemma 1.2.2 Suppose R is a commutative ring and let $f_1, f_2, \ldots, f_r$ be elements of R. Then $D(f_1) \cup D(f_2) \cup \ldots \cup D(f_r) = Spec(R)$ if and only if the ideal $Rf_1 + Rf_2 + \cdots + Rf_r = R$.

We leave the proof of this Lemma as an exercise.

Corollary 1.2.3 Suppose R is a commutative ring and assume that $Spec(R) = D(f) \cup D(g)$ for some f and g in R. Let M be an R-module.

(a) Suppose that M_f and M_g are finitely generated. Then M is finitely generated.

(b) Let $m_1, \ldots, m_r$ be elements in M such that their respective images generate both M_f and M_g. Then $m_1, \ldots, m_r$ generate M.

Proof. The corollary follows from Corollary 1.2.2.

Corollary 1.2.4 Suppose R is a commutative ring. Then a sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M'$$

of R-modules and R-linear maps is exact if and only if the induced sequence

$$M'_m \longrightarrow M_m \longrightarrow M'_m$$

is exact for all maximal ideals m of R.

Proof. See the book of Kunz ([K1]), Chapter IV.

Corollary 1.2.5 Let $f : M \longrightarrow N$ be an R-linear map.

1. Then f is injective if and only if f_m is injective for all m in $\max R$.
2. Similarly, f is surjective if and only if f_m is surjective for all m in $\max R$.

Proof. It is immediate from Corollary 1.2.4.

Example 1.2.2 Suppose D is a Dedekind domain that is not a PID. Let I be an ideal that is not principal. Then I_m is one generated for all m in $\max R$. This is probably the simplest example to illustrate that the local number of generators and the global number of generators are not always the same. *Deriving the global number of generators from the local number of generators is one of our main interests in these notes.*

Definition 1.2.1 (Zariski Topology) For a noetherian commutative ring R, the Zariski Topology on $Spec(R)$ is defined by declaring $D(f)$ as the basic open sets, for f in R. Equivalently, the closed sets in $Spec(R)$ are $V(I)$, where I is an ideal in R.

Exercise 1.2.1 Let R be a commutative noetherian ring. Then $Spec(R)$ is connected if and only if R has no idempotent element other than 0 and 1.

1.3 Homomorphisms of Modules and Flatness

The main theorem in this section is about the commutativity of the tensor product for a flat extension with the module of homomorphisms of modules. Of particular interest are the cases of polynomial extensions and localizations.

Notations 1.3.1 Suppose R is a commutative ring and M and N are two R-modules. We shall denote the set of all R-linear maps from M to N by $Hom_R(M,N)$ or simply by $Hom(M,N)$. Note that $Hom(M,N)$ is also an R-module in a natural way.

Definition 1.3.1 Suppose R, M, N are as in Notations 1.3.1 and let S be a multiplicative subset of R. We define a natural map

$$\varphi : S^{-1}Hom_R(M,N) \longrightarrow Hom_{R_S}(M_S, N_S)$$

by defining $\varphi(f/t) : M_S \longrightarrow N_S$ as

$$\varphi(f/t)(m/s) = f(m)/st$$

for f in $Hom(M,N)$, m in M and s,t in S.

Exercise 1.3.1 Let R, M, N, S be as in Definition 1.3.1 and let M be finitely generated. Prove that the natural map in Definition 1.3.1 is injective. Further, if M is finitely presented then the natural map is an isomorphism.

In fact, Definition 1.3.1 and Exercise 1.3.1 are, respectively, the particular cases of Definition 1.3.2 and Theorem 1.3.1 that follow.

Definition 1.3.2 Let $i : R \longrightarrow A$ be a flat homomorphism of rings. Let M and N be two R-modules. Define the natural map

$$\varphi : Hom_R(M,N) \otimes A \longrightarrow Hom_A(M \otimes A, N \otimes A)$$

by setting $\varphi(f \otimes t)(m \otimes s) = f(m) \otimes st$ for f in $Hom(M,N)$, m in M, and s,t in A.

Theorem 1.3.1 *Let $i : R \longrightarrow A$ be a flat homomorphism of commutative rings and let M, N be two R-modules with M being finitely presented. Then the natural map*

$$\varphi : Hom_R(M,N) \otimes A \longrightarrow Hom_A(M \otimes A, N \otimes A)$$

is an isomorphism.

Proof. First assume that $M \approx R^n$ is free. In that case we have the commutative diagram :

$$\begin{array}{ccc} Hom(M,N) \otimes A & \xrightarrow{\varphi} & Hom(M \otimes A, N \otimes A) \\ \downarrow\wr & & \downarrow\wr \\ N^n \otimes A & \xrightarrow{\varphi'} & (N \otimes A)^n . \end{array}$$

Here N^n denotes the direct sum of n copies of N and φ' is the natural identification. Since the vertical maps are isomorphisms, φ is also an isomorphism.

In the general case, since M is finitely presented, there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

of R-linear maps. This sequence will induce the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & Hom(M,N)\otimes A \rightarrow & Hom(R^n,N)\otimes A \rightarrow & Hom(R^m,N)\otimes A \\ & & \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & Hom(M',N') \rightarrow & Hom(A^n,N') \rightarrow & Hom(A^m,N') \end{array}$$

where $M' = M \otimes A$ and $N' = N \otimes A$.

Since the operation of taking $Hom(\ ,N)$ is left exact and $R \longrightarrow A$ is flat the first row of this diagram is exact. For similar reasons, the last row is also exact. By the case when M is free, the last two vertical maps are isomorphisms. Hence the first vertical map is also an isomorphism. This completes the proof of Theorem 1.3.1 .

Remark 1.3.1 All the rings we consider now onwards will be assumed to be noetherian and commutative. That is why any finitely generated module will also be a finitely presented module.

Remark 1.3.2 Let $A = R[X]$ be the polynomial ring over a noetherian commutative ring R. Let M and N be two finitely generated R-modules. It follows that

$$Hom(M \otimes R[X], N \otimes R[X]) \approx Hom(M,N) \otimes R[X].$$

Remark 1.3.3 Let R be a noetherian commutative ring and S be a multiplicative subset of R. For finitely generated R-modules M and N, we have

$$Hom_{R_S}(M_S, N_S) \approx Hom_R(M,N) \otimes_R R_S \approx \ (Hom_R(M,N))_S\,.$$

1.4 Definition of Projective Modules

Before we define projective modules, we want to discuss the splitting properties of exact sequences.

Definition 1.4.1 Suppose R is a commutative ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules and R-linear maps. We say that the sequence *splits* if there is an R-linear map $\zeta : M'' \longrightarrow M$ such that $go\zeta = Id_{M''}$.

Lemma 1.4.1 Suppose that R is a commutative ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of R-modules and R-linear maps. Then the following conditions are equivalent :

1. The sequence splits.
2. $M = M' \oplus N$ for some submodule N of M such that the restriction $g|_N : N \longrightarrow M''$ is an isomorphism.
3. There is a map $t : M \longrightarrow M'$ such that $tof = Id_{M'}$.
4. The natural map

   $$\varphi \ : \ Hom(M'', M) \longrightarrow Hom(M'', M''),$$

   that sends a map $h : M'' \longrightarrow M$ to goh, is surjective.

Proof. It is easy to see that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To see that (1) implies (4) let $\zeta : M'' \longrightarrow M$ be a split i.e. $go\zeta = Id_{M''}$. We have the following

$$\begin{array}{ccc} M & \xrightarrow{g} & M'' \\ \zeta\uparrow & \nearrow Id & \\ M'' & & \end{array}$$

commutative diagram. Given a map $h : M'' \longrightarrow M''$, we have $\varphi(\zeta oh) = go\zeta oh = h$. Hence φ is surjective. This establishes (4).

To see (4) implies (1), let $\varphi(\zeta) = Id$. Then ζ is a split of g. This completes the proof of the Lemma.

Corollary 1.4.1 Suppose R is a commutative noetherian ring and let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated R-modules and R-linear maps. Then the sequence splits if and only if the induced exact sequences

$$0 \longrightarrow M'_m \longrightarrow M_m \longrightarrow M''_m \longrightarrow 0$$

split for all m in $\max(R)$.

Proof. It is immediate from Lemma 1.4.1 and Corollary 1.2.5.

Now we are ready to define projective modules.

Definition 1.4.2 Suppose R is a commutative ring and let P be an R-module. We say that P is a *projective* R-module if one of the following equivalent conditions hold :

1. Given a surjective R-linear map $f : M \longrightarrow N$ and an R-linear map $g : P \longrightarrow N$ there is an R-linear map $h : P \longrightarrow M$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & h\swarrow & \downarrow g & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

commutes.

2. Every exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ of R-modules and R-linear maps splits.

3. There is an R-module Q such that $P \oplus Q$ is free.

4. The functor $M \longrightarrow Hom_R(P, M)$ from the category of R-modules to itself is exact.

Proof.

(1) ⇒ (2) follows by looking at the diagram

$$\begin{array}{ccccc} & & P & & \\ & h\swarrow & \downarrow Id & & \\ M & \longrightarrow & P & \longrightarrow & 0. \end{array}$$

(2) ⇒ (3) We can find a surjective map $f : F \longrightarrow P$, where F is free. Take Q = kernel (f). Then $P \oplus Q \approx F$ is free.

(3) ⇒ (4) Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of R-modules and R-linear maps. It is a general fact that

$$0 \longrightarrow Hom(P, M') \longrightarrow Hom(P, M) \longrightarrow Hom(P, M'')$$

is exact. So, we need only to show that the map

$$Hom(P, M) \longrightarrow Hom(P, M'')$$

is surjective. Let $F = P \oplus Q$ be free and let $h : P \longrightarrow M''$ be any R-linear map. If $h_0 : F \longrightarrow M''$ is the map such that $h_0|_P = h$ and $h_0|_Q = 0$ then there is an R-linear map $h_0' : F \longrightarrow M$ such that $goh_0' = h_0$. Let $h' = h_0|_P$, then $goh' = h$.

(4) ⇒ (1) is obvious.

此为试读，需要完整PDF请访问：www.ertongbook.com