Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Schottky Groups and Mumford Curves

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Introduction

The idea of investigating the p-adic version of classical uniformizations of curves is due to John Tate who showed that an elliptic curve over a p-adic field K whose j-invariant has absolute value greater than 1 can be analytically uniformized. While Tate's original paper has never been published there are good accounts of his work available, see [34].

The generalization of the above result of Tate to curves of higher genus has been given by David Mumford in 1972 in a work called "Analytic construction of degenerating curves over complete local rings".

The main result of the paper states that there is a one-to-one correspondence between

a) conjugacy classes of Schottky groups $\Gamma \subset PGL_2(K)$

b) isomorphism classes of curves C over K which are the generic fibers of normal schemes over the valuation ring $\overset{\circ}{K}$ of K whose closed fiber is a split degenerate curve.

In these Notes we call the curves that Mumford has associated to p-adic Schottky groups Mumford curves. Manin has called them Schottky-Mumford curves in [26].

When Mumford received the Fields medal in 1974 his discovery was praised by Tate when he described the work of Mumford. He said: "Next I want to mention briefly p-adic uniformization. Motivated by the study of the boundary of moduli varieties for curves, i.e. of how nonsingular curves can degenerate, Mumford was led to introduce p-adic Schottky groups, and to show how one can obtain certain p-adic curves of genus ≥ 2 transcendentally as the quotient by such groups of the p-adic projective line minus a Cantor set. The corresponding theory for genus 1 was discovered by the author, but the generalization to higher genus was far from obvious. Besides its significance for moduli, Mumford's construction is of interest in itself as a highly nontrivial example of "rigid" p-adic analysis".

While Mumford worked with formal schemes over the valuation ring of K, several authors, stimulated by Mumford's investigation, developed his construction within the framework of analytic spaces over K.

Manin-Drinfeld [27] and Myers [29] introduced the notion of automorphic forms and made clear that the Jacobian variety of a p-adic Schottky curve can be constructed analytically as an analytic torus given by a period matrix which is symmetric and positive definite. A good account of the analytic theory of Schottky curves has been given by Manin in 1974, see [26].

In recent years a number of new results on Mumford curves have been obtained by different authors as D. Goss, F. Herrlich and the authors of these Notes. It is the purpose of this work to give an introduction into the theory of Mumford curves presenting the major results and describing a variety of explicit examples.

We will employ two different approaches to the topic, one of which relies on p-adic function theory methods and the machinery of automorphic forms. The advantage of this approach lies in the fact that it is quite elementary and we have tried to be as much down to earth as possible.

The second approach works with methods stemming from algebraic, formal and affinoid geometry and exploiting the reduction of spaces. Here it is not always possible to entirely avoid more advanced and less accessible constructions.

Discontinuous subgroups Γ of the group $PGL_2(K)$ of fractional linear transformations of the projective line $\mathbb{P}(K)$ where K is a non-archimedean valued complete algebraically closed field are introduced in Chap. I and used throughout the Notes. The condition "discontinuous" for a subgroup Γ means that the closure of every orbit of Γ is compact and Γ has ordinary points. In the non-elementary cases we associate to Γ a canonical tree T on which Γ acts. Using the tree one shows for a finitely generated Γ the existence of a normal subgroup Γ_o of Γ of finite index which is a Schottky group. All Schottky groups can be constructed from a fundamental domain consisting of the complement of 2g open disks.

In Chap.II automorphic forms relative to a Schottky group Γ with constant factors of automorphy are constructed and determined as products of the basic forms $\Theta(a, b; z)$. This allows to prove that the field of Γ-invariant meromorphic functions on the domain Ω of ordinary points for Γ is an algebraic function field of one variable whose set of places S coincides with the orbit space Ω/Γ.

In the first section of Chap. III some basic material on affinoid algebras, affinoid and analytic spaces is presented. Especially, reductions of analytic spaces are introduced. In the second part the construction of Ω/Γ (Γ Schottky group, Ω the set of its ordinary points) as analytic space is given. It is shown that Ω/Γ is in fact a non-singular complete curve of genus g. One further obtains that Ω/Γ has a split degenerate reduction.

In Chap. IV domains Ω in $\mathbb{P}(K)$ are characterized among the non-singular one dimensional analytic spaces by the property: "Ω has an analytic reduction $\bar{\Omega}$ consisting of genus zero curves with a tree as intersection graph".

A complete non-singular curve X which has a split degenerate reduction (i.e. the reduction consists of genus zero curves and only nodes as singularities) is shown to have a universal covering $\Omega \to X$. The space Ω has a reduction of the type explained above and it follows that Ω is a domain in $\mathbb{P}(K)$. Moreover the automorphism group Γ (which is the fundamental group of X) of the covering $\Omega \to X$ turns out to be a Schottky group with Ω as set of ordinary points. This amounts to Mumford's theorem: A curve X has a split degenerate reduction if and only if X can be parametrized by a Schottky group.

The main result of Chap. V is the existence of an analytic reduction $\bar{X}$ of a complete non-singular curve X (of genus > 1) satisfying: $\bar{X}$ has only nodes as singularities and $\bar{X}$ has a finite group of automorphisms. Such a reduction is called stable and is uniquely determined by X. This result is very close to Deligne-Mumford's result [5] on the existence of stable algebraic reductions. As a corollary one finds: X is a Mumford curve if and only if X has a finite covering by affinoid subsets of $\mathbb{P}(K)$.

In Chap. VI we present an analytic construction of the Jacobian variety $\mathcal{J}(S)$ for a Mumford curve S together with the canonical mapping ϕ of the curve into its Jacobian and show that $\mathcal{J}(S)$ is an analytic torus $(K^*)^g$ modulo a lattice with a polarization defined by a period matrix. The Riemann theta function $\vartheta(u_1,\ldots, u_g)$ on the algebraic torus $(K^*)^g$ associated to a square root of the period matrix for Γ is well-defined. We can prove that the divisor of $\vartheta(c \cdot u(z))$, $c \in (K^*)^g$, is of degree g if $\vartheta(c \cdot u(z))$ does not vanish identically on Ω, where $u : \Omega \to (K^*)^g$ is a lift of the canonical mapping $\phi : S \to \mathcal{J}(S)$. As in the complex Riemann vanishing theorem $\vartheta(u_1,\ldots, u_g) = 0$ is the equation for a translate of the hypersurface $\phi(S^{g-1}) \subset \mathcal{J}(S)$.

The starting point of the discussion in Chap. VII on automorphisms is the result that the automorphism group Aut S of a Mumford curve $S = S(\Gamma)$ is canonically isomorphic to the factor group N/Γ where N is the normalizer of Γ in $PGL_2(K)$. We describe various results the most striking of which states that the order of Aut S is less than or equal to $12(g-1)$ if the ground field K has characteristic zero and the characteristic of the residue field is different from 2, 3, 5.

In Chap. VIII we consider the curve T associated to a finitely generated discontinuous group N which does contain transformations of finite order and show how one can describe the divisor class group of degree 0 by automorphic forms with respect to N. The genus of T turns out to be the $\mathbb{Z}$-rank of the commutator factor group of N.

In the first part of Chap. IX we show that the group $H(\mathbb{Z}[\frac{1}{p}])^*$ of invertible Hurwitz quaternions with coefficents in $\mathbb{Z}[\frac{1}{p}]$ is a discrete subgroup of $GL_2(k)$ where k is a finite extension of $\mathbb{Q}_p$. Its image Λ in $PGL_2(k)$ is a discontinuous group. The genera of the Mumford curves parametrized by Λ and the congruence subgroup $\Lambda(2)$ are calculated. The geometry of the curves and their stable reductions is made explicit.

In the second part Whittaker groups are considered. They are subgroups of index 2 of groups generated by elliptic transformations of order 2. They parametrize hyperelliptic curves.

In Chap. X we work with the Laurent series field $k = \mathbb{F}_q((\frac{1}{t}))$ and the discontinuous group $\Gamma(1) = PSL(2, \mathbb{F}_q[t])$ which shares many features with the classical modular group $PSL(2, \mathbb{Z})$. The quotient space with respect to $\Gamma(1)$ is the affine line and can be completed by adjoining a parabolic point. The algebra of modular forms for $\Gamma(1)$ is determined.

We like to thank Dr. F. Herrlich for contributing ideas in the preparation of Chap VII and for his help in proof reading. We are grateful to Prof. Dr. S. Bosch who read part of the manuscript and suggested improvements. Also we like to express our gratitude to Mrs. Marianne Puhlvers for the excellent job of typing the manuscript and her patience with the authors.

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Chapter I Discontinuous groups

Introduction: The field k is supposed to be complete with respect to a non-archimedean valuation. By K we denote a complete and algebraically closed field containing k. We work with the projective line over k as analytic variety. However, in this chapter almost no function theory is needed and it suffices to consider $\mathbb{P}^1(k)$ and $\mathbb{P}^1(K)$ as topological spaces. The projective linear group PGL(2, k) acts in the ususal way on $\mathbb{P} = \mathbb{P}^1(K)$. A subgroup Γ of PGL(2, k) is called discontinuous if the closure of every orbit of Γ in $\mathbb{P}$ is compact and Γ has ordinairy points.

Let Γ be discontinuous and let $\mathcal{L}$ be its set of limit points. As in the complex case, $\mathcal{L}$ is compact, nowhere dense. Further $\mathcal{L}$ is perfect if $\mathcal{L}$ contains more than two points.

Unlike the complex case $\mathbb{P} - \mathcal{L}$ is always connected. Another feature which differs from the classical case is that a parabolic element of infinite order does not generate a discontinuous group. (§1). In §2 one associates to $\mathcal{L}$ (and more generally to a compact set X in $\mathbb{P}$) an infinite tree T. The group Γ acts on this tree and for a finitely generated Γ the quotient T/Γ is a finite graph. This tree is in fact the same tree introduced by D. Mumford [28]. Using the action of Γ on T one shows the following structure theorem (§3): If Γ is finitely generated then Γ has a normal subgroup Γ_o of finite index such that Γ_o is a finitely generated free group.

A finitely generated free, discontinuous group is called a Schottky group. Again, using the action on the tree, one shows in §4 that every Schottky group has a nice fundamental domain F:

$F = \mathbb{P}$ - (2g open disks). Let us, for convenience, assume that $\infty \in F$ and call the open disks $B_1, B_2, \ldots, C_1, \ldots, C_g$. Then the disks are in "good position", which means that the corresponding closed disks are disjoint. Moreover Γ has free generators $\gamma_1, \ldots, \gamma_g$ satisfying γ_i maps $\mathbb{P} - B_i$ onto C_i^+, and $\mathbb{P} - B_i^+$ onto C_i.

§1 Groups acting on $\mathbb{P}^1$

(1.1) In what follows k denotes a field which is complete with respect to a non-archimedean valuation $| \, |$. This means that there is given a map $| \, | : k \to \mathbb{R}$ with the properties:

1) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$.

2) $|xy| = |x||y|$.

3) $|x + y| \leq \max(|x|, |y|)$.

4) there is an $x \in k$ with $|x| \neq 0, 1$.

5) k is complete with respect to the metric $d(x, y) = |x - y|$.

The most interesting examples are possibly $\mathbb{Q}_p$, the field of p-adic numbers and $\mathbb{F}_p((t))$, the field of Laurent-series in t with coefficients in the finite field $\mathbb{F}_p$.

The field of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the valuation $| \, |_p$ (or metric $d(x, y) = |x - y|_p$) defined by $|p^m \frac{t}{n}| = p^{-m}$ if $m \in \mathbb{Z}$ and $(p, t) = (p, n) = 1$.

The valuation on $\mathbb{F}_p((t))$, which consists of the expressions $\sum_{n >> -\infty} a_n t^n$ and is the quotient field of the formular power series ring $\mathbb{F}_p[\![t]\!]$, is defined by $|\Sigma a_n t^n| = \max\{p^{-n} | a_n \neq 0\}$.

In this section we collect some of the properties of (non-archimedean) valued fields. The valuation ring k^o of k is given by $\{\lambda \in k \,|\, |\lambda| \leq 1\}$. Its unique maximal ideal k^{oo} equals $\{\lambda \in k \,|\, |\lambda| < 1\}$ and $\bar{k} = k^o / k^{oo}$ is called the residue field of k. The value group $|k^*| = \{|\lambda| \,|\, \lambda \in k, \lambda \neq 0\}$ of k is a subgroup of $\mathbb{R}_{>0}$. We say that the valuation of k is discrete if $|k^*| \cong \mathbb{Z}$; in this case k^o is a Noetherian ring. The valuation is called dense if $|k^*|$ is a dense subgroup of $\mathbb{R}_{>0}$;

in this case k^o is not Noetherian. By $\sqrt{|k^*|}$ is meant $\{a \in \mathbb{R}_{>0} |$ for some $n \geq 1$, $a^n \in |k^*|\}$.

For every field extension ℓ of k there is a valuation on ℓ which extends the valuation on k. This extension is unique if ℓ is an algebraic field-extension of k. In particular, the algebraic closure of k has a unique valuation and the completion of this field is again algebraically closed. We denote by $K \supset k$ an algebraically closed field which is complete with respect to a valuation extending the valuation of k. As we have seen, K exists.

Now and then we will work with maximally complete fields k. That is, k has the property that every sequence $B_1 \supset B_2 \supset B_3 \supset \dots$ of disks (open or closed) in k has a non-empty intersection. This property is equivalent to: for every valued field extension $\ell \supset k$ one has $|\ell^*| \supsetneq |k^*|$ or $\bar{\ell} \supsetneq \bar{k}$.

A field k is called a <u>local field</u> if k is locally compact. This is equivalent to $|k^*| \cong \mathbb{Z}$ and $\bar{k}$ is finite. Every local field is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ (and conversely).

Finally we recall that a complete field k has the <u>Hensel-property</u>, i.e.: if $F \in k^o[t]$ is a monic polynomial and if its image $\bar{F} \in \bar{k}[t]$ is a product of two monic polynomials f_1, f_2 with g.c.d. 1, then $F = F_1F_2$ where F_i are monic polynomials with $\bar{F}_i = f_i$ $(i = 1,2)$.

(1.2) The <u>projective line</u> over k is denoted by $\mathbb{P}^1(k)$. As usual, each point p of $\mathbb{P}^1(k)$ represent a line $L \subset k^2$ through $(0,0)$. If $L = \{\lambda x_o, x_1) | \lambda \in k\}$ then we will write $p = [x_o, x_1]$. The field k is identified with $\mathbb{P}^1(k) - \{[0, 1]\}$ by means of the map $\lambda \to [1,\lambda]$ and the point $[0, 1]$ will be denoted by ∞. This identification leads to writing $z \in k \cup \{\infty\}$ for the elements of $\mathbb{P}^1(k)$.

Let $\sim$ denote the equivalence relation on $k^2 - \{(0, 0)\}$ given by $(x, y) \sim (x', y')$ if $(x, y) = \lambda(x', y')$ for some $\lambda \in k^*(= k - \{0\})$. Then $\mathbb{P}^1(k) = k^2 - \{(0, 0)\}/\sim$ and $\mathbb{P}^1(k)$ inherents a topology from k. Further $\mathbb{P}^1(k)$ is compact if and only if k is locally compact.

We abbreviate in the sequel $\mathbb{P}^1(K)$ by $\mathbb{P}(K)$ or $\mathbb{P}$.

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, k)$ = invertible 2×2-matrices over k, we consider the "fractional linear" automorphism ϕ of $\mathbb{P}^1(k)$ given by $z \mapsto \frac{az + b}{cz + d}$, or in the homogeneous coordinates "$[x_o, x_1]$" ϕ is given by $[x_o, x_1] \mapsto [cx_1 + dx_o, ax_1 + bx_o]$. The group of automorphisms of $\mathbb{P}^1(k)$ thus obtained is $PGL(2, k) = GL(2, k)/\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} | \lambda \in k^*\}$. In more than one aspect (namely algebraically and analytically) those are the only automorphisms of $\mathbb{P}^1(k)$. ALso GL(2, k) and PGL(2, k) inherit in an obvious way a topology from k. They are never compact. But if k is locally compact then GL(2, k) and PGL(2, k) have interesting maximal compact subgroups. We will return to this in §2.

(1.3) Let Γ be a subgroup of PGL(2, k). An element $p \in \mathbb{P}$ is called a limit point of Γ if there exists $q \in \mathbb{P}$ and an infinite sequence $\{\gamma_n | n \geq 1\} \subset \Gamma$ (i.e. $\gamma_n \neq \gamma_m$ id $n \neq m$) with $\lim \gamma_n(q) = p$. If Γ is not discrete in PGL(2, k) then there exists a sequence $\{\gamma_n\}$ in Γ with $\lim \gamma_n = \gamma \in PGL(2, k)$. So $\lim \gamma_n(\gamma^{-1}(p)) = p$ for all $p \in \mathbb{P}$ and every point of $\mathbb{P}$ is a limit point for Γ. Let $\mathcal{L}$ denote the set of all limit points of Γ.

We will call Γ a discontinuous group if

(a) $\mathcal{L} \neq \mathbb{P}$.

(b) $\overline{\Gamma p}$ (= the closure of the orbit of p) is compact for all $p \in \mathbb{P}$.

Condition (b) is superfluous if k is locally compact. In the sequel we will use the following terminology: an element $\gamma \in PGL(2, k)$ represented by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, k)$ is called elliptic, parabolic or hyperbolic according to the following three cases:

- the eigenvalues of A are different but have the same absolute value -
- the eigenvalues are equal - or
- the eigenvalues have different absolute value.

In general the eigenvalues of A are not in k but in some finite extension of k which carries a unique valuation extending the valuation on k. Obviously the choice of k is unimportant in the definition above. Further if γ is elliptic of parabolic then $\frac{(a+d)^2}{ad-bc}$ has absolute value ≤ 1. If γ is hyperbolic then $|\frac{(a+d)^2}{ad-bc}| > 1$.

Let $GL(2, k^o)$ denote the 2×2-matrices over k^o with determinant invertible in k^o. Further $PGL(2, k^o)$ will denote the image of $GL(2, k^o)$ in $PGL(2, k)$.

(1.4) <u>Lemma:</u> <u>Let</u> $\gamma \in PGL(2, k)$.

(1) γ <u>is elliptic or parabolic if and only if a conjugate of</u> γ^2 <u>lies in the subgroup</u> $PGL(2, k^o)$ <u>of</u> $PGL(2, k)$.

(2) γ <u>is hyperbolic if and only if</u> γ <u>is conjugated to an element of</u> $PGL(2, k^o)$ <u>represented by a matrix</u> $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, $q \in k$, $0 < |q| < 1$.

Proof:

(1) If γ^2 (or γ^n for some $n \geq 1$) is conjugated to an element in $PGL(2, k^o)$, then clearly $|\frac{(a+d)^2}{ad-bc}| \leq 1$ for a representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of γ. So γ is elliptic or parabolic.

Conversely, suppose that γ is parabolic of elliptic. If γ represented by $B \in GL(2, k)$ and γ^2 by B^2 then there is a $\lambda \in k^*$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A = B^2$ satisfies $|ad - bc| = 1$. Then $|a + d| \leq 1$.

Put $N = k^o \oplus k^o \subset k \oplus k$ and $M = N + A(N)$. Then M is a finitely generated k^o-module, invariant under A since $A^2 - (a + d) A + (ad - bc) = 0$. Let $\{e_1, \ldots, e_n\}$ be a minimal set of generators of M as k^o-module. If there exists a non-trivial relation $\lambda_1 e_1 + \ldots + \lambda_n e_n = 0$, $\lambda \in k$, then we may assume $\max |\lambda_i| = 1$ and also that one of λ_i equals 1. This contradicts the minimality of n. So $n = 2$ and $\{e_1, e_2\}$ is a free base of the k^o-module M. Let $C : k^2 \to k^2$ be the k-linear map given by $C(1, 0) = e_1$, $C(0, 1) = e_2$. Then $A \in C\,GL(2, k^o)C^{-1}$ and γ^2 lies in a conjugate of the subgroup $PGL(2, k^o)$ of $PGL(2, k)$.

(2) Clearly $\begin{pmatrix} q & o \\ o & 1 \end{pmatrix}$, $0 < |q| < 1$, and all its conjugates are hyperbolic. On the other hand, if γ is hyperbolic then γ can be represented by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 1$ and $|ad - bc| < 1$. The characteristic polynomial P of A is $X^2 - X + (ad + bc) \in k^o[X]$. The reduction $\bar{P}$ in $\bar{k}[X]$ has two roots $\bar{0}, \bar{1} \in \bar{k}$. Since k has the Hensel property, there are roots λ_o, λ_1 of P in k^o with $\bar{\lambda}_o = \bar{0}$, $\bar{\lambda}_1 = \bar{1}$. So the eigenvalues of A has the required form $\begin{pmatrix} q & o \\ o & 1 \end{pmatrix}$, $q \in k$, $0 < |q| < 1$.

(1.5) Examples: Let Γ be generated by one element $\gamma \in PGL(2, k)$.

(1) If γ is hyperbolic then Γ is discontinuous. The set $\mathcal{L}$ of limit points of Γ consists of the two points of $\mathbb{P}^1(k)$ corresponding to the two eigenvectors of γ. These points are the two fixed points of γ.

(2) If γ is elliptic or parabolic and Γ is discontinuous then γ has finite order. In $PGL(2, K)$ the element γ is either conjugated to $\delta_1 : z \mapsto \lambda z$ and $|\lambda| = 1$, or to $\delta_2 : z \mapsto z + b$. The group generated by δ_1 is discontinuous if and only if $\overline{\{\lambda^n | n \in \mathbb{Z}\}}$ is compact and 1 is not a limit point of this set. This means that γ should be a root of unity. So δ_1 and γ have finite order. The group generated by δ_2 is discontinuous if and only if $\overline{\{nb | n \in \mathbb{Z}\}}$ is compact and does not have 0 as limit point. That means that $nb = 0$ for some $n \neq 0$. So γ_2 and γ have finite order. We note that γ has two fixed points if γ is elliptic and has one fixed point if γ is parabolic. We have also shown the following result:

(3) If γ is parabolic ($\neq$ id) and if Γ is discontinuous then k has characteristic $p \neq 0$ and γ has order p.

(1.6) A subgroup Γ of $PGL(2, k)$ is called a Schottky group if

(a) Γ is fintely generated.

(b) Γ has no elements ($\neq 1$) of finite order.

(c) Γ is discontinuous.

According to (1.5) condition (b) can be replaced by: every $\gamma \in \Gamma$, $\gamma \neq 1$, is hyperbolic. We start now the investigation of $\mathcal{L}$ = the limit points of a discontinuous group Γ. We may (and will) assume that $\infty \notin \mathcal{L}$.

Proposition: Γ is a discontinuous group and $\infty \notin \mathcal{L}$. Then

(1.6.1) Represent any $\gamma \in \Gamma$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, k)$ (a, b, c, d depending on γ). For any $\delta > 0$ the set $\{\gamma \in \Gamma \,|\, |c|^2 \leq \delta |ad - bc|\}$ is finite. Moreover Γ is finite or countable.

(1.6.2) For $a \in \mathbb{P}$, $\mathcal{L}(a) \subseteq \overline{\Gamma a}$ denotes the set of $b \in \mathbb{P}$ for which there exists an infinite sequence $\{\gamma_n\} \subset \Gamma$ with $\lim \gamma_n(a) = b$.

Given three different points $a_1, a_2, a_3 \in \mathbb{P}$. Then there exists an i with $\mathcal{L}(a_i) = \mathcal{L}$.

(1.6.3) $\mathcal{L} = \mathcal{L}(\infty) = \overline{\Gamma(\infty)} - \Gamma(\infty)$ is compact. $\mathcal{L}$ has no interior. $\mathcal{L}$ is perfect if $\mathcal{L}$ contains more than two elements.

(1.6.4) Suppose that k is a local field. Then any discrete subgroup Λ of $PGL(2, k)$ is discontinuous and has a set of limit points $\mathcal{L} \subset \mathbb{P}(k)$.

Proof: Any infinite sequence in Γ (or Λ in case (1.6.4)) has a subsequence $\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ such that $\frac{a_n}{c_n}, \frac{b_n}{d_n}, \frac{d_n}{c_n}$ (equal to $\gamma_n(\infty)$, $\gamma_n(0)$, $-\gamma_n^{-1}(\infty)$) are convergent. In case of the group Λ we can change the coordinates such that all limits are $\neq \infty$.

$$\text{Then } \lim \begin{pmatrix} \frac{a_n}{c_n} & \frac{b_n}{c_n} \\ 1 & \frac{d_n}{c_n} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}.$$ From the discreteness of Γ (or Λ) it follows that $ad = b$.

For $q \in \mathbb{P}$ we find $\lim \gamma_n(q) = a$ unless $q = -d$ and the sequence $\frac{d_n}{c_n}$ is constant.

(1) If the sequence γ_n satisfies $|c_n|^2 \leq \delta|a_n d_n - b_n c_n|$ then we obtain the contradiction $ad - b = (\frac{a_n}{c_n}\frac{d_n}{c_n} - \frac{b_n}{c_n}) \neq 0$. This proves the first statement of (1.6.1). The second one follows at once.

(2) It follows from our considerations above that $\mathcal{L} = \mathcal{L}(a_1) \cup \mathcal{L}(a_2)$ if $a_1 \neq a_2$. Moreover if $a_1 \notin \mathcal{L}$ then for any infinite sequence γ_n as above we have $\lim \gamma_n(a_1) = a$ since a_1 is unequal to $-d \in \mathcal{L}$.
It follows that $\mathcal{L} = \mathcal{L}(a_1)$. Let now a_1, a_2, a_3 be three different points in $\mathcal{L}$. Then $a_3 \in \mathcal{L}(a_1) \cup \mathcal{L}(a_2)$, say $a_3 \in \mathcal{L}(a_1)$.
Then $\mathcal{L} = \mathcal{L}(a_3) \cup \mathcal{L}(a_1) \subseteq \mathcal{L}(a_1) \cup \mathcal{L}(a_1) = \mathcal{L}(a_1) \subseteq \mathcal{L}$.

(3) Suppose that $\mathcal{L}$ contains at least 3 points. Then $\mathcal{L} = \mathcal{L}(a)$ for some $a \in \mathcal{L}$. Then clearly $\mathcal{L}(a) = \overline{\Gamma a}$. Hence $\mathcal{L}$ is compact and perfect.
Further $\mathcal{L}(\infty) = \overline{\Gamma(\infty)} - \Gamma(\infty) = \mathcal{L}$ since ∞ is the fixed point of only finitely many elements of Γ. So $\mathcal{L}$ has no interior points.

(4) Let $q \in \mathbb{P}$ and let a sequence in $\Lambda(q)$ be given. We have to show the existence of a convergent subsequence with limit in $\mathbb{P}^1(k)$. We take a subsequence $\gamma_n(q)$, with $\{\gamma_n\}$ as above. Then $\lim \gamma_n(q) = a \in \mathbb{P}^1(k)$ or $q = -d$ and $\lim \gamma_n(q) = \infty \in \mathbb{P}^1(k)$. So we have shown that $\overline{\Lambda(q)}$ is compact and $\mathcal{L} \subseteq \mathbb{P}^1(k)$.

(1.7) Examples:

(1) Suppose that the discontinuous group Γ contains no hyperbolic elements, then either (a) Γ is finite.
or (b) the characteristic of k is $p \neq 0$; Γ_o the subset of parabolic elements is an infinite normal subgroup of Γ isomorphic to a discrete subgroup of k; Γ/Γ_o is a finite group of roots of unity in k^*. Further $\mathcal{L}$ consists of one point.