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J. Bellissard M. Degli Esposti G. Forni
S. Graffi S. Isola J. N. Mather

Transition to Chaos in Classical and Quantum Mechanics

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Transition to Chaos in Classical and Quantum Mechanics

Lectures given at the 3rd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, July 6-13, 1991

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FOREWORD

This volume collects the texts of two series of 8 lectures, and the expanded version of a seminar, given at the C.I.M.E. Session on "Transition to Chaos in Classical and Quantum Systems", which took place at the Villa "La Querceta" in Montecatini, Italy, from July 6 to July 13, 1991.

The purpose of the Session was to give a broad survey of the mathematical problems and techniques, as well as of some of the most relevant physical motivations, which arise in the study of the stochastic behaviour, if any, of deterministic dynamical systems both in classical and quantum mechanics.

The transition to chaos in the most relevant and widely studied examples of classical dynamical systems, the area preserving maps, is thoroughly covered in the first series of lectures, delivered by Professor John Mather and written in collaboration with Dr. Giovanni Forni. In particular the reader can find in this text an up-to-date version of the well known Aubry-Mather theory. The lectures of Professor Jean Bellissard cover in turn, in addition to his algebraic approach to the classical limit, the behaviour of the quantum counterpart of the above systems, with particular emphasis on localization, and on qualitative as well as quantitative properties of the spectra of the relevant Schrödinger operators in classically chaotic regions. They can be therefore considered an exhaustive introduction to the mathematical aspects of the so-called "quantum chaos". The third series of lectures, delivered by Professor Anatole Katok, covered the basic stochastic properties of classical dynamical systems and some of their most recent developments. Unfortunately Professor Katok could not find the time to write up the text of his course.

A very prominent role in describing the chaotic behaviour of classical dynamical systems is played, as discussed also in Professor's Katok lectures, by the proliferation and equidistribution of the unstable periodic orbits of increasing period. An overview of recent results in this direction, and of their intimate connection to the problem of the classical limit of the quantized toral symplectomorphisms, is contained in an outgrowth of a seminar held by M.Degli Esposti, written in collaboration with S.Isola and the Editor.

Bologna, April 1994

Sandro Graffi

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One considers a spinning particle submitted to rotate around a fixed axis. Let $\theta \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be its angle of rotation, $L \in \mathbf{R}$ its angular momentum, I its moment of inertia, μ its magnetic moment, and B a uniform magnetic field parallel to the axis of rotation. Its kinetic energy is given by :

$$\mathcal{H}_0 = \frac{L^2}{2I} + \mu BL \ , \tag{1}$$

We assume that this system is kicked periodically in time according to the following Hamiltonian :

$$\mathcal{H} = \frac{L^2}{2I} + \mu BL + k\cos(\theta) \sum_{n\in\mathbf{Z}} \delta(t - nT) \ . \tag{2}$$

where T is the period of the kicks, and k is a coupling constant representing the kicks strength. Here δ is the Dirac measure. Classically the motion is provided by the solution of the Hamilton-Jacobi equations :

$$\frac{d\theta}{dt} = \frac{\partial\mathcal{H}}{\partial L} \qquad \frac{dL}{dt} = -\frac{\partial\mathcal{H}}{\partial\theta} \ . \tag{3}$$

Between two kicks, $\partial\mathcal{H}/\partial\theta = 0$, so that L is constant whereas θ varies linearly in time. When the kick is applied, L changes suddenly according to $L(nT + 0) = L(nT - 0) + k\sin(\theta)$. If we set :

$$A_n = T\left(\frac{L(nT-0)}{I} + \mu B\right) \qquad \theta_n = \theta(nT - 0) \ , \tag{4}$$

the equation of motion can be expressed as :

$$A_{n+1} = A_n + K\sin(\theta_n) \qquad \theta_{n+1} = \theta_n + A_{n+1} \bmod 2\pi \ , \tag{5}$$

where K is the dimensionless coupling strength namely :

$$K = \frac{kT}{I} \ . \tag{6}$$

The phase space is the cylinder $\mathcal{C} = \mathbf{T}\times\mathbf{R}$, if A is considered as a real number. If we set

$$f(\theta, A) = (\theta', A') \qquad \theta' = \theta + A + K\sin(\theta) \qquad A' = A + K\sin(\theta) \ , \tag{7}$$

the solution of the equation of motion can be written as :

$$(\theta_{n+1}, A_{n+1}) = f(\theta_n, A_n) \ . \tag{8}$$

f is an analytic diffeomorphism of the cylinder $\mathcal{C}$, which is area preserving, namely $d\theta' \wedge dA' = d\theta \wedge dA$, and a twist map, namely $\partial\theta'/\partial A > 0$, which preserves the ends (see the course of John Mather in this issue). We remark that f also commutes with the translation $A \mapsto A + 2\pi$ of the action variable A in such a way that it also defines a map of the 2-torus $\mathbf{T}^2$.

The orthodox way of quantizing this model consists in choosing the Hilbert space $\mathcal{K} = L^2(\mathbf{T}, d\theta/2\pi)$ as the state space, and replacing L and θ by operators as follows :

$$\mathbf{L} = \frac{\hbar}{i}\frac{\partial}{\partial\theta} \qquad \mathcal{V} = \text{multiplication by } \mathcal{V}(\theta) \ , \tag{9}$$

whenever $\mathcal{V}$ is a continuous 2π -periodic function of the variable θ . Quantum Mechanics requires using a new parameter $\hbar$, the Planck constant which gives rise to a new dimensionless parameter :

$$\gamma = \frac{\hbar T}{I} = 4\pi\frac{\nu_{\rm QM}}{\nu_{\rm CL}} \ , \tag{10}$$

where $\nu_{\rm CL} = 1/T$ is the kicks frequency, whereas $\nu_{\rm QM}$ is the eigenfrequency of the free quantum rotor in a zero magnetic field. To compute the motion, we need to solve Schrödinger's equation, namely, we look for a path $t \in \mathbf{R} \mapsto \psi_t \in \mathcal{K}$ such that :

$$i\hbar\dot{\psi}_t = H(t)\psi_t \qquad H(t) = \frac{\mathbf{L}^2}{2I} + \mu B\mathbf{L} + k\cos(\theta)\sum_{n\in\mathbf{Z}}\delta(t-nT) \ . \tag{11}$$

The δ-kicks may create a technical difficulty. To overcome it let us consider a smooth approximation δ_ϵ of δ given by a non negative L^1-function on $\mathbf{R}$ supported by $[0,\epsilon]$, with integral equal to 1. The solution can be given in term of a convergent Dyson expansion. Then letting ϵ converge to zero, we get the following result (see Appendix 1) :

Theorem 1 *The solution of (11) is given by the following evolution equation :*

$$\psi_{T-0} = F^{-1}\psi_{0-} \qquad F^{-1} = e^{-iA^2/2\gamma}e^{-iK\cos\theta/\gamma}e^{i\hat{y}} \tag{12}$$

where

$$A = T\left(\frac{L}{I} + \mu B\right) \ , \qquad \hat{y} = (\mu B)^2\frac{TI}{\hbar} \ . \tag{13}$$

Let us also introduce the dimensionless magnetic field x :

$$x = -\mu BT \qquad \Rightarrow \qquad \hat{y} = \frac{x^2}{2\gamma} \ . \tag{14}$$

The operators of the form $\mathcal{V}$ whenever $\mathcal{V}(\theta)$ is a continuous 2π -periodic function of the variable θ, can be obtained as the norm limit of polynomials in the operator

$$U = e^{i\theta} \ . \tag{15}$$

In much the same way, one can quantize the action in the torus geometry by considering the operator :

$$V = e^{-iA} \ . \tag{16}$$

U and V are two unitary operators satisfying the following commutation rule :

$$UV = e^{i\gamma}VU \ . \tag{17}$$

The C^*-algebra generated by these two operators is the non commutative analog of the space of continuous functions on the 2-torus. By analogy with the commutative case, this algebra will be seen as the space of continuous functions on a virtual space, the "quantal phase space". Any such function will be the norm limit of polynomials of the form :

$$a = \sum_{\mathbf{m}\in\mathbf{Z}^2,|m|\leq N} a(\mathbf{m}) U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2/2} , \tag{18}$$

where the $a(m)$'s are complex numbers. We denote by $\mathcal{A}_\gamma$ the norm closure of this algebra. Whenever $\gamma = 0$, this algebra coincides with the space $\mathcal{C}(\mathbf{T}^2)$ of continuous functions on the 2-torus. One remarks that $\cos(\theta) \in \mathcal{A}_\gamma$, but there is no way of writing $F_0 = \exp(iA^2/\gamma)$ as an element of $\mathcal{A}_\gamma$ since it is not periodic with respect to A. Therefore $F_0 \notin \mathcal{A}_\gamma$ in general. However, the following properties hold :

$$\text{(i)} F_0 V F_0^{-1} = V \qquad \text{(ii)} F_0 U F_0^{-1} = U V^{-1} e^{i\gamma/2} , \tag{19}$$

so that, setting $\beta_0(a) = F_0 a F_0^{-1}$ for $a \in \mathcal{A}_\gamma$, β_0 defines an automorphism of $\mathcal{A}_\gamma$, which coincides for $\gamma = 0$ with the free rotation f_0 in $\mathbf{T}^2$, namely :

$$f_0(\theta, A) = (\theta + A, A) , \tag{20}$$

In particular if $\gamma \neq 0$, $a \in \mathcal{A}_\gamma$, we get :

$$\beta(a) = FaF^{-1} = e^{iK\cos(\theta)/\gamma} \beta_0(a) e^{-iK\cos(\theta)/\gamma} \in \mathcal{A}_\gamma , \tag{21}$$

which means that β is an automorphism of $\mathcal{A}_\gamma$.

At last, β admits a classical limit as $\gamma \mapsto 0$, namely the automorphism of $\mathcal{C}(\mathbf{T}^2)$ corresponding to the standard map (see section 3 below). For if $\mathcal{V} = K\cos(\theta)$, let us denote by $\mathcal{L}_v$ the "Liouville operator" defined by :

$$\mathcal{L}_v(a) = \frac{\mathcal{V}a - a\mathcal{V}}{i\gamma} , \tag{22}$$

the limit of $\mathcal{L}_v(a)$ as $\gamma \mapsto 0$ coincides with the Poisson bracket of $\mathcal{V}$ with a, and β can be written as :

$$\beta = e^{-\mathcal{L}_v} \circ \beta_0 . \tag{23}$$

To summarize, we have obtained an algebraic framework describing the quantal observables which is completely analogous to the classical description of the system, and which converges to the classical analog as $\gamma \mapsto 0$. In this framework,

(i) the observable algebra $\mathcal{A}_\gamma$ is the non commutative analog of the space $\mathcal{C}(\mathbf{T}^2)$ of continuous functions on the classical phase space $\mathbf{T}^2$.

(ii) the quantal evolution is described through the automorphism β of $\mathcal{A}_\gamma$ which admits the standard map as a classical limit.

Before leaving this section, let us describe the complementary point of view, given in wave Mechanics by the Feynman path integral, which happens to be exact and finite dimensional in this case.

Lemma 1 *If $\psi \in \mathcal{C}^\infty(\mathbf{T})$, then the following formula holds :*

$$\left(e^{-iA^2/2\gamma}\psi\right)(u) = e^{-i\pi/4} \int_{-\infty}^{+\infty} \frac{du'}{\sqrt{2\pi\gamma}} e^{i(u'-u-x)^2/2\gamma} e^{-ix^2/2\gamma} \psi(u') . \tag{24}$$

Proof : From (9)&(13), we get $A = -i\gamma\partial/\partial\theta - x$. If $\psi \in \mathcal{C}^\infty(\mathbf{T})$, let $(\psi_n)_{n\in\mathbf{Z}}$ its Fourier series, so that :

$$\left(e^{-iA^2/2\gamma}\psi\right)(\theta) = \sum_{n\in\mathbf{Z}} e^{-i(\gamma n - x)^2/2\gamma}\psi_n e^{in\theta} = \sum_{n\in\mathbf{Z}} \int_{-\pi}^{+\pi} \frac{d\theta'}{2\pi} e^{in(\theta-\theta')-i(\gamma n-x)^2/2\gamma}\psi(\theta') \ .$$

To compute the distribution kernel coming into this sum, we use the Poisson summation formula :

$$\sum_{n\in\mathbf{Z}} e^{in(\theta-\theta'+x)-i\gamma n^2/2} = \frac{e^{-i\pi/4}}{\sqrt{2\pi\gamma}} 2\pi \sum_{l\in\mathbf{Z}} e^{i(\theta-\theta'+x+2\pi l)^2/2\gamma} \ .$$

Now we perform the change of variables $u' = \theta' + 2\pi l$, $u = \theta$, and the sum over $l \in \mathbf{Z}$ will give rise to an integral over $\mathbf{R}$ with respect to u', leading to (24).

Using (12)&(24), we immediately get the following Feynman path integral representation :

Corollary 1 *For any $t \in \mathbf{N}$ and $\psi \in \mathcal{C}^\infty(\mathbf{T})$, we get :*

$$\left(F^{-t}\psi\right)(u) = \int_{\mathbf{R}^t} \frac{du_1\cdots du_t}{(2\pi\gamma)^{t/2}} e^{-it\pi/4} e^{i\left(\sum_{s=1}^{t}(u_s-u_{s-1}-x)^2/2-K\cos(u_s)\right)/\gamma}\psi(u_t) \ , \tag{25}$$

where $u_0 = u$, and the right-hand-side defines a convergent oscillatory integral which is periodic of period 2π with respect to u.

Remark : The expression contained in the phase factor

$$S(u_1,\cdots,u_t;u_0,x) = \sum_{1\le s\le t-2} \left(\frac{(u_s-u_{s-1}-x)^2}{2} - K\cos(u_s)\right) \ , \tag{26}$$

is nothing but the "Percival" Lagrangean or the "Frenkel-Kontorova" energy functional used by Aubry and Mather to describe the trajectories of the standard map. For indeed the stationnary points of such a Lagrangean are finite sequences $(u_s)_{1\le s\le t}$ satisfying the recursion relation :

$$2u_s - u_{s+1} - u_{s-1} + K\sin(u_s) = 0 \ , \ (1 \le s \le t-1) \ , \ u_t - u_{t-1} - x + K\sin(u_t) = 0 \ .$$

In particular if we set $p_s = u_s - u_{s-1}$ (for $1 \le s \le t$) we get $u_{s+1} = u_s + p_{s+1}$ for $0 \le s \le t-1$, and $p_{s+1} = p_s + K\sin(u_s)$ for $1 \le s \le t-1$, $x = p_t + K\sin(u_t)$, namely we recover the standard map (5) in $\mathbf{R}^2$ now instead of $\mathbf{T}^2$, for a trajectory $(\theta_0, A_0), \cdots, (\theta_t, A_t)$ such that $\theta_0 = u_0 \bmod 2\pi$, and $A_{t+1} = x \bmod 2\pi$.

2 The Rotation Algebra

2.1 The Polynomial Algebra $\mathcal{P}_I$

In this section we define properly the algebra $\mathcal{A}_\gamma$ and we will describe without proof its most important properties. We refer the reader to [BaBeFl] for more details. Actually given an interval I of $\mathbf{R}$, we will rather consider the algebra $\mathcal{A}_I$ which is

roughly speaking the set of continuous sections of the continuous field $\gamma \in I \mapsto \mathcal{A}_\gamma$. The semiclassical limit will be included whenever I contains $\gamma = 0$.
Let I be a compact subset of $\mathbf{R}$. The polynomial algebra $\mathcal{P}_I$ is defined as follows :
- the elements of $\mathcal{P}_I$ are the sequences $(a(\mathbf{m}))_{\mathbf{m}\in\mathbf{Z}^2}$ with finite support, where for each $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}^2$, $a(\mathbf{m}) : \gamma \in I \mapsto a(\mathbf{m}, \gamma) \in \mathbf{C}$ is a complex continuous function on I.
- $\mathcal{P}_I$ admits a natural structure of $\mathcal{C}(I)$-module by setting, for $a, b \in \mathcal{P}_I$, and $l \in \mathcal{C}(I)$:

$$(a + b)(\mathbf{m}) = a(\mathbf{m}) + b(\mathbf{m}) \qquad \lambda a(\mathbf{m}; \gamma) = \lambda(\gamma) a(\mathbf{m}; \gamma) \ . \tag{27}$$

- any element $a \in \mathcal{P}_I$ admits an adjoint $a^\star$ defined by :

$$a^\star(\mathbf{m}; \gamma) = \overline{a(-\mathbf{m}; \gamma)} \ , \tag{28}$$

where $\overline{z}$ denotes the complex conjugate of z in $\mathbf{C}$.
-if $a, b \in \mathcal{P}_I$, their product is defined by :

$$(ab)(\mathbf{m}; \gamma) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} a(\mathbf{m}'; \gamma) b(\mathbf{m} - \mathbf{m}'; \gamma) e^{i\gamma \mathbf{m}' \wedge (\mathbf{m} - \mathbf{m}')} \ , \tag{29}$$

where we have set if $\mathbf{m}', \mathbf{m}'' \in \mathbf{Z}^2$:

$$\mathbf{m}' \wedge \mathbf{m}" = m'_1 m''_2 - m'_2 m''_1 \ . \tag{30}$$

- the topology on $\mathcal{P}_I$, is the direct sum topology obtained from the uniform norm on $\mathcal{C}(I)$.
Denoting by $\mathcal{P}_\gamma$ the algebra $\mathcal{P}_I$ whenever $I = \{\gamma\}$ it follows that $\mathcal{P}_\gamma = \mathcal{P}_{\gamma+4\pi}$. Moreover setting $\alpha(a) = ((-)^{m_1 m_2} a(m))_{\mathbf{m}\in\mathbf{Z}^2}$, α defines a $*$-isomorphism between $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma+2\pi}$. Thus, as far as $\mathcal{P}_\gamma$ is concerned, one will consider that γ is defined mod. 2π. The same definition holds if we replace I by the torus $\mathbf{T}$ namely the continuous functions on I by the continuous 2π-periodic functions on $\mathbf{R}$. We will denote by $\mathcal{P}$ the corresponding algebra.

The following elements in $\mathcal{P}_I$ are remarkable :

$$\mathbf{I}(\mathbf{m}; \gamma) = \delta_{\mathbf{m},\mathbf{0}} \qquad U(\mathbf{m}; \gamma) = \delta_{\mathbf{m},(1,0)} \qquad V(\mathbf{m}; \gamma) = \delta_{\mathbf{m},(0,1)} \ . \tag{31}$$

For indeed, $\mathbf{I}$ is the identity of $\mathcal{P}_I$ whereas U, V, are unitaries namely $UU^\star = U^\star U = VV^\star = V^\star V = \mathbf{I}$, and obey to the commutation rules (17). Moreover, $\mathcal{P}_I$ is algebraically generated by U, V as a $\mathcal{C}(I)$-algebra, namely if $a \in \mathcal{P}_I$, it can be written as :

$$a = \sum_{\mathbf{m}' \in \mathbf{Z}^2} a(\mathbf{m}) U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2/2} \ .$$

It will be convenient to introduce the "Weyl operators" as follows :

$$W(\mathbf{m}) = U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2/2} \ . \tag{32}$$

From the interpretation given in the previous section, it follows that $\mathcal{P}_I$ is the set of trigonometric polynomials over the "non-commutative" 2-torus. In particular if $I = \{0\}$, we recover the convolution algebra, which by Fourier transform is exactly the algebra of usual trigonometric polynomials.

The "evaluation" homomorphism η_γ is defined as the map from $\mathcal{P}_I$ into $\mathcal{P}_\gamma$ by :

$$\eta_\gamma(a) = (a(\mathbf{m}; \gamma))_{\mathbf{m}\in\mathbf{Z}^2} \ . \tag{33}$$

It is immediate to check that η_γ is a $*$-homomorphism, namely it is linear, and preserves the product and the adjoint.

2.2 Canonical calculus

Using the analogy with the space of trigonometric polynomials on the 2-torus, we now define some rules for the differential calculus.
The integral is given by the trace defined by :

$$\tau(a) = a(0) \in \mathcal{C}(I) \; . \tag{34}$$

We will denote by $\tau_\gamma(a)$ the value of $\tau(a)$ at γ. The trace τ is a linear module map from $\mathcal{P}_I$ into $\mathcal{C}(I)$ satisfying :

(i) positivity : $\tau(a^* a) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} |a(\mathbf{m})|^2 \geq 0$, $a \in \mathcal{P}_I$,
(ii) normalization : $\tau(\mathbf{I}) = 1$,
(iii) trace property : $\tau(ab) = \tau(ba)$, $a, b \in \mathcal{P}_I$.

We remark that the value of $\tau(a)$ at $\gamma = 0$ is the 0^{th} Fourier coefficient of $\eta_0(a)$, namely the integral of its Fourier transform :

$$\tau(a)|_{\gamma=0} = \int_{\mathbf{T}^2} \frac{d\theta dA}{4\pi^2} a_{\rm cl}(\theta, A) \; . \tag{35}$$

where $a_{\rm cl}$ is the Fourier transform of $\eta_0(a)$.
The angle average, is defined by the element $\langle a \rangle$ in $\mathcal{P}_I$ given by :

$$\langle a \rangle(\mathbf{m}) = \delta_{m_1,0} a(0, m_2) \; . \tag{36}$$

The map $a \mapsto \langle a \rangle$ is a module-map taking values in the commutative subalgebra $\mathcal{D}_I$ generated by V as a $\mathcal{C}(I)$-module. The usual Fourier transform permits to associate with any element b of $\mathcal{D}_I$ a continuous function of $(\gamma, A) \in I \times \mathbf{T}$ denoted by b_{av} as follows :

$$b_{av}(\gamma, A) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} b(0, m_2; \gamma) e^{-i m_2 A} \; . \tag{37}$$

The mapping $b \in \mathcal{D}_I \mapsto b_{av} \in \mathcal{C}(I \times \mathbf{T})$, is a $*$-homomorphism, namely $(bc)_{av} = b_{av} c_{av}$ and $(b^*)_{av} = b^*_{av}$. We will say that $b \in \mathcal{D}_I$ is positive whenever b_{av} is positive. Using these definitions, the angle averaging satisfies :

$$\begin{array}{rll} \text{(i)} & \text{positivity property :} & \langle a^* a \rangle \geq 0 \; , \; a \in \mathcal{P}_I \\ \text{(ii)} & \text{projection property :} & \langle \langle a \rangle \rangle = \langle a \rangle \; , \\ \text{(iii)} & \text{normalization :} & \langle \mathbf{I} \rangle = 1 \; , \\ \text{(iv)} & \text{conditional expectation :} & \langle ab \rangle = \langle a \rangle b \; , \; \langle ba \rangle = b \langle a \rangle \; , \text{ if } b \in \mathcal{D}_I \; , \; a \in \mathcal{P}_I \; . \end{array} \tag{38}$$

A differential structure is defined on $\mathcal{P}_I$ through the data of two $*$-derivations ∂_θ and ∂_A given by :

$$(\partial_\theta a)(\mathbf{m}) = i m_1 a(\mathbf{m}) \qquad (\partial_A a)(\mathbf{m}) = i m_2 a(\mathbf{m}) \; . \tag{39}$$

These two derivations ∂_μ (if $\mu = \theta, A$) actually commute and satisfy :

$$\begin{array}{rcl} \text{(i)} & \text{they are } \mathcal{C}(I)\text{−linear} & \\ \text{(ii)} & \partial_\mu(a^*) = (\partial_\mu a)^* & a \in \mathcal{P}_I \; , \\ \text{(iii)} & \partial_\mu(ab) = (\partial_\mu a)\, b + a\, (\partial_\mu b) & a, b \in \mathcal{P}_I \; , \\ \text{(iv)} & \partial_\theta U = iU \; , \; \partial_\theta V = 0 \; , \; \partial_A U = 0 \; , \; \partial_A V = -iV \; . & \end{array} \tag{40}$$

Moreover one can exponentiate them, namely defining by $\{\rho_{\theta,A}; (\theta, A) \in \mathbf{T}^2\}$ as the 2-parameter group of $*$-automorphisms given by :

$$\rho_{\theta,A}(a)(\mathbf{m}) = e^{i(m_1\theta - m_2 A)} a(\mathbf{m}) \ , \tag{41}$$

we get :

$$\partial_\mu a = \left(\frac{\partial \rho_{\theta,A}(a)}{\partial \mu}\right)_{\theta=A=0} \qquad \mu = \theta, A \ . \tag{42}$$

Actually $\rho_{\theta,A}$ is a module-$*$-homomorphism such that $(\theta, A) \in \mathbf{T}^2 \mapsto \rho_{\theta,A}(a) \in \mathcal{P}_I$ is continuous and :

$$\rho_{\theta,A} \circ \rho_{\theta',A'} = \rho_{\theta+\theta',A+A'} \ , \tag{43}$$

If $a, b \in \mathcal{P}_I$ their Poisson (or Moyal [Bou]) bracket $\{a, b\}$ is defined as follows :

$$\{a, b\}(\mathbf{m}; \gamma) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} a(\mathbf{m}'; \gamma) b(\mathbf{m} - \mathbf{m}'; \gamma) \frac{2}{\gamma} \sin\left(\frac{\gamma}{2} \mathbf{m}' \wedge (\mathbf{m} - \mathbf{m}')\right) \ , \tag{44}$$

where we set $(\sin x)/x = 1$ for $x = 0$. In particular that for $\gamma = 0$, it coincides with the usual Poisson bracket, namely :

$$\{a, b\}_{\text{cl}} = \{a_{\text{cl}}, b_{\text{cl}}\} = \partial_\theta a_{\text{cl}} \partial_A b_{\text{cl}} - \partial_A a_{\text{cl}} \partial_\theta b_{\text{cl}} \ , \tag{45}$$

From (44), the right-hand-side defines a continuous function of γ on I, so that the Poisson bracket $\{a, b\}$ still belongs to $\mathcal{P}_I$. The "Liouville operator" associated to $w \in \mathcal{P}_I$ is the module map defined by :

$$L_w(a) = \{w, a\} \ , \ a \in \mathcal{P}_I \ . \tag{46}$$

The properties of this operator are the following :

$$\begin{array}{lccc} \text{(i)} & L_w \text{ is } \mathcal{C}(I)\text{-linear} & & \\ \text{(ii)} & L_w(a^*) = L_{w^*}(a)^* & w, a \in \mathcal{P}_I \ , & \\ \text{(iii)} & L_w(ab) = L_w(a)b + aL_w(b) & w, a, b \in \mathcal{P}_I \ , & \\ \text{(iv)} & [L_w, L_{w'}] = L_{\{w,w'\}} \text{ (Jacobi's identity)} & w, w' \in \mathcal{P}_I \ . & \end{array} \tag{47}$$

We also remark that

$$\tau\left(\rho_{\theta,A}(a)\right) = \tau(a) \quad \tau(\{a, b\}) = 0 \quad a, b \in \mathcal{P}_I \ , (\theta, A) \in \mathbf{T}^2 \ , \tag{48}$$

which is equivalent to the "integration by parts formula" :

$$\tau(\partial_\mu a \cdot b) = -\tau(a \cdot \partial_\mu b) \qquad \tau\left(L_w(a) \cdot b\right) = -\tau\left(a \cdot L_w(b)\right) \ , \tag{49}$$

2.3 The Rotation Algebra $\mathcal{A}_I$

In order to get all continuous functions on our non commutative torus, we ought to define the non commutative analog of the uniform topology on $\mathcal{P}_I$. This can be done by remarking that in the commutative case, the uniform topology is defined through a C^*-norm, namely a norm on the algebra which satisfies :

$$\|ab\| \leq \|a\| \|b\| \qquad \|a^* a\| = \|a\|^2 \ . \tag{50}$$

The importance of this relation comes from the fact that such a norm is actually entirely defined by the algebraic structure, namely it is given by the spectral radius of a^*a. Therefore, the algebraic structure is sufficient and the uniform topology becomes natural.
To construct such a norm, one uses the representations of $\mathcal{P}_I$. A "representation" of $\mathcal{P}_I$ is a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a separable Hilbert space, and π is a $*$-homomorphism from $\mathcal{P}_I$ into the algebra $\mathcal{B}(\mathcal{H}_\pi)$ of bounded linear operators on $\mathcal{H}_\pi$. The formulæ (17)&(18) give an example of representation for which $\mathcal{H}_\pi = L^2(\mathbf{T}, d\theta/2\pi)$. In particular $\pi(U)$, $\pi(V)$ will be unitary operators on $\mathcal{H}_\pi$ so that if $a \in \mathcal{P}_I$, one gets (if $\|f\|_I$ denotes the sup norm in $\mathcal{C}(I)$) :

$$\|\pi(a)\| \leq \sum_{\mathbf{m}\in\mathbf{Z}^2} \|a(\mathbf{m})\|_I < \infty \ . \tag{51}$$

Two representations $(\pi, \mathcal{H}_\pi)$ and $(\pi', \mathcal{H}_{\pi'})$ are equivalent whenever there is a unitary operator S from $\mathcal{H}_\pi$ into $\mathcal{H}_{\pi'}$ such that for every $a \in \mathcal{P}_I$:

$$S\pi(a)S^{-1} = \pi'(a) \ . \tag{52}$$

Up to unitary equivalence, one can always assume that $\mathcal{H}_\pi = \ell^2(\mathbf{N})$, so that the family of all equivalence classes of representations of $\mathcal{P}_I$ is a set denoted by $\mathrm{Rep}(\mathcal{P}_I)$. We remark that the norm $\|\pi(a)\|$ depends only upon the equivalence class of π. We then define a seminorm on $\mathcal{P}_I$ by :

$$\|a\|_I = \sup\{\|\pi(a)\|; \pi \in \mathrm{Rep}(\mathcal{P}_I)\} \ . \tag{53}$$

This notation agrees with the sup-norm on $\mathcal{C}(I)$ if $a \in \mathcal{C}(I)$. Then one has [BaBeFl] :

Proposition 1 *The mapping $a \in \mathcal{P}_I \mapsto \|a\|_I \in \mathbf{R}_+$ is a C^*-norm.*

Remark : The only non trivial fact in this statement is that it is a norm, namely that $\|a\|_I = 0$ implies $a = 0$.

Definition 1 *The algebra $\mathcal{A}_I$ (resp. $\mathcal{A}$) is the completion of $\mathcal{P}_I$ (resp. $\mathcal{P}$) under the norm $\|\cdot\|_I$ (resp. $\|\cdot\|_{\mathbf{T}}$). $\mathcal{A}$ is called the "universal rotation algebra".*

Proposition 2 *1)-Any representation of $\mathcal{P}_I$ extends in a unique way to a representation of $\mathcal{A}_I$*
2)-If $\mathcal{B}$ is any C^*-algebra, and β is a $*$-homomorphism from $\mathcal{P}_I$ to $\mathcal{B}$, then β extends in a unique way as a $*$-homomorphism from $\mathcal{A}_I$ to $\mathcal{B}$.
3)-Any pointwise continuous group of $*$-automorphisms of $\mathcal{P}_I$ extends in a unique way as a norm pointwise continuous group of $*$-automorphisms of $\mathcal{A}_I$.
4)-The trace τ and the angle average $\langle\cdot\rangle$ satisfy :

$$\|\tau(a)\|_I \leq \|a\|_I \quad \|\langle a\rangle\|_I \leq \|a\|_I \quad a \in \mathcal{P}_I \ , \tag{54}$$

and therefore they extend uniquely to $\mathcal{A}_I$.
5)-The norm $\|\cdot\|_I$ satisfies :

$$||a||_I = \sup_{\gamma\in I} ||\eta_\gamma(a)|| \quad a \in \mathcal{P}_I \ . \tag{55}$$

In practice the explicit computation of the norm does not require the knowledge of every representation. It is enough to have a faithfull family, namely a family $\{\pi_j\}_{j\in J}$ where J is a set of indices, such that $\pi_j(a) = 0$ for all j's implies $a = 0$. In other words $\cap_{j\in J}\mathrm{Ker}(\pi_j) = \{0\}$. We recall that the spectrum $\mathrm{Sp}(a)$ of an element a of a C^*-algebra with unit $\mathcal{A}$, is the set of complex numbers z such that $z\mathbf{I} - a$ is non invertible in $\mathcal{A}$.

Proposition 3 *Let $(\pi_j)_{j\in J}$ be a faithfull family of representations of the C^*-algebra $\mathcal{A}$, then :*

$$\|a\|_I = \sup_{j\in J}\|\pi_j(a)\| \quad \mathrm{Sp}(a) = \mathrm{closure}\{\cup_{j\in J}\mathrm{Sp}(\pi_j(a))\}\ . \tag{56}$$

In particular if π is faithfull (namely if J contains only one point), $\|a\|_I = \|\pi(a)\|$ and $\mathrm{Sp}(a) = \mathrm{Sp}(\pi(a))$.

2.4 Smooth functions in $\mathcal{A}_I$

Beside $\mathcal{P}_I$, one can define many dense subalgebras of $\mathcal{A}_I$ playing the role of various subspaces of smooth functions.

(i) For $N \in \mathbf{N}$, the algebra $\mathcal{C}^N(\mathcal{A}_I)$ of N-times differentiable elements of $\mathcal{P}_I$ is the completion of $\mathcal{A}_I$ under the norm :

$$\|a\|_{\mathcal{C}^N,I} = \sum_{0\leq n,n';n+n'\leq N}\frac{1}{n!}\frac{1}{n'!}\|\partial_\theta^n\partial_A^{n'}(a)\|_I\ . \tag{57}$$

(ii) $\mathcal{C}^\infty(\mathcal{A}_I) = \cap_{N\geq 0}\mathcal{C}^N(\mathcal{A}_I)$. It coincides with the set of elements $a = (a(\mathbf{m}))_{\mathbf{m}\in\mathbf{Z}^2}$ with rapidly decreasing Fourier coefficients. It is a nuclear space, similar to the Schwartz space on the torus. Its dual space $S(\mathcal{A}_I)$ is a space of non commutative tempered distributions which can be very useful in investigating unbounded elements.

(iii) $\mathcal{H}^s(\mathcal{A}_I)$ is the Sobolev space, namely the completion of $\mathcal{P}_I$ under the Sobolev norm :

$$\|a\|_{\mathcal{H}^s,I} = \left(\tau(a^*a) + \tau(a^*(-\Delta)^{s/2}a)\right)^{1/2} \qquad \Delta = \partial_\theta^2 + \partial_A^2\ , \tag{58}$$

where $-\Delta$ is the Laplacean on the non commutative torus. The imbedding $\mathcal{H}^{s'}(\mathcal{A}_I) \mapsto \mathcal{H}^s(A_I)$ is compact if $s' > s$ and $\mathcal{C}^\infty(\mathcal{A}_I) = \cap_{s\geq 0}\mathcal{H}^s(\mathcal{A}_I)$, showing that $\mathcal{C}^\infty(\mathcal{A}_I)$ is a nuclear space.

(iv) An element of $\mathcal{A}_I$ is holomorphic in some domain D of $(\mathbf{T}+i\mathbf{R})^2$ if the continuous mapping $(\theta, A) \in \mathbf{T}^2 \mapsto \rho_{\theta,A}(a) \in \mathcal{A}_I$, can be extended as a holomorphic function on D. A special interesting case consists in considering the algebra $\mathcal{A}_I(r)$ for $r > 0$, obtained by completing $\mathcal{P}_I$ with the norm :

$$\|a\|_{r,I} = \sup_{\gamma\in I}\sum_{\mathbf{m}\in\mathbf{Z}^2}|a(\mathbf{m};\gamma)|e^{r|m|_1}\ , \tag{59}$$

where $|m|_1 = |m_1| + |m_2|$. Then $\mathcal{A}_I(r)$ becomes a Banach $*$-algebra of holomorphic elements in the strip $D(r) = \{|\mathrm{Im}\theta| < r\ ,\ |\mathrm{Im}A| < r\}$.

(v) Let us consider now the case for which I is an open interval, and let $\mathcal{P}_I^\infty$ be the subalgebra of $\mathcal{P}_I$ the elements of which have Fourier coefficients given by $\mathcal{C}^\infty$-functions on I. Let us define the operator ∂_γ on $\mathcal{P}_I^\infty$ by :

$$\partial_\gamma a = \left(\frac{\partial a(\mathbf{m})}{\partial\gamma}\right)_{\mathbf{m}\in\mathbf{Z}^2}\ . \tag{60}$$