INTRODUCTION TO PROJECTIVE GEOMETRY AND MODERN ALGEBRA

BY ROBERT A. ROSENBAUM, *Wesleyan University*

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PREFACE

The late nineteenth-century development of algebra and geometry was marked by continual interplay between the two subjects. Algebraic methods were applied to the study of geometry, and geometry was helpful in interpreting algebraic results. Indeed, many concepts now regarded as of primarily algebraic interest had their genesis in geometric problems, and today's "n-dimensional geometry" is a branch of algebra couched in geometric language solely for reasons of analogy. This book returns to the spirit of the historical development as providing a natural and effective approach to these topics, exploiting their relationships whenever possible.

This course is designed for undergraduates at a stage when it may bridge the gap between the usual intuitive introduction to calculus and the more rigorous and abstract treatment of advanced mathematics courses. For some students an abrupt change from "intuition" to "rigor" may be a traumatic experience, accentuated by instructions to *prove* theorems when it is not clear what one is permitted to *assume*. We attempt to effect a smooth transition from plausibility to proof, indicating at every stage our presuppositions and level of rigor. This is not, then, intended as a tightly organized presentation of projective geometry and linear algebra; other related topics of interest (e.g., groups) are included. Highly structured presentations have greater significance to a student already somewhat familiar with their subject matter. An acquaintance with the main concepts and methods of modern algebra and geometry is needed both by students who will go on to more specialized courses in these subjects and by those in other fields.

The starred exercises in the problem sets are not necessarily difficult. The star is used to indicate that the problem is especially significant, or that the result is likely to be encountered again later in the book. Hence all the starred problems should at least be read, if not worked.

There are many gaps in the textual exposition which are left to the reader to fill. Attention is called to such gaps by the reference, "See Problem 00 at the end of this section." The student is invited to attempt to fill the gap before turning to the cited problem at the end of the section. If he is unsuccessful, he may find a hint as to how to proceed by reading the cited problem, for such hints are given in the case of a number of nonroutine problems.

Serious use of this device can, we believe, encourage initiative in the student and give him both a firmer grasp of the material and greater enjoyment than would come from reading a presentation complete in every detail. Because of these features, the amount of material covered is somewhat greater than the number of pages in the book may indicate.

Chapters 1, 2, and 3 may be covered as quickly or as slowly as the background and interests of the students dictate. For students with a substantial background in algebra, the main topics in the book can be covered in one semester. Most classes with little college experience in algebra and geometry (beyond analytic geometry) will find ample material for a full year course.

The idea for this book originated when I was a visiting professor at Swarthmore, many years ago. The initial drafts were written at Reed, during tenure of a fellowship of the Fund for the Advancement of Education. A Cottrell Grant from The Research Corporation and grants from the research funds of Wesleyan University were helpful in developing the manuscript. Colleagues at several colleges have made useful suggestions, have caught errors, and have offered interesting exercises; T. H. M. Crampton, R. G. Long, A. P. Mattuck, G. M. Merriman, and Hing Tong should be especially mentioned. I owe thanks to all these individuals and institutions. But it transcends my power to express my indebtedness to my principal colleague—Louise Johnson Rosenbaum.

Middletown, Conn. R. A. R.
January 1963

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CHAPTER 1

Geometrical Introduction (I)

Mathematics has been traditionally divided into three categories: geometry, algebra, and analysis. There are no clear-cut dividing lines separating these disciplines, and each is useful in, and in some instances essential to, the solving of problems in the other two fields. Some analysis (i.e., material concerned with limits and limiting processes) will appear in this book, but its role will be a subordinate one. Geometry and algebra will be presented as closely related subjects, each interesting in itself, each helpful in obtaining results and clarifying concepts in the other field, and both furnishing methods which can fruitfully be applied in the physical sciences, in engineering, and in some of the social sciences.

Of these three branches of mathematics, geometry has the longest history as a formal discipline. This first chapter will be devoted to reviewing some familiar subject matter and methods of geometry, and to introducing some new types of problems and new methods.

1–1 CLASSIFICATION OF GEOMETRIES

There are many sorts of geometries, with two principal modes of classification: by subject matter and by method. One division on the basis of subject matter, for example, is that between plane and solid geometry. More significant subject-matter divisions, giving rise to a "hierarchy" of geometries, will receive considerable attention in later chapters.

A classification on the basis of method, namely, the synthetic and the analytic, is familiar to all students, at least in fact if not by name. In the synthetic approach, the proof proceeds by logical argument to the desired conclusion from geometrical hypotheses and already proven theorems. In the analytic method, the geometrical data of the problem are translated into algebraic terms through the medium of a coordinate system, laws of algebra are utilized to transform the data and to draw (algebraic) conclusions from them, and finally the results are translated back into geometrical language, presumably as the desired result.

Two remarks should be made about the foregoing characterizations of the synthetic and analytic methods. (i) The analytic method has no necessary connection with "analysis," one of the three branches of mathematics mentioned in the introduction to this chapter. (ii) Deductive reasoning from hypothesis to conclusion is not restricted to synthetic geometry. A rigorous approach to analytic geometry would also start with axioms (synonyms: postulates, assumptions), permitting the translation from geometrical language to algebraic and back again, and justifying the algebraic manipulations which we perform. Indeed, the current view of mathematicians is that *every* mathematical system is one in which each conclusion is obtained by a chain of logical argument from certain axioms assumed at the start. The characteristic feature of "pure" synthetic geometry is that almost all axioms refer to geometrical entities, and that a coordinate system (with its attendant arithmetic and algebra) does not enter. More will be said about axiomatized systems in Section 2–1.

As an exercise in comparison of methods, we will sketch analytic and synthetic proofs that the altitudes of a triangle are concurrent. A set of lines is *concurrent* if all the lines of the set pass through one point. The point of concurrency of the altitudes is called the *orthocenter* of the triangle.

It will be especially instructive if you close the book before reading the outlines of proofs given below, and try to work out the analytic and synthetic proofs yourself.

Analytic. Let the vertices of the triangle be $A(a, 0)$, $B(b, 0)$, $C(0, c)$. Then the line BC has slope $-c/b$, and AC has slope $-c/a$. Hence the altitudes through A and B have slopes b/c and a/c, respectively. Therefore, the altitude through A has the equation $bx - cy = ab$, while the altitude through B has the equation $ax - cy = ab$. Solving these equations simultaneously gives $x = 0$ as the abscissa of the point of intersection of the altitudes through A and B. But the altitude through C clearly lies along the y-axis, i.e., the third altitude is the line $x = 0$. □

Synthetic. Through each of the vertices A, B, C of the given triangle, draw a line parallel to the opposite side of the triangle. In this way we construct a triangle, $A'B'C'$, say. The altitudes of ABC are the perpendicular bisectors of the sides of $A'B'C'$. But the perpendicular bisectors of the sides of a triangle are known to meet in a point (the center of the circumscribed circle, or *circumcenter*, of the triangle). □

Comments. (i) The position of the coordinate axes relative to the triangle is virtually standard; any student, after only a brief introduction to analytic geometry, would make the choice suggested above, or another equally convenient. After the axes have been chosen, the rest of the analytic procedure is also standard.

But there is a trick to the synthetic proof. An individual might be quite experienced, ingenious, and perceptive, and still not hit upon the easy approach.

This is a common situation; the synthetic method often requires considerable ingenuity and power of visualization on the part of the student, while the analytic method, once a convenient position has been chosen for the coordinate axes, usually proceeds in fairly routine fashion. The synthetic method often provides a short and elegant proof if essential features are noted and correctly interpreted, or if appropriate construction lines are drawn. The analytic method has the advantage of being "sure," albeit sometimes "slow."

(ii) Note that both proofs involve much background material. To consider only one item of the analytic proof, we observe that the formula for the slope of a line comes from the concept of similar triangles. Likewise, one step of the synthetic proof is based on the concurrency of the perpendicular bisectors of the sides of a triangle, an easily obtained result, usually proved early in a geometry course.

PROBLEM SET 1-1

Try each of the following problems by both the synthetic and the analytic methods. In some cases one of the methods will prove to be decidedly simpler than the other.

1. Prove that the diagonals of a parallelogram bisect each other, and the converse.

2. Prove that the medians of a triangle meet in a point (called the *centroid* of the triangle) which is two-thirds the distance from each vertex to the midpoint of the opposite side.

*3. Prove that the bisectors of the angles of a triangle are concurrent. (The point of concurrency is the center of the inscribed circle, abbreviated "incenter.")

*4. Let A_1, A_2, A_3, A_4, P, Q be 6 distinct points on a circle. Show that

$$\frac{\sin \angle A_3PA_1}{\sin \angle A_3PA_2}\cdot\frac{\sin \angle A_4PA_2}{\sin \angle A_4PA_1} = \frac{\sin \angle A_3QA_1}{\sin \angle A_3QA_2}\cdot\frac{\sin \angle A_4QA_2}{\sin \angle A_4QA_1}.$$

5. Let A, B, C, D be any four points, not all on one line and not necessarily all lying in a plane. Let P, Q, R, S be the midpoints of the segments AB, BC, CD, DA, respectively. What can be said about the figure $PQRS$?

6. (a) Suppose that a variable line through a fixed point P meets a fixed circle in A and B. The point P may be inside, on, or outside the circle. Show that the product $PA \cdot PB$ is constant.

(b) If P is outside the circle, with a tangent from P touching the circle at T, show that $PA \cdot PB = PT^2$.

7. Given a fixed line l and a fixed point A not on l. Point P moves so that its distance from l always equals the distance AQ, where Q is the foot of the perpendicular dropped from P to l. What is the locus of P?

8. (a) In a plane, what is the locus of a point, the sum of the squares of whose distances from two fixed points is constant?

(b) Same as (a), in three dimensions.

9. What is the locus of the midpoint of a line segment of constant length whose end points move on two fixed intersecting perpendicular lines?

10. (a) Show that the locus of the midpoint of a line segment of constant length whose end points move on two fixed intersecting nonperpendicular lines is an ellipse.

(b) What is the locus of a point which divides a line segment of length l in the ratio $r: (l - r)$ if the ends of the line segment move on two fixed, intersecting lines?

(c) Show that in 9(b), the locus is a circle if and only if the lines are perpendicular and $r = l/2$.

11. What is the locus of the midpoint of a line segment of constant length whose endpoints move on two fixed, perpendicular, nonintersecting lines?

12. Show that in a plane, the locus of the center of a circle tangent to two fixed unequal circles which are external to each other consists of both branches of two hyperbolas whose foci are the centers of the fixed circles.

13. Let A, B, C, D be consecutive vertices of a parallelogram and let X, Y be arbitrary points on AB, CD, respectively.

(a) Let AY and DX meet at P, and BY and CX meet at Q. Show that the line PQ bisects the area of the parallelogram.

(b) Let AY and CX meet at R; BY and DX meet at S. What can be said about the line RS?

14. Let D be an arbitrary point of the altitude AH of triangle ABC. Let BD meet AC at E, and CD meet AB at F. Show that angle AHE equals angle AHF. Are there any cases which need special treatment? (From a Putnam Prize Exam.)

15. Prove that, if two medians of a triangle are equal in length, then the triangle is isosceles.

16. In triangle ABC (Fig. 1–1), AD and BE meet on the bisector of angle C, and $AD = BE$. Show that the triangle is isosceles.

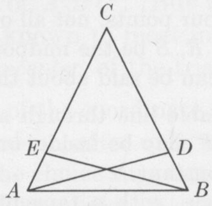

FIGURE 1–1

17. In triangle ABC with $AC = AB$, let D be the midpoint of BC, let E be the foot of the perpendicular from D to AC, and let F be the midpoint of DE. Show that AF is perpendicular to BE. (From *The American Mathematical Monthly*.)

18. In triangle ABC, D lies on BC and E on AC. $AC = BC$, the measure of angle C is 20°, that of angle DAB is 50°, and that of angle EBA is 60°. Prove that the measure of angle DEB is 30°.

*19. The points P, Q, R, S lie on the sides AB, BC, CD, DA of the quadrilateral $ABCD$. Show that if PQ and RS meet on AC, then PS and QR meet on BD.

20. Given a triangle ABC and a point P not on any side of the triangle. Let M_1, M_2, M_3 be the centroids of triangles PAB, PBC, PCA. Prove that for a fixed triangle ABC and for the position of P arbitrary, the triangle $M_1M_2M_3$ has a fixed size and shape. Does P have to be in the plane of triangle ABC? Can you describe the size and location of triangle $M_1M_2M_3$ relative to the size and location of triangle ABC?

21. (a) In triangle ABC, let P, Q, R be points on AB, BC, CA such that $AP/AB = \frac{1}{3} = BQ/BC = CR/CA$. Show that the area of the triangle whose sides are CP, AQ, BR is $\frac{1}{7}$ the area of triangle ABC.
(b) If the fraction $\frac{1}{3}$ in part (a) is changed to $1/n$, what does the fraction $\frac{1}{7}$ become?

22. (a) Suppose that a secant l meets a circle in A and B, the midpoint of the chord AB being M. Let P_1Q_1 and P_2Q_2 be chords of the circle through M. Suppose that P_1Q_2 and P_2Q_1 meet l in G and H, respectively, and that P_1P_2 and Q_1Q_2 meet l in R and S, respectively. Show that $MG = MH$ and that $MR = MS$. (Brooks)
(b) With reference to part (a), let the tangents at P_1 and Q_1 meet l in U and V. Show that $MU = MV$. (Morgan)

23. Given a parallelogram $ABCD$ with a circle passing through A. Let the circle meet AB, AC, AD in P, Q, R, respectively. Prove that $AB \cdot AP + AD \cdot AR = AC \cdot AQ$. (Morgan)

24. In Fig. 1–2, PX and PY are the tangents to the circle from the arbitrary point P; XY is also tangent to the circle (at A), and AB is a diameter of the circle. Show that $XC = YA$. Is your proof valid for all positions of P outside the circle? (Morgan)

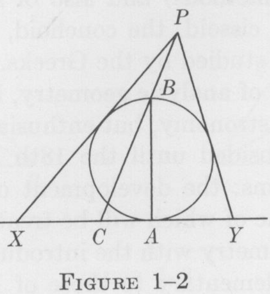

FIGURE 1–2

*25. In a plane, l, l', m are three concurrent lines, and O, P are two points not lying on any of the lines. To each point X of l we make correspond the point X' of l' such that OX and PX' meet on m. Show that all the lines XX' are concurrent at a point Q. What can be said about the location of Q?

*26. Let the circles $\mathcal{C}_1, \mathcal{C}_2$ intersect in P, P'; circles $\mathcal{C}_2, \mathcal{C}_3$ intersect in Q, Q'; and circles $\mathcal{C}_3, \mathcal{C}_1$ intersect in R, R'. Show that the lines PP', QQ', RR' are concurrent. Must the circles all lie in one plane?

27. *Definition.* Two lines are *skew* if they do not lie in a plane.

(a) Let l, m be skew lines, with P, Q any points on l, and P', Q' any points on m. Show that the lines PP', QQ' are skew.

(b) Let each pair of lines: l, m; m, n; n, l be skew. Are there lines which intersect all three lines l, m, n? Describe fully.

1–2 "MODERN" GEOMETRY

The theorems used in the foregoing set of problems are all standard results of Euclidean geometry, the sort of propositions known to the Greeks over 2000 years ago and proved by them by the synthetic method. Virtually no advance over the geometry of the Greeks was made until the introduction of the analytic method (Descartes, 1637).

The preceding sentences represent an oversimplification of the situation. Some of the Greek geometers used a sort of coordinate system, but not in a systematic fashion, probably because of their poorly developed algebraic notation.

"Descartes' merits," writes Struik in his book, *A Concise History of Mathematics,* "lie above all in his consistent application of the well developed algebra of the early Seventeenth Century to the geometrical analysis of the Ancients, and, by this, in an enormous widening of its applicability . . . the first analytic geometry of conic sections which is fully emancipated from Apollonios appeared only with Euler's *Introductio* (1748)."

Analytic geometry permitted easy and systematic study of the conic sections (which had already been investigated by the Greeks by what we now consider laborious methods) and also of some higher plane curves, such as the cycloid, the cissoid, the conchoid, and the limaçon, some of which likewise had been studied by the Greeks.

After the introduction of analytic geometry, its methods became standard for certain work in astronomy, but enthusiasm for geometry as a subject for study again subsided until the 18th and 19th centuries. The revival took several forms: the development of new geometries (i.e., of new subject matter) some of which will be treated in this book; a marked extension of analytic geometry with the introduction of algebraic methods different from and supplementary to those of Descartes (this will be an important concern for us); and a renewal of interest in the methods and general subject matter of Euclid, leading to the discovery of many beautiful theorems (principally relating to simple figures like the triangle and the circle) which had not been suspected by the Greeks or by any mathematicians in the intervening 2000 years.

Listed below are a few of the many theorems discovered in the 18th and 19th centuries. (Some of them had also been known to the Greeks.) These particular theorems have been selected because of their relationship to each other and to other theorems of more general nature, which will appear later in our work. It is suggested that you attempt to prove them, by synthetic or analytic means. If you are unsuccessful, look up a proof in one of the texts listed after the problems. (It will be surprising if you succeed in proving more than a few of these theorems, but you should at least understand their statements.)

PROBLEM SET 1-2

1. The circumcenter, the orthocenter, and the centroid of a triangle are collinear. (A set of points is *collinear* if all the points of the set lie on a straight line.) The distance from the centroid to the orthocenter is equal to twice the distance from the centroid to the circumcenter. (Euler line, 18th century)

2. Let R be the radius of the circumscribed circle of a triangle, r the radius of the inscribed circle, and d the distance between the circumcenter and the incenter. Then (Euler, 18th century)

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}.$$

*3. The lines joining the vertices of a triangle (Fig. 1-3) to a given point not on the sides of the triangle determine on the sides of the triangle six segments such that the product of three nonconsecutive segments is equal to the product of the remaining three (Ceva, 17th century):

$$AC' \cdot BA' \cdot CB' = C'B \cdot A'C \cdot B'A \qquad \text{or} \qquad \frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

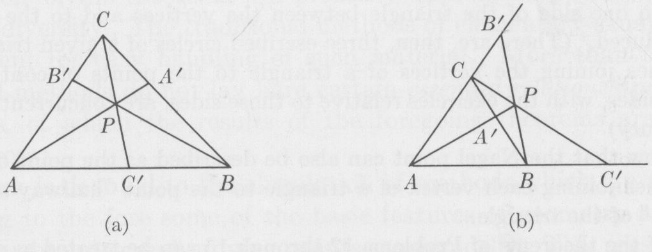

FIGURE 1-3

*4. *Discussion of directed distances.* It is often convenient to choose one of the two "senses" on a line as *positive,* in which case the opposite sense is *negative.* Thus, if the arrowhead denotes the positive sense in Fig. 1–4, AB is positive and CB is negative. Whichever is the choice of positive sense on the line, if B is the midpoint of the segment AC, then $CA/AB = -2$. With this convention for directed distances, it is possible to state the following converse of Ceva's theorem.

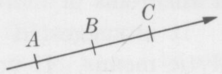

FIGURE 1–4

Given a triangle ABC with A', B', C' on BC, CA, AB, respectively, such that $AC' \cdot BA' \cdot CB' = C'B \cdot A'C \cdot B'A$, where the equality is valid both in magnitude and in sign. Then AA', BB', CC' are concurrent.

*5. Let a transversal t meet the sides BC, CA, AB of triangle ABC in A', B', C'. Then, considering directed distances (Menelaus, 1st century, A.D.)

$$AC' \cdot BA' \cdot CB' = -C'B \cdot A'C \cdot B'A, \qquad \text{or} \qquad \frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = -1.$$

*6. Given a triangle ABC with A', B', C' on BC, CA, AB, respectively, such that $AC' \cdot BA' \cdot CB' = -C'B \cdot A'C \cdot B'A$. Then A', B', C' are collinear. (Converse of Menelaus' theorem)

Each of the theorems of Problems 7 through 11 can be treated as a corollary of the converse of Ceva's theorem.

7. The medians of a triangle are concurrent.

8. The internal angle bisectors of a triangle are concurrent.

9. The altitudes of a triangle are concurrent.

10. The lines joining each of the vertices of a triangle to the point of contact on the opposite side of the inscribed circle are concurrent. (Gergonne, 19th century)

11. (a) *Definition.* An *escribed circle* (or, *excircle*) of a triangle is a circle tangent to one side of the triangle between the vertices and to the other two sides produced. (There are, then, three escribed circles of a given triangle.)

The lines joining the vertices of a triangle to the points of contact of the opposite sides, with the excircles relative to those sides, are concurrent. (Nagel, 19th century)

(b) Show that the Nagel point can also be described as the point of intersection of lines joining each vertex of a triangle to the point "halfway around the perimeter" of the triangle.

Each of the theorems of Problems 12 through 15 can be treated as a corollary of the converse of Menelaus' theorem.

12. The external angle bisectors of a triangle meet the opposite sides in three collinear points.

13. Two interior angle bisectors and the bisector of the exterior angle at the third vertex meet their respective opposite sides in collinear points.

14. The tangents to the circumcircle of a triangle at its vertices meet the opposite sides of the triangle in three collinear points. What can be said concerning the special cases involving parallelism?

15. (a) Let A, B, C be three points of a line l, and X, Y, Z three points on another line m, coplanar with l. Then the points of intersection of AY and XB, of AZ and XC, and of BZ and YC are collinear. (Theorem of Pappus, 4th century, or Pascal, 17th century)

(b) Can you use the result of (a) to solve easily Problem 13 (a), Section 1–1?

16. (a) In a given plane, let A, B, C be fixed collinear points, and l, m fixed lines. Let x, y, z be lines through A, B, C, respectively, with x and y meeting on l, and y and z meeting on m. Show that the locus of the intersection of z and x is a line. (Euclid)

(b) Same as part (a), except that A, B, C are *not* collinear. Then the locus of the intersection of z and x is a conic. (Maclaurin, 18th century)

(c) Generalization of part (a): In a given plane, let $A_1, A_2, \ldots, A_n$ be fixed collinear points, and $l_1, l_2, \ldots, l_{n-1}$ be fixed lines. Let $x_1, x_2, \ldots, x_n$ be lines through $A_1, A_2, \ldots, A_n$, respectively, with x_1 and x_2 meeting on l_1, x_2, and x_3 meeting on $l_2, \ldots, x_{n-1}$, and x_n meeting on l_{n-1}. Show that the locus of the intersection of each pair not already mentioned is a line. (Pappus)

(d) Obtain a result related to part (c) as part (b) is related to part (a).

The following references may be helpful.

N. A. COURT, *College Geometry*. New York: Barnes and Noble, Inc., 1952.

R. A. JOHNSON, *Advanced Euclidean Geometry*. New York: Dover Publications, 1929.

D. J. STRUIK, *A Concise History of Mathematics*. New York: Dover Publications, 1948.

Drawings of the figures associated with the foregoing theorems will impress a reader with the simplicity and beauty of the results, and a study of the proofs of the theorems will probably impress him with their difficulty and diversity. The traditional methods of elementary geometry are not sufficient for easy handling of such material. More than that, the traditional methods do not lay bare certain *essential relationships* of configurations, of which the results of the foregoing theorems are merely special cases.

This book is devoted to the elucidation of methods which are powerful in bringing to the fore some of the basic features of geometry and algebra, and in obtaining and proving results economically. This presentation begins in Chapter 4; the remainder of Chapter 1 and Chapter 2 involve