Mechanics of Solids

Volume III

Theory of Viscoelasticity, Plasticity, Elastic Waves, and Elastic Stability

Editor: C. Truesdell

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VOLUME III

Theory of Viscoelasticity, Plasticity, Elastic Waves, and Elastic Stability

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Preface.

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The mechanical response of solids was first reduced to an organized science of fairly general scope in the nineteenth century. The theory of small elastic deformations is in the main the creation of CAUCHY, who, correcting and simplifying the work of NAVIER and POISSON, through an astounding application of conjoined scholarship, originality, and labor greatly extended in breadth the shallowest aspects of the treatments of particular kinds of bodies by GALILEO, LEIBNIZ, JAMES BERNOULLI, PARENT, DANIEL BERNOULLI, EULER, and COULOMB. Linear elasticity became a branch of mathematics, cultivated wherever there were mathematicians. The magisterial treatise of LOVE in its second edition, 1906 – clear, compact, exhaustive, and learned – stands as the summary of the classical theory. It is one of the great "gaslight works" that in BOCHNER's words[1] "either do not have any adequate successor[s] ... or, at least, refuse to be superseded ...; and so they have to be reprinted, in ever increasing numbers, for active research and reference", as long as State and Society shall permit men to learn mathematics by, for, and of men's minds.

Abundant experimentation on solids was done during the same century. Usually the materials arising in nature, with which experiment most justly concerns itself, do not stoop easily to the limitations classical elasticity posits. It is no wonder that the investigations LOVE's treatise collects, condenses, and reduces to symmetry and system were in the main ill at ease with experiment and unconcerned with practical applications. In LOVE's words, they belong to "an abstract conceptual scheme of Rational Mechanics". He concluded thus his famous Historical Introduction:

The history of the mathematical theory of Elasticity shows clearly that the development of the theory has not been guided exclusively by considerations of its utility for technical Mechanics. Most of the men by whose researches it has been founded and shaped have been more interested in Natural Philosophy than in material progress, in trying to understand the world than in trying to make it more comfortable. From this attitude of mind it may possibly have resulted that the theory has contributed less to the material advance of mankind than it might otherwise have done. Be this as it may, the intellectual gain which has accrued from the work of these men must be estimated very highly. The discussions that have taken place concerning the number and meaning of the elastic constants have thrown light on most recondite questions concerning the nature of molecules and the mode of their interaction. The efforts that have been made to explain optical phenomena by means of the hypothesis of a medium having the same physical character as an elastic solid body led, in the first instance, to

[1] SALOMON BOCHNER: "Einstein between centuries", Rice Univ. Stud. 65 (3), 54 (1979).

the understanding of a concrete example of a medium which can transmit transverse vibrations, and, at a later stage, to the definite conclusion that the luminiferous medium has not the physical character assumed in the hypothesis. They have thus issued in an essential widening of our ideas concerning the nature of the aether and the nature of luminous vibrations. The methods that have been devised for solving the equations of equilibrium of an isotropic solid body form part of an analytical theory which is of great importance in pure mathematics. The application of these methods to the problem of the internal constitution of the Earth has led to results which must influence profoundly the course of speculative thought both in Geology and in cosmical Physics. Even in the more technical problems, such as the transmission of force and the resistance of bars and plates, attention has been directed, for the most part, rather to theoretical than to practical aspects of the questions. To get insight into what goes on in impact, to bring the theory of the behaviour of thin bars and plates into accord with the general equations - these and such-like aims have been more attractive to most of the men to whom we owe the theory than endeavours to devise means for effecting economies in engineering constructions or to ascertain the conditions in which structures become unsafe. The fact that much material progress is the indirect outcome of work done in this spirit is not without significance. The equally significant fact that most great advances in Natural Philosophy have been made by men who had a first-hand acquaintance with practical needs and experimental methods has often been emphasized; and, although the names of Green, Poisson, Cauchy show that the rule is not without important exceptions, yet it is exemplified well in the history of our science.

LOVE's treatise mentions experiment rarely and scantly. Its one passage concerning experiment in general, § 63, in effect warns its reader to have a care of experimental data because of their indirectness.

In an irony of history the ever-increasing use of mathematical notation in physical science, to the point that now often works on experiment are dominated by their authors' seemingly compulsive recourse to mathematical formulae interconnected by copied or adapted bits of old mathematical manipulation, LOVE's treatise is sometimes in reproaches upon modern "pure" or "abstract" researchers held up as a model of practical, applied theory.

Experiment on the mechanical properties of solids became in the later nineteenth century a science nearly divorced from theory. Nevertheless, no great treatise on experiment fit to be set beside LOVE's on theory ever appeared. Even such books of experiment as were published seem to have in the main taken positions either dominated by theory, usually crude, verbose, and ill presented, or flatly opposed to theory.

The modern reader will cite as objections against the foregoing coarse summary many individual masterpieces that do not support it: brilliant comparisons of theory with experiment by ST. VENANT, independent experiments of fundamental importance by WERTHEIM, CAUCHY's marvellously clear mathematical apparatus for conceiving stress and arbitrarily large strains and rotations, theories of internal friction and plasticity proposed by BOLTZMANN, ST. VENANT, and others. If he is searching for antecedents of what has happened in the second half of the twentieth century, he is abundantly right in citing these and other achievements of the nineteenth while passing over the work of the ruck, but in that century's gross product of solid mechanics they are exceptions that prove rules.

In planning this volume on the mechanics of solids for the *Encyclopedia of Physics* I designed

1) To provide a treatise on experimental mechanics of solids that, not dominated by mathematical theory and not neglecting the work of the eighteenth and nineteenth centuries in favor of recent, more popular, and more costly forays, should be comparable in authority, breadth, and scholarship with LOVE's.
2) To provide treatises on basic, mathematical theory that would stand at the level of LOVE's while in their own, narrower scope supplanting it by compact and efficient development of fundamentals, making use of modern, incisive, yet elementary mathematics to weave together old and recent insights and achievements.
3) To illustrate the power of modern mathematical theory and modern experiment by articles on selected topics recently developed for their intellectual and practical importance, these two qualities being closer to each other than to some they may seem.

I encouraged the authors to meet the standard established by LOVE in just citation and temperate respect for the discoverers.

The reader will be able to form his own judgment of such success and failure as did accrue.

On the first head, experiment in general, the reader will find the treatise by Mr. BELL, filling all of Part 1. While it is not primarily a historical work, the historian S. G. BRUSH pronounced it in 1975 "the most important new publication by a single author" on the history of physics.

On the second head, the reader should not expect to find the basic ideas of solids treated *ab novo* or in isolation. The general and unified mechanics of EULER and CAUCHY, in which fluids, solids, and materials of other kinds are but instances, has come into its own in our day. No wise scientist now can afford to shut out solids when studying fluids or to forget the nature and peculiarities of fluids when studying solids. The two are but extreme examples in the class of systems comprised by mechanics. Articles in Parts 1 and 3 of Volume III of the *Encyclopedia: The Classical Field Theories* and *The Non-Linear Field Theories of Mechanics,* are cited so often by the authors writing in Volume VIa as to make it fatuous to deny that they provide the basic concepts, structures, and mathematical apparatus for the articles on theoretical mechanics of solids. In particular *The Non-Linear Field Theories* goes into such detail regarding large mechanical deformation as to allow most of the text in Volume VIa to concentrate upon small strain.

This much understood, we see that while Mr. BELL's volume provides, at last, a monument of exposition and scholarship on experiment, the articles by Messrs. GURTIN, CARLSON, FICHERA, NAGHDI, ANTMAN, and FISHER & LEITMAN, by Mrs. GEIRINGER, and by Mr. TING together provide a modern treatise on mathematical theories of the classical kinds. The survey of theories of elastic stability by Messrs. KNOPS & WILKES, now justly regarded as the standard reference for its field, necessarily considers deformations that need not be small.

Coming finally to application, in which theory and experiment complement one another, the reader will find major examples in the articles by Messrs. CHEN; NUNZIATO, WALSH, SCHULER and BARKER; and THURSTON. Many more topics of application might have been included. I regret that I could not secure articles about them. The most serious want is a survey of applications of linear elasticity to problems of intrinsic or applied interest that have arisen in this century and that illustrate the power of new mathematical analysis in dealing with special problems. A long article of that kind, a veritable treatise, was twice contracted and twice defaulted. Fortunately the gap thus left has been abundantly and expertly filled by Mr. VILLAGGIO, *Qualitative Methods in Elasticity,* Leyden, Noordhoff, 1977.

Baltimore, December, 1983 C. TRUESDELL

Contents.

The Linear Theory of Viscoelasticity.

By

MARSHALL J. LEITMAN and GEORGE M. C. FISHER.

A. Introduction.

1. Plan and scope of this article. It has long been known that material behavior is not always elastic. Indeed, many substances exhibit the property of hereditary response. That is, the present state of stress depends not only upon the present state of deformation, but also upon previous states. This property is revealed experimentally in the phenomena of creep, stress relaxation, and the intrinsic attenuation of propagating waves. The nonlinear mechanical theories of materials with memory have been developed to characterize such behavior.[1] More specifically, the linearly viscoelastic material is a model which seeks to characterize hereditary effects within the context of an infinitesimal linearized theory. A theory based on this model should, in the absence of hereditary effects, reduce to the classical linearized theory of elasticity.

The linear theories of elasticity and viscoelasticity have much in common. Indeed, there is considerable formal similarity in their developments. This is not illusory since many results from the elastic theory have direct generalizations to the viscoelastic theory.

In this article we focus not only upon the formal similarities but also upon the intrinsic differences of the two theories. We attempt to isolate those conceptual and mathematical difficulties which arise over and above those inherent in elastic problems. The greatest of these arises directly from the inclusion of hereditary effects. In elastic theories the relationship between stress and strain is finite-dimensional, whereas, in a hereditary theory this relationship is generally infinite-dimensional. Even within the context of a linear theory, the additional complications may be formidable. It will be seen that the ramifications of this assertion permeate the entire body of the theory.

We develop our subject entirely within the context of an infinitesimal linearized theory. In this respect our presentation is phenomenological and follows

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Acknowledgements. We are greatly indebted to our teacher Professor M. E. GURTIN for his friendship and guidance since our student days. To Professor C. TRUESDELL we express our gratitude for the opportunity to write this article and for his patience and help in its preparation. We should like to thank Professor A. J. LOHWATER of Case Western Reserve University and J. S. COURTNEY-PRATT, F. E. FROEHLICH, F. T. GEYLING, and R. N. THURSTON, all of Bell Telephone Laboratories, for their encouragement, support, and valuable discussions. We also wish to acknowledge the cheerful and essential assistance given to us by Mrs. N. G. ROSELLI in the typing of the many versions of the manuscript. Last, but certainly not least, we thank our wives CAROLYN LEITMAN and ANN FISHER for their help, encouragement, and great patience.

[1] GREEN and RIVLIN [1957, *2*], NOLL [1958, *16*], COLEMAN and NOLL [1961, *2*], [1964, *5*], COLEMAN [1964, *4*], COLEMAN and MIZEL [1966, *5*].

the ideas set down first by BOLTZMANN[2] and later by VOLTERRA.[3] Mathematically, our formulation is patterned after the more recent work of KÖNIG and MEIXNER[4] and GURTIN and STERNBERG.[5] We have been greatly influenced by the investigations of the non-linear theories of mechanics and the concept of fading memory. Most significant of these are the fundamental studies of GREEN and RIVLIN, NOLL, COLEMAN and NOLL, COLEMAN, and COLEMAN and MIZEL.[6]

We have attempted throughout to adopt a style which may be rendered precise, in the mathematical sense, without overburdening the reader with cumbersome notation. This article is intended as a sequel to the Linear Theory of Elasticity (LTE) by GURTIN, which appears in the preceding part of this volume. The completeness of that article enables us to use it as a primary reference.

Our survey is divided into four chapters. In Chap. A we provide some requisite technical background and notation. Since we adopt most of the notation and results developed in LTE, we are able to concentrate on those mathematical concepts which are central to the viscoelastic theory but absent in the elastic theory. A self-contained formulation of the foundations of the linear theory is presented in Chap. B. We do not approach this theory as a linear approximation to a non-linear theory but proceed directly within the phenomenological framework of a linear theory. In Chap. C we present the fundamental mathematical results of the quasi-static theory. By the term "quasi-static" we mean that all inertial effects are systematically neglected. We include the inertial effects in Chap. D, which concerns the full dynamic theory and the propagation of waves in viscoelastic media. Throughout, we concentrate on presentation of results without usually giving detailed proofs.

We are cognizant of certain omissions which might properly belong in an article of this type. However, our selection of topics has been based upon limitations of space coupled with a desire to give precedence to fundamental mathematical aspects of the theory.

We have made only casual reference to the vast experimental literature. For more information in this regard, the reader is referred to the reviews of FERRY and KOLSKY.[7]

We have paid little attention to the differential models for viscoelastic behavior—finite networks of springs and dashpots. These models are somewhat special and a separate treatment using them is unnecessary. We do, however, include a discussion of their place in the general linear theory. We take as fundamental the classical Boltzmann model and the notion of linear superposition.

Our treatment of material symmetry is brief. It is readily seen that the concept is the same for both elastic and viscoelastic materials. There is an extensive treatment of this topic in LTE which obviates a parallel treatment in this article. The only type of symmetry singled out for special consideration is that of isotropy. Furthermore, we do not separately consider the problems associated with constrained materials such as incompressible media. Our discussion is restricted to general compressible viscoelastic solids. The extension to constrained solids and fluids may be effected along lines of the general theories of mechanics.[8]

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[2] BOLTZMANN [1874, *1*], [1878, *1*].
[3] VOLTERRA [1909, *1*], [1913, *1*].
[4] KÖNIG and MEIXNER [1958, *11*].
[5] GURTIN and STERNBERG [1962, *10*].
[6] Cited in footnote 1, p. 1.
[7] FERRY [1970, *4*], KOLSKY [1963, *10*], [1969, *3*].
[8] Cf. TRUESDELL and NOLL [1965, *29*].

Only the linearization based upon infinitesimal deformations is considered. It is possible, however, to develop a linear theory based upon finite deformations. This has been done, for example, by COLEMAN and NOLL.[9]

For brevity we have omitted completely the important discussion of thermodynamics for viscoelastic materials. A sound basis for this theory is presented in the work of COLEMAN.[10] Other work of interest includes that of BIOT and SCHAPERY[11] as well as the research concerning free energy, recoverable work, and related work bounds as exemplified by the articles of DAY, BREUER, MARTIN and PONTER, and BREUER and ONAT.[12] Furthermore, BIOT's[13] well known development of the concept of hidden variables has led VALANIS[14] to an interesting discussion of the viscoelastic potential function. COLEMAN and GURTIN[15] have generalized the internal state variables approach in the context of modern thermodynamics. THURSTON[16] has identified a set of hidden variables as ensemble-averaged occupation numbers which obey a relaxation type of differential equation and has subsequently derived, from microscopic concepts, the macroscopic theory of linear viscoelasticity as the appropriate linearization.

We do not mention the notion of thermorheologically simple materials in the sense of SCHWARZL and STAVERMAN.[17] Although this concept seems quite useful, especially for experimental investigations, the subject is still in a state of growth and the reader is referred to the works of LEADERMAN, FERRY, MUKI and STERNBERG, and MORLAND and LEE[18] for additional information.

Stability analysis for functional differential equations as applied to materials with memory is still a subject in its infancy. Even at this stage, however, we can refer to the works of COLEMAN and MIZEL and DAFERMOS,[19] who shed some light on the subject.

Throughout, we have not examined solutions to particular problems in linear viscoelasticity. This has meant the exclusion of, among others, the general boundary value problem, which includes the contact problem, the problem of ablating boundaries, numerical techniques, and the effect of boundaries in wave propagation.

Finally, we observe that no precise relationship between the quasi-static and dynamic theories has yet been revealed.

2. Notation. Vectors, tensors, and linear transformations. The primary source for notation and basic mathematical notions is the article in the preceding part of this volume entitled The Linear Theory of Elasticity (LTE) by M. E. GURTIN. We use the notation in Subchaps. B.I and B.II of LTE virtually without change. However, we depart from this procedure for Subchap. B.III of LTE (Functions of position and time) since our requirements in this regard are somewhat different. *All definitions and notation not explicitly established in this article are established by* GURTIN *in LTE.*

[9] COLEMAN and NOLL [1961, *2*], [1964, *5*].
[10] COLEMAN [1964, *4*].
[11] BIOT [1958, *3*], [1965, *2*], SCHAPERY [1962, *20*], [1964, *23*], [1965, *26*].
[12] DAY [1970, *2*], BREUER [1969, *1*], MARTIN and PONTER [1966, *17*], BREUER and ONAT [1963, *1*], [1964, *1*].
[13] BIOT [1965, *2*].
[14] VALANIS [1968, *6*].
[15] COLEMAN and GURTIN [1967, *2*].
[16] THURSTON [1968, *5*].
[17] SCHWARZL and STAVERMAN [1952, *2*].
[18] LEADERMAN [1958, *12*], MUKI and STERNBERG [1961, *5*], MORLAND and LEE [1960, *13*].
[19] COLEMAN and MIZEL [1968, *2*], DAFERMOS [1970, *1*].