PURCELL

3rd edition

calculus with analytic geometry

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CALCULUS WITH ANALYTIC GEOMETRY, 3rd edition
Edwin J. Purcell

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preface

This book presents a first course in calculus and analytic geometry. The author has tried to write it in a simple and straightforward style, with ample explanations, an abundance of illustrative examples, and carefully graded exercise sets, so that it will be unusually suitable for a reader of average ability to study alone or with minimum help from a teacher. There is enough material for three semesters.

Starting with simple first principles, each new concept is motivated by a natural, intuitive introduction. Seven basic concepts are stressed: function, limit of a function, continuity, derivative, antiderivative, definite integral, and infinite series. Effort is made to impress on the reader that a mastery of these ideas is indispensable in acquiring a genuine understanding of calculus.

At the same time, there is an abundance of material dealing with the degree of accuracy of computed results and with other aspects of the computational work that is so important for progress in science and technology.

Since ϵ, δ methods are necessary for a proper definition of the limit of a function, a very thorough treatment of inequalities and absolute values precedes it. However, the use of ϵ, δ methods is minimized in later work by utilizing limit theorems wherever possible.

Set notation is introduced early. It is employed when clearly advantageous, but not slavishly.

Vectors in two- and three-dimensional space are presented with a firm mathematical basis and are applied widely. Vectors do not supplant a sound foundation in Cartesian plane analytic geometry but complement it and make possible a more concise formulation of some of the theorems that were first derived in the classical

Cartesian manner. In three-dimensional analytic geometry, vectors are used from the outset. This avoids duplication of effort and contributes to a better understanding of both subjects.

To make this third edition more readable for the increasing numbers of students with less preparation for calculus than formerly, the following improvements have been made.

Many new illustrative examples with complete solutions clarify the theorems, definitions, and techniques. Students who study them should be able to do most of the exercises.

The exercise sets have been reworked and excessive algebraic manipulation has been eliminated. Each set now starts with easy variations of the illustrative examples, progresses through exercises of increasing difficulty, and concludes with some to challenge the stronger students.

Each chapter begins with an intuitive preview of the main ideas to be discussed and their relation to what has gone before. This helps the reader to see the developing calculus as a whole rather than as a series of isolated processes.

At the end of each chapter there is a set of review exercises. A student often finds it easy to work the exercises in a particular section because the method has just been explained. But a set of miscellaneous review exercises based on the material of an entire chapter causes the student to review the chapter and gain a better understanding of it.

Many proofs have been simplified and some of the more tedious ones have been moved to the appendix.

Stronger students will find that the logical development of this third edition and the careful statement of its theorems maintain the integrity of earlier editions. The organization has been improved by moving infinite series forward to Chapter 14, so that the first fifteen chapters now constitute a two semester course in single variable calculus. Chapters 16 to 19 treat the calculus of two or more variables.

The preliminary material has been shortened in order to start the actual calculus sooner. The chapter on the definite integral has been rewritten; it is simpler, more direct, and easier to understand. The treatment of trigonometric functions has been much improved.

There are new sections on Lagrange multipliers and on surface area.

Throughout this book the principal definitions and theorems are prominently labeled, numbered, and displayed, both for easy reference and to keep the main structure of the material before the reader's eyes. The number 7.3.4, for example, refers to the fourth numbered definition or theorem in Section 3 or Chapter 7. The number 14.6 refers to Section 6 of Chapter 14. Fig. 11-5 indicates the fifth figure in Chapter 11.

The most used theorems and definitions are printed in color to enable the student to concentrate his efforts to advantage.

I wish to thank many users of the earlier editions, both faculty and students, for their comments, criticism and encouragement. My thanks are also due to my wife, Bernice Lee Purcell, who made the index and typed the manuscript.

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1 preliminaries

For students who are familiar with the preliminary concepts discussed in this chapter, a careful reading of most of it will suffice. However, *inequalities* and *absolute value* are so important in calculus that a mastery of Sections 1.5 and 1.6 is indispensable for success in studying this book. These topics are seldom covered adequately in high school, and a large proportion of the exercises in Sections 1.5 and 1.6 should be worked.

1.1 INTRODUCTION

Prior to the seventeenth century, algebra and geometry were studied as separate, unconnected subjects. The Greeks had perfected elementary geometry two thousand years ago, and in the centuries that followed the Hindus and Arabs cultivated algebra. Their algebra dealt with numbers, whereas Euclidian geometry was concerned with points, lines, planes, and the like.

There seemed little connection between algebra and geometry until the seventeenth century when two French mathematicians, René Descartes (1596–1650), who was also a philosopher, and Pierre de Fermat (1601–1655), invented a method, now called *analytic geometry*, that uses algebraic operations and equations to solve geometric problems; their method also shed new light on algebra by exhibiting its equations as geometric curves.

The basis for analytic geometry was Descartes' coordinate system, which associated the numbers of algebra with the points of geometry. By means of Cartesian coordinates, large parts of algebra and geometry were seen to be two aspects of the same thing, somewhat as two different languages may express the same meaning. For instance the algebraic statement "Two distinct equations of the first degree in two variables have a single common solution or none" is equivalent to the geometric theorem "Two distinct lines in the same plane intersect in a single point or are parallel."

The names generally associated with the invention of *calculus* are Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). Newton, an Englishman, developed calculus as a tool for his investigations in physics and astronomy. The German, Leibniz, was a universal genius who, independently of Newton and almost simultaneously, also developed calculus.

Calculus is based on the properties of numbers, and by using a Cartesian coordinate system, much of calculus can be presented in geometric terms. Thus the recently discovered analytic geometry was an ideal prelude to the invention of calculus.

Calculus, unlike the mathematics that preceded it, is the study of change and growth. The two basic processes of calculus are *differentiation* and *integration*. Differentiation gives the instantaneous rate of change of a varying quantity, and integration measures the total effect of continuous change. The key to Newton's and Leibniz's success in developing the calculus was their insight into the intimate relation between differentiation and integration as inverse processes, somewhat as multiplication and division of numbers are inverse operations.

Many scattered ideas from calculus were known to predecessors of Newton and Leibniz, even as far back as Archimedes (287–212 B.C.), who, without any algebra, succeeded in finding the areas of circles and regions under a parabola. For circles, he computed the areas of inscribed regular polygons of more and more sides. As the number of sides increased, the areas of the polygons increased and approached the area of the circle as a limit (Fig. 1-1). This is an example of integration.

The area of a circle is the limit of the area of an inscribed regular polygon of n sides as the number of sides, n, increases indefinitely

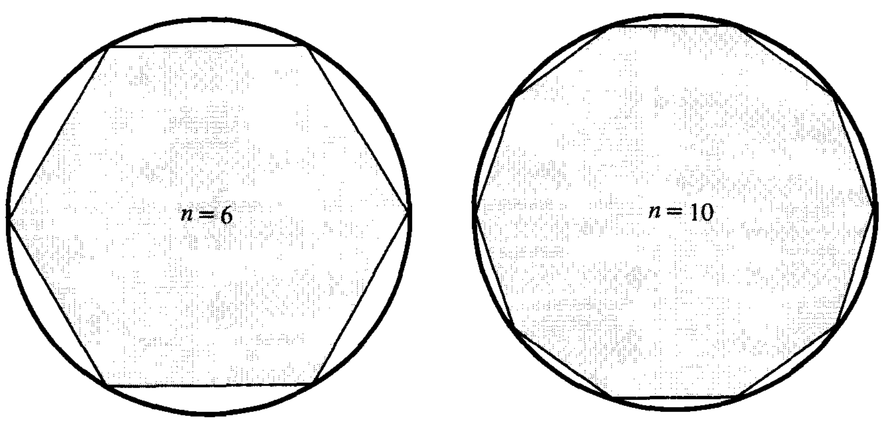

Figure 1-1

In the generation just before Newton, the problem of finding maximum and minimum values of a function was solved for some individual cases by finding the points on its graph where the tangent line is horizontal (Fig. 1-2). This led to a method for determining the direction of the tangent line to a curve at any point on the curve.

Let P be an arbitrarily chosen point on a curve and draw the secant line through P and a neighboring point Q on the curve (Fig. 1-3). Draw the vertical and horizontal

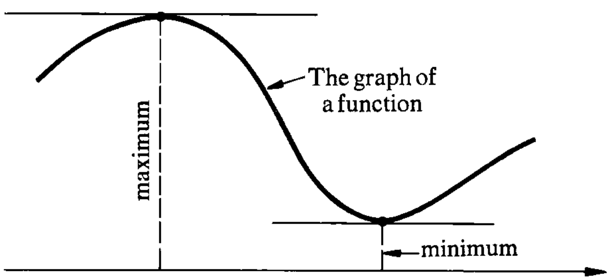

Figure 1-2

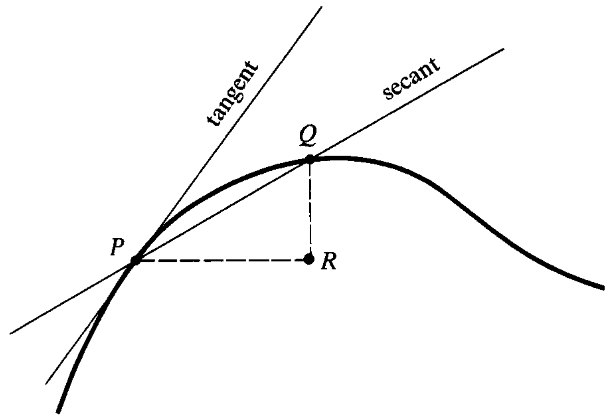

Figure 1-3

line segments, RQ and PR; the ratio of their lengths, RQ/PR, is a measure of the steepness of ascent of a point on the secant and hence of the direction of the secant. Keeping P fixed on the curve, allow Q to approach P along the curve. This causes the secant PQ to rotate about P and approach the position of the tangent line at P. The limit of the ratio RQ/PR as Q approaches P along the curve gives the direction of the tangent line at P. This is an example in differentiation.

Notice that in both examples the word *limit* was used. Limit is the most important concept in calculus and is what distinguishes calculus from all previous mathematics.

Thus Newton and Leibniz were not the first to differentiate or integrate. In particular, Isaac Barrow, Newton's teacher at Cambridge, understood the area problem and the tangent problem and probably knew that they were inverse to each other. The importance of Newton and Leibniz in calculus resulted from their consolidation of the known fragments into a general method, incorporating what is now known as the fundamental theorem, which is applicable to very large classes of functions, both algebraic and transcendental. Leibniz also devised a good notation, much of which is still being used.

Today calculus is essential in engineering and the physical sciences, and is being used more and more in biology and such social sciences as economics, sociology, and psychology. Without calculus, one could not design radar systems or cyclotrons, to name just a few. Calculus is used to determine the orbits of earth satellites and the paths for space travel.

Calculus is generally considered to be one of the greatest intellectual achievements of mankind.

1.2 SETS

A **set** is a collection of things. Some examples of sets are the letters in our alphabet, all American citizens, the positive integers, and the positions on a baseball team.

The **elements** of a set are the objects belonging to the set; they may or may not be material. In this book we shall be chiefly concerned with sets of real numbers and sets of points. The statement ***a* is an element of the set *S*** is symbolized by

$$a \in S,$$

and ***a* is not an element of *S*** is symbolized by $a \notin S$.

A set is **defined** when its description is sufficient to enable us to determine whether any arbitrary object belongs to the set. For instance, if S is the set of all integers greater than $\frac{4}{5}$, then $7 \in S$, $\frac{9}{4} \notin S$, and $-3 \notin S$. It is essential that if a is any object whatever, the definition of a set will enable us to give the unqualified answer "Yes" or "No" to the question "Does a belong to the set?" Thus "all beautiful women" fails to define a set because the decision of membership would be a matter of opinion.

When the number of elements of a set is finite, we can define the set by listing its elements. For example, the set consisting of the numbers π, $\sqrt{2}$, and 8 can be written $\{\pi, \sqrt{2}, 8\}$. Other sets are $\{a, b, c, d\}$ and $\{2, 3, 5, 7, 11\}$.

A different kind of set is

$$S = \{\{-1, 6\}, \{8, 16, 24\}, \{z, w\}\}.$$

This is a **set of sets** (or a collection of sets) whose three elements are the sets $\{-1, 6\}$, $\{8, 16, 24\}$, and $\{z, w\}$. Notice that $-1 \notin S$, although $\{-1, 6\} \in S$.

If the number of elements of a set is not finite, or if it is not convenient to list all the elements of a set, some rule that enables us to determine whether any given object belongs to the set will suffice. To illustrate, "the set of all numbers that can be expressed in the form $2n$, where n is an integer" defines the set of even integers.

The symbol

$$\{x \mid \cdots\}$$

means "the set of elements x such that"; the three dots here stand for some statement or statements about the elements of the set that clearly define the set. For example, $\{x \mid x \text{ is a real number and } 2x^2 - 5x - 3 = 0\}$ is the set of real numbers x such that $2x^2 - 5x - 3 = 0$ is true; in other words, it is the set consisting of the real roots of $2x^2 - 5x - 3 = 0$, which is the set $\{3, -\frac{1}{2}\}$. Again, $\{x \mid x \text{ is a positive integer and } x \text{ is less then } 10\}$ is the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Another example is $\{x \mid x \text{ is a negative integer and } x^2 - x - 6 = 0\}$, which is the set $\{-2\}$.

Notice that $\{x \mid x \text{ is a real number and } x^2 + 1 = 0\}$ contains no elements at all. It is the **empty set** and is represented by $\varnothing$. Thus $\varnothing = \{x \mid x \text{ is a real number}, x^2 + 1 = 0\}$, $\varnothing = \{\theta \mid \sin\theta = 1.5\}$, and $\varnothing = \{y \mid y \text{ is a living person and } y \text{ signed the Declaration of Independence}\}$.