Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Number Theory

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Number Theory

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INTRODUCTION

This is the fourth volume of papers presented at the New York Number Theory Seminar. Since 1982 the Seminar has been meeting every Tuesday afternoon during the academic year at the Graduate School and University Center of the City University of New York. The goal of the Seminar is to provide a forum for the exposure of new results in number theory and allied fields in the New York metropolitan area. Mathematicians who plan to be in New York and would like to attend or lecture in the Seminar are encouraged to contact the organizers.

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SOME CONJECTURED RELATIONSHIPS BETWEEN THETA FUNCTIONS AND EISENSTEIN SERIES ON THE METAPLECTIC GROUP

BY DANIEL BUMP AND JEFFREY HOFFSTEIN

This research was supported by NSF Grants # DMS 8612896 and # DMS 8519916. We would like to express our thanks to S. J. Patterson for many helpful discussions.

Suzuki [**6**] considered the Fourier coefficients of a theta function on the four-fold cover of $GL(2)$. Despite courageous efforts, he was only able to obtain partial information about these coefficients. This failure was explained by the work of Kazhdan and Patterson [**2**], who showed that if $r < n-1$, the methods of Hecke theory only yield partial information about the Fourier coefficients of theta functions on the n-fold cover of $GL(r)$, owing to the fact that the local representations do not have unique Whittaker models. Nevertheless, in the special case $r = 2$, $n = 4$, Patterson [**4**], [**5**] was able to formulate a conjecture which would specify the unknown coefficients up to sign, as square roots of Gauss sums. This work inspired a further paper of Suzuki [**7**], but unfortunately it is not clear to us precisely what is proved in this latter paper.

We shall review the evidence of Patterson, and develop further evidence of our own. Essentially, our theme is that if Patterson's conjecture is true, there are identities between various Dirichlet series which arise as Fourier coefficients of Eisenstein series or as Rankin-Selberg convolutions. By comparing the functional equations and the locations of the poles of these Dirichlet series, we become convinced that the conjecture must be true. Indeed, in some sense the conjecture must be true "on average" owing to the locations of these poles.

We shall finally state a very general conjecture asserting the equality of two Rankin-Selberg convolutions of theta functions. These convolutions may also be interpreted (conjecturally) as a Fourier coefficient of an Eisenstein series on the metaplectic group, generalizing a key relation in our discussion of Patterson's conjecture. This conjecture allows us to predict many values of (and relations between values of) Fourier coefficients of theta functions beyond what is predicted by the theory of Kazhdan and Patterson. We also conjecture that the Rankin-Selberg convolution of a metaplectic cusp form with a theta function may be interpreted as the Fourier coefficient of an Eisenstein series.

Because the forms which we shall consider are automorphic with respect to congruence subgroups, which have multiple cusps, the various Dirichlet series which we shall consider will need congruence conditions. These congruence conditions also complicate the functional equations of these Dirichlet series. As one sees for example in [**6**], keeping track of these nuances involves some rather tedious bookkeeping. We shall not state these congruence conditions explicitly, because they are a distraction, and because we have not worked out all the details arising from them. Nor shall we state the functional equations precisely, or compute the Gamma factors which go with them. *Thus many of the formulas contained herein should be taken as suggestive rather than strictly truthful.* We hope that this transgression will be excused on the grounds that it allows us to tell our story more freely.

Let F be a number field which, for simplicity, we assume to be totally complex. Let n be a fixed positive integer, and assume that the group μ_n of n-th roots of unity in C is contained in F. Let $\left(\frac{a}{b}\right)$ be the n-th power residue symbol for the field F, which is defined for coprime a, b in the ring $\mathcal{O}$ of integers in F, and takes values in μ_n. Among the properties of this symbol which we shall need, it satisfies

$$\left(\frac{aa'}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{a'}{b}\right), \qquad \left(\frac{a}{bb'}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{b'}\right),$$

$$\left(\frac{a}{b}\right) = \left(\frac{a'}{b}\right) \qquad \text{if } a \equiv a' \bmod b,$$

and the reciprocity law

$$\left(\frac{a}{b}\right) = (b,a)\left(\frac{b}{a}\right),$$

where (b, a) is a "Hilbert symbol".

We shall eventually be concerned with the particular case where $F = Q(i)$. In this case, any ideal which is prime to $\lambda = 1 + i$ has a unique generator which is congruent to 1 mod λ^3. We shall always use that generator. Thus if p is a prime, we always assume that $p \equiv 1 \bmod \lambda^3$. In this case (for such a and b), the Hilbert symbol

$$(a, b) = (-1)^{\frac{1}{2}(Na-1)\,\frac{1}{2}(Nb-1)}.$$

Kubota proved that

$$\kappa\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c}{d}\right)$$

defines a character of the congruence subgroup $\Gamma(n^2)$ of $SL(2, \mathcal{O})$. Furthermore, if we embed $SL(2, \mathcal{O})$ into $SL(r, \mathcal{O})$ by sending a 2×2 matrix into the upper right hand corner of an $r \times r$ matrix (with ones elsewhere on the diagonal), then the Kubota character κ extends to a character κ of a congruence subgroup of $SL(r, \mathcal{O})$. We shall be concerned with *metaplectic forms*, which are automorphic forms on such a congruence subgroup, formed with the Kubota character.

Firstly, let us consider automorphic forms on $SL(2)$ which satisfy

$$\phi(\tau) = \kappa(\gamma)\,\phi(\gamma\tau)$$

for γ in the congruence subgroup $\Gamma(n^2)$ and τ in the homogeneous space, which is product of hyperbolic 3-spaces. As was noted by Hecke, Wohlfahrt and Shimura, the theory of Hecke operators for such forms is different from the theory of Hecke operators for nonmetaplectic forms. Let us review such a theory (a more careful treatment may be found in Bump and Hoffstein [**1**]). We assume that ϕ has a Fourier expansion

$$\phi(\tau) = \sum_m a(m)\, Nm^{-1/2}\, W(m\tau),$$

where W is a "Whittaker function" (essentially a product of K-Bessel functions).

The Hecke operators are double cosets whose elementary divisors are n-th powers. If p is a prime of $\mathcal{O}$ which does not divide n, decompose the double coset

$$\Gamma\xi\Gamma = \bigcup_i \xi_i, \qquad \xi_i = \gamma_i \xi \delta_i,$$

where

$$\xi = \begin{pmatrix} 1 & \\ & p^n \end{pmatrix}, \qquad \gamma_i, \delta_i \in \Gamma.$$

Then define

$$(T_{p^n}\phi)(\tau) = \sum_i \kappa(\gamma_i)\, \kappa(\delta_i)\, \phi(\xi_i \tau).$$

As in Bump and Hoffstein [**1**], one may explicitly compute the decomposition of the double coset, and consequently, the effect of the Hecke operators on the Fourier coefficients. In the case $n = 4$, the m-th Fourier coefficient of $T_{p^4}\phi$ is (to oversimplify somewhat)

$$A(m) = Np^2\, a\left(\frac{m}{p^4}\right) + Np\,(m,p)\, g_3\left(\frac{m}{p^2}, p\right)\, a\left(m/p^2\right))$$
$$+ Np\,(m,p^2)\, g_2\left(\frac{m}{p}, p\right)\, a(m) + Np\,(m,p^3)\, g_1(m,p)\, a(mp^2) + Np^2\, a(mp^4).$$

Here the Gauss sums are defined by

$$g_r(m,p) = \sum_{k \bmod p} \left(\frac{k}{p}\right)^r e\left(\frac{mk}{p}\right),$$

in terms of the fourth power residue symbol, where e is a certain fixed character of F mod $\mathcal{O}$. We interpret $a(m) = 0$ and $g_r(m,p) = 0$ if m is not integral.

The *theta function* θ is a residue of a certain Eisenstein series, which is automorphic with respect to the fourth power Kubota symbol. It is an eigenfunction of the Hecke operators with $T_{p^4}\theta = Np^2\, \lambda_p\, \theta$, where $\lambda_p = Np^{1/2} + Np^{-1/2}$, and so its Fourier coefficients $\tau(m)$ satisfy

(1) $(Np^{1/2} + Np^{-1/2})\tau(m)$

$$= \tau\left(\frac{m}{p^4}\right) + Np^{-1}\ (m,p)\, g_3\left(\frac{m}{p^2}, p\right)\ \tau\left(\frac{m}{p^2}\right)$$

$$+ Np^{-1}\ (m,p^2)\, g_2\left(\frac{m}{p}, p\right)\ \tau(m) + Np^{-1}\ (m,p^3)\, g_1(m,p)\ \tau(mp^2) + \tau(mp^4).$$

Furthermore, we have the following *Periodicity Theorem*, a simple but rather deeper fact, which was proved in complete generality by Kazhdan and Patterson [**2**]. This is the fact that $\tau(m)$ depends only on m modulo fourth powers:

(2) $$\tau(h^4 m) = Nh^{1/2}\,\tau(m).$$

These relations tell us quite a bit about $\tau(m)$. If the theta function is normalized so that $\tau(1) = 1$, with $m = 1$, (1) and (2) imply that

$$Np^{-1/2} + Np^{1/2} = Np^{-1}\, g_1(1,p)\tau(p^2) + Np^{1/2},$$

and so

$$\tau(p^2) = Np^{-1/2}\overline{g_1(1,p)}.$$

Similarly, taking $m = p^3$ in (1) gives

$$\tau(p^3) = 0.$$

On the other hand, taking $m = p$ in (1) reduces to a tautology, since the quadratic Gauss sum $g_2(1,p) = Np^{1/2}$. The relations (1) and (2) do not imply anything about the values of $a(p)$. Still, we may sometimes show that $a(m) = 0$ for squarefree m (the coefficients are *not* expected to be multiplicative!) For example, taking $m = pq$, where q is a different prime from p, we have

$$(Np^{1/2} + Np^{-1/2})\tau(pq) = Np^{-1}\, g_2(q,p)\,\tau(pq) + Np^{1/2}\,\tau(pq).$$

Since the quadratic Gauss sum

$$g_2(q,p) = \left(\frac{q}{p}\right)^2 Np^{1/2},$$

this implies that $\tau(pq) = 0$ if p is a quadratic nonresidue modulo q, and more generally, it may be shown that if m is squarefree, then

$$\tau(m) = 0 \qquad \text{if} \qquad \left(\frac{m_1}{m_2}\right)^2 = -1$$

for any factorization $m = m_1 m_2$.

These relations are essentially those found by Suzuki. To go beyond this, Patterson considered the Rankin-Selberg convolution of θ with itself. This is an integral of θ^2 against a (quadratic metaplectic) Eisenstein series. This integral represents the Dirichlet series

$$\zeta(4s-1) \sum \tau(m)^2 \, Nm^{-s},$$

where ζ is the Dedekind zeta function of the field. This Dirichlet series has a functional equation under $s \mapsto 1-s$, and a pole at $s = \frac{3}{4}$. On the other hand, the Dirichlet series (first considered by Kubota)

$$\psi(s) = \sum g_1(1,m) \, Nm^{-s}$$

occurs in the Fourier coefficients of the quartic metaplectic Eisenstein series on $GL(2)$—the precise coefficient, which has a functional equation under $s \mapsto 1-s$, is $\zeta(8s-3)\,\psi(2s)$, with a pole at $s = \frac{5}{8}$. Consequently, $\zeta(4s-1)\,\psi(s+\frac{1}{2})$ also has a functional equation under $s \mapsto 1-s$, and a pole at $s = \frac{3}{4}$. Now Patterson made the remarkable observation that the assumption that

$$\sum \tau(m)^2 \, Nm^{-s} = \zeta(4s-1) \, \overline{\psi\left(s+\tfrac{1}{2}\right)^2} \tag{3}$$

is consistent with everything which is known about $\tau(m)$. For example, after multiplying both sides by $\zeta(4s-1)$, both sides have the same pole and functional equation (Patterson checked that the Gamma factors are the same). Moreover, the properties of τ which were found by Suzuki are consistent with this conjecture: the factor $\zeta(4s-1)$ causes the coefficients on the right to be periodic, as predicted by the Periodicity Theorem (2), and, for example, if m is square-free and admits a factorization $m = m_1 m_2$ with

$$\left(\frac{m_1}{m_2}\right)^2 = -1,$$

then cancellations cause the coefficient of Nm^{-s} on the right to vanish. On the other hand, the other squarefree coefficients will not vanish—if the squarefree m admits no such factorization into $m_1 m_2$, the conjecture implies that

$$\tau(m)^2 = 2^k \, Nm^{-1/2} \, \overline{g_1(1,m)}.$$

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Here k is the number of prime factors of m. Thus Patterson's conjecture determines all the Fourier coeffients of θ, at least *up to sign*.

Furthermore, Patterson considered the convolution of θ with its *complex conjugate*. This is the integral of $|\theta|^2$ against a nonmetaplectic Eisenstein series. It represents the Dirichlet series

$$\zeta(2s) \sum |\tau(m)|^2 \, Nm^{-s}, \tag{4}$$

which has analytic continuation and a functional equation with respect to $s \mapsto 1-s$. There is a pole at $s = 1$ (there are also poles at $s = 0$, $\frac{1}{4}$ and $\frac{3}{4}$). The location of the pole is consistent with the *magnitude* of $\tau(m)^2$ predicted by the conjecture—for squarefree m, the conjecture predicts that $|\tau(m)|$ would be 2^k with probability 2^{1-k} and otherwise zero, where k is the number of prime factors of m. We shall see later that if the conjecture is true, (4) is equal to a Dirichlet series which comes up in another context, and which does in fact have a functional equation and a simple pole at $s = 1$.

To go beyond this evidence of Patterson, let us consider an Eisenstein series on $GL(4)$. Specifically, let us define a function $I(\tau, s)$, where τ lies in $GL(4, \mathbb{C})/ZU(4)$ (Z being the center of $GL(4, \mathbb{C})$), and s is a complex parameter. Namely, any element of this homogeneous space has a representative of the form

$$\tau = \begin{pmatrix} y_1 y_2 y_3 & y_2 y_3 x_1 & y_3 x_4 & x_6 \\ & y_2 y_3 & y_3 x_2 & x_5 \\ & & y_3 & x_3 \\ & & & 1 \end{pmatrix}.$$

Then we let

$$I(\tau, s) = \theta \begin{pmatrix} y_1 & x_1 \\ & 1 \end{pmatrix} \theta \begin{pmatrix} y_2 & x_2 \\ & 1 \end{pmatrix} |y_1 y_2^2 y_3|^{2s},$$

where the y_i are positive real numbers. Let $\Gamma_0(4)$ be the subgroup of matrices in $\Gamma(4)$ such that 2×2 block in the lower left hand corner consists of zeros. Then we have the following Eisenstein series:

$$E^*(\tau, s) = \zeta(8s-7)\, \zeta(8s-6)\, E(\tau, s),$$

$$E(\tau, s) = \sum_{\Gamma_0(4)\backslash\Gamma(4)} \kappa(\gamma)\, I(\gamma\tau, s).$$

This Eisenstein series has a functional equation with respect to $s \mapsto 2-s$, with poles at $s = \frac{5}{4}, \frac{9}{8}, 1, \frac{7}{8}$, and $\frac{3}{4}$. Let us consider the Fourier coefficients. Specifically, let

$$w_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.$$

The Fourier coefficients $D(s;n_1,n_2,n_3)$ are defined by

$$\int_{\mathbf{C}/\mathcal{O}}\cdots\int_{\mathbf{C}/\mathcal{O}} E\left(w_1\begin{pmatrix}1 & x_1 & x_4 & x_6\\ & 1 & x_2 & x_5\\ & & 1 & x_3\\ & & & 1\end{pmatrix}\tau\right) e(-n_1x_1-n_2x_2-n_3x_3)\,dx_1\cdots dx_6$$

$$= D(s;n_1,n_2,n_3)\,Nn_1^{-3/2}\,Nn_2^{-2}\,Nn_3^{-3/2}\,W\left(\begin{pmatrix}n_1n_2n_3 & & & \\ & n_2n_3 & & \\ & & n_3 & \\ & & & 1\end{pmatrix}\tau\right),$$

Where $D(s;n_1,n_2,n_3)$ is a certain Dirichlet series involving the coefficients $\tau(m)$. Full details of the determination of these Dirichlet series will be given elsewhere, but here we recapitulate the basic idea. (Indeed, it is necessary to do this simply in order to state the definition of $D(s,n_1,n_2,n_3)$.) A coset in $\Gamma_0\backslash\Gamma$ is given by the following data: If γ is a matrix with the i,j-th entry being equal to c_{ij}, let A_{ij}, for $1\le i<j\le 4$ be the minor $c_{3i}c_{4j}-c_{3j}c_{4i}$. Then the coset of γ in $\Gamma_0\backslash\Gamma$ is associated with the six numbers A_{12}, A_{13}, A_{14}, A_{23}, A_{24} and A_{34}, which are coprime, and which satisfy

$$A_{12}A_{34}-A_{13}A_{24}+A_{14}A_{23}=0. \tag{5}$$

Conversely, given six coprime integers subject to the condition (5), there exists a coset having those numbers as minors. In computing the Fourier coefficients of the Eisenstein series, it is important to chose the coset representatives in a particular way. Specifically, let the A_{ij} be given. The coefficients c_{ij} are to be reconstructed as follow: let A_4 be the greatest common divisor of A_{34}, A_{24} and A_{14}. Find r, s and t so that $A_4=rA_{34}+sA_{24}+tA_{14}$, and let $A_3=sA_{23}+tA_{13}$, $A_2=-rA_{23}+tA_{12}$, $A_1=-rA_{13}-sA_{12}$. Also, let A_{234} be the greatest common divisor of A_{34}, A_{24} and A_{23}. Find R, S and T so that $A_{234}=RA_{34}+SA_{24}+TA_{23}$, and let $A_{134}=SA_{14}+TA_{13}$, $A_{124}=-RA_{14}+TA_{12}$ and $A_{123}=-RA_{13}-SA_{12}$. It may be shown that A_1, A_2, A_3 and A_4 are coprime, and that A_{123}, A_{124}, A_{134} and A_{234} are coprime. Furthermore, we may choose the coset representative γ so as to have bottom row (A_1,A_2,A_3,A_4), and so that the bottom row of the involute ${}^\iota\gamma=w_1\,{}^t\gamma^{-1}w_1$ has bottom row $(-A_{123},A_{124},-A_{134},A_{234})$. This done, we may now describe $D(s;n_1,n_2,n_3)$: In fact, this is the Dirichlet series

$$\sum_{A_{34}}\ \sum_{\substack{A_{24},A_{14},A_{23},A_{13}\bmod A_{34}\\ A_{34}|A_{13}A_{24}-A_{14}A_{24}\\ A_{12},A_{13},A_{14},A_{23},A_{24},A_{34}\text{ coprime}\\ A_{234}^2|n_1A_{34},\,A_4^2|n_3A_{34}}} \left\{\kappa(\gamma)N(A_4A_{234}A_{34}^{-1})\,\tau\left(\frac{n_1A_{34}}{A_{234}^2}\right)\tau\left(\frac{n_3A_{34}}{A_4^2}\right)\right.$$
$$\left. e\left(n_1\frac{A_{134}}{A_{234}}+n_2\frac{A_{24}}{A_{34}}+n_3\frac{A_3}{A_4}\right)\right\}NA_{34}^{-2s}. \tag{6}$$

For the moment, we are only concerned with the coefficient where $n_1 = n_2 = n_3 = 1$. We see that

$$\varsigma(8s-7)\,\varsigma(8s-6)\,D(s;1,1,1)$$

has a functional equation with respect to $s \mapsto 2-s$, with a simple pole at $s = \frac{5}{4}$.

We have shown how Patterson deduced from the location of the pole of the Dirichlet series (4) that, on the average, the $\tau(m)$ have the same *magnitude* as predicted by his conjecture. Now let us show that the location of the pole of (6), with $n_1 = n_2 = n_3 = 1$, shows that on the average, the $\tau(m)$ have the right *arguments*. Thus, we seek to show that $\tau(p)^2 g_1(1,p)$ is, on the average, about $Np^{1/2}$. Indeed, we may calculate the coefficients in (6) more explicitly, and interestingly enough, like the series (4) and the left side of (3), they only involve the *squares* of the $\tau(p)$. Let us restrict ourselves to describing the p-part of the series (6) (with $n_1 = n_2 = n_3 = 1$), in other words, the sum of the coefficients of Np^{-ks}. Of course, one must compute all the coefficients, which we have done, but for the moment considering just the p-part will be sufficient to show what is happening. It is convenient to make the following change of variables: let $w = 2s - \frac{3}{2}$. Then the p-part is

$$1 + g_1(1,p)\,\tau(p)^2\,Np^{-\frac{1}{2}-w} + 2g_1(1,p)\,\tau(p^2)\,Np^{-\frac{1}{2}-2w} + g_1(1,p)\,\tau(p)^2\,Np^{-\frac{1}{2}-3w} + Np^{-4w}.$$

This is to have a functional equation with respect to $w \mapsto 1-w$, and a simple pole at $w=1$. Since $|\tau(p)|$ is, on the average, constant (from the location of the pole of (4)), if the argument of $\tau(p)^2$ was not approximately the same as $\overline{g_1(1,p)}$, the pole of (6) would be to the *left* of $w = 1$. Thus the location of the pole shows that on the average, the argument of the $\tau(m)^2$ is consistent with the conjecture. Actually if m is squarefree, and one assumes the conjecture, then the coefficient of Nm^{-w} would be 2^k with probability 2^{1-k}, where k is the number of prime factors of m, and zero otherwise.

Now let us show that, if the conjecture is true, then the Dirichlet series (4) and (6) may actually be identified with known Dirichlet series having the correct functional equations and poles. Firstly, assuming the conjecture, the following identity may be established:

$$\varsigma(4w-1)\,\varsigma(4w)\,D\big(\tfrac{1}{2}(w+\tfrac{3}{2});1,1,1\big) = \varsigma(2w)\sum|\tau(m)|^2\,Nm^{-w}. \tag{7}$$

It follows from the general theory of Eisenstein series that the left hand side has simple poles at $s = 0$ and 1, and at $s = \frac{1}{4}$, and $\frac{3}{4}$. (The Eisenstein series itself also has a pole at $s = \frac{1}{2}$, but only the degenerate Fourier coefficients have poles—the left hand side of (7) has no pole at $s = \frac{1}{2}$.) The right hand side has the same poles.

We shall further show that assuming Patterson's conjecture, the above two Dirichlet series may be realized as the Fourier coefficient of an Eisenstein series on the two-fold cover of $GL(3)$. Specifically, if

$$\tau = \begin{pmatrix} y_1y_2 & y_2x_1 & x_3 \\ & y_2 & x_2 \\ & & 1 \end{pmatrix}, \qquad y_i > 0,$$

let us define

$$I_{\nu_1,\nu_2}(\tau) = y_1^{4\nu_1+2\nu_2}\, y_2^{2\nu_1+4\nu_2},$$

$$E(\tau,\nu_1,\nu_2) = \varsigma(6\nu_1 - 1)\,\varsigma(6\nu_2 - 1)\,\varsigma(6\nu_1 + 6\nu_2 - 3) \sum_{\Gamma_\infty(4)\backslash\Gamma(4)} \kappa(\gamma)^2\, I_{\nu_1,\nu_2}(\gamma\tau).$$

The Kubota symbol is squared to indicate that this Dirichlet series is made with *quadratic* symbols. The Eisenstein series $E(\tau,\nu_1,\nu_2)$ has functional equations with respect to

$$(\nu_1,\nu_2) \mapsto (\tfrac{2}{3} - \nu_2, \tfrac{2}{3} - \nu_1),$$

$$(\nu_1,\nu_2) \mapsto (\nu_1 + \nu_2 - \tfrac{1}{3}, \tfrac{2}{3} - \nu_2),$$

$$(\nu_1,\nu_2) \mapsto (\tfrac{2}{3} - \nu_1, \nu_1 + \nu_2 - \tfrac{1}{3}),$$

$$(\nu_1,\nu_2) \mapsto (1 - \nu_1 - \nu_2, \nu_1),$$

$$(\nu_1,\nu_2) \mapsto (\nu_2, 1 - \nu_1 - \nu_2).$$

The leading n_1, n_2-th Fourier coefficient is the Dirichlet series

$$R(\nu_1,\nu_2) = \varsigma(6\nu_1 - 1)\,\varsigma(6\nu_2 - 1)\,\varsigma(6\nu_1 + 6\nu_2 - 3) \sum_{C_1,C_2} H(C_1,C_2) NC_1^{-3\nu_1} NC_2^{-3\nu_2},$$

$$H(C_1,C_2) = \sum_{\substack{A_1,B_1 \bmod C_1 \\ A_2,B_2 \bmod C_2 \\ A_1,B_1,C_1 \text{ coprime} \\ A_2,B_2,C_2 \text{ coprime} \\ A_1C_2+B_1B_2+C_1A_2=0}} \kappa(\gamma)^2\, e\left(\frac{n_1B_1}{C_1} + \frac{n_2B_2}{C_2}\right),$$

where γ is a matrix having the bottom row (A_1,B_1,C_1), and whose involute has bottom row (A_2,B_2,C_2). Incidentally, it is possible to realize this Dirichlet series also as the Mellin transform of the quadratic Eisenstein series on $GL(2)$, but the $GL(3)$ interpretation seems to give more information.

We shall only be concerned here with the case $n_1 = n_2 = 1$. Then, assuming Patterson's conjecture, we may show that

$$\varsigma(2w) \sum |\tau(m)|^2\, Nm^{-w} = R(\tfrac{w}{3} + \tfrac{1}{6}, \tfrac{w}{3} + \tfrac{1}{6}).$$

By (7), this identifies both Dirichlet series (4) and (6). This equation is consistent with the functional equations and the locations of the poles.

Now let us present an identity which is closely related to Patterson's conjecture, and which may be the key to both proving the conjecture, and to generalizing it. There exists a theta series θ_3 on the 4-fold cover of $GL(3)$ with known Fourier coefficients. Restricting ourselves strictly to those coefficients parametrized by powers of a prime, it is sufficient to describe the coefficients $\tau(p^{k_1}, p^{k_2})$ where $0 \leq k_1, k_2 < 4$, because of the periodicity theorem. The only nonvanishing such coefficients are

$$\tau(1,1) = 1, \qquad \tau(1,p) = \tau(p,1) = Np^{-1/2}\,\overline{g_1(1,p)},$$

$$\tau(p,p^2) = \tau(p^2,p) = \overline{g_1(1,p)}, \qquad \tau(p^2,p^2) = Np^{-1/2}\,\overline{g_1(1,p)}^2.$$

Now, denoting the theta function on the 4-fold cover of $GL(2)$ for definiteness as θ_2, the *Rankin-Selberg convolution* of θ_2 with $\overline{\theta}_3$ is the Dirichlet series

$$L(s, \theta_2 \times \overline{\theta}_3) = \sum_{m_1, m_2} \tau(m_1)\,\overline{\tau(m_1, m_2)}\,N(m_1 m_2^2)^{-s}.$$

It has a functional equation with respect to $s \mapsto 1 - s$. Patterson's conjecture is closely related to the formula

$$L(s, \theta_2 \times \overline{\theta}_3) = \zeta(4s - \tfrac{3}{2})\,L(s, \overline{\theta}_2), \tag{8}$$

where $L(s, \overline{\theta}_2)$ is the "Mellin transform" $\sum \overline{\tau(m)}\,Nm^{-s}$.

In this form, there seems to be some hope of proving the conjecture, by a variation of the method originally used by Patterson [**3**] to determine the Fourier coefficients of the cubic theta function on $GL(2)$. For the left-hand side automatically has a functional equation by the theory of Rankin-Selberg convolutions. One might hope to show then by the converse theorem that there exists an automorphic form ϕ on the four-fold cover of $GL(2)$ such that

$$L(s, \theta_2 \times \overline{\theta}_3) = \zeta(4s - \tfrac{3}{2})\,L(s, \phi).$$

Then, by the method used by Patterson in [**3**] or otherwise, one would hope to show that $\phi = \overline{\theta}$.

Now let us propose a very general conjecture which includes both (7) and (8) as special cases. Let θ_r denote a theta function on the n-fold cover of $GL(r)$, where n is fixed throughout the following discussion. If $r' \leq r \leq n - 1$, then we conjecture that

$$(9) \quad L(s, \theta_r \times \overline{\theta}_{n-r'})$$

$$= \zeta\left(ns - \tfrac{n-r+r'}{2}\right)\,\zeta\left(ns - \tfrac{n-r+r'+2}{2}\right) \cdots \zeta\left(ns - \tfrac{n+r-r'-2}{2}\right)\,L(s, \theta_{r'} \times \overline{\theta}_{n-r}).$$

Moreover, we conjecture that (9) may be identified as the Fourier coefficient of an Eisenstein series of parabolic type r, r' on $GL(r + r')$.

This conjecture is consistent with everything which we know, and seems almost certainly true, although we have no idea how it should be proved. It implies a great deal about the Fourier coefficients of the theta functions, and when the Fourier coefficients of the Eisenstein series are further investigated (work in progress in collaboration with Solomon Friedberg), we believe that a complete and satisfactory generalization of Patterson's conjecture will be at hand. What is lacking at this time is the generalization of (6) to the case where r or r' is greater than 2.

Finally, it should be mentioned that this conjecture has an analog for cusp forms. If ϕ is a cusp form on the n-fold cover of $GL(r)$, we conjecture that the Rankin-Selberg convolution $L(s, \phi \times \overline{\theta}_{n-r'})$ is equal to the Fourier coefficient of an Eisenstein series on the n-fold cover of $GL(r+r')$ involving ϕ. Since the Fourier coefficients of θ_{n-1} are essentially known, if $r' = 1$, this conjecture is probably provable with the present state of knowledge. For this, it is not necessary to assume that $r \leq n-1$.

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