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Combinatorics with Emphasis on the Theory of Graphs

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Preface

Combinatorics and graph theory have mushroomed in recent years. Many overlapping or equivalent results have been produced. Some of these are special cases of unformulated or unrecognized general theorems. The body of knowledge has now reached a stage where approaches toward unification are overdue. To paraphrase Professor Gian-Carlo Rota (Toronto, 1967), "Combinatorics needs fewer theorems and more theory."

In this book we are doing two things at the same time:

A. We are presenting a unified treatment of much of combinatorics and graph theory. We have constructed a concise algebraically-based, but otherwise self-contained theory, which at one time embraces the basic theorems that one normally wishes to prove while giving a common terminology and framework for the development of further more specialized results.

B. We are writing a textbook whereby a student of mathematics or a mathematician with another specialty can learn combinatorics and graph theory. We want this learning to be done in a much more unified way than has generally been possible from the existing literature.

Our most difficult problem in the course of writing this book has been to keep A and B in balance. On the one hand, this book would be useless as a textbook if certain intuitively appealing, classical combinatorial results were either overlooked or were treated only at a lev l of abstraction rendering them beyond all recognition. On the other hand, we maintain our position that such results can all find a home as part of a larger, more general structure.

To convey more explicitly what this text is accomplishing, let us compare combinatorics with another mathematical area which, like combinatorics, has been realized as a field in the present century, namely topology. The basic unification of topology occurred with the acceptance of what we now call a "topology" as the underlying object. This concept was general enough to encompass most of the objects which people wished to study, strong enough to include many of the basic theorems, and simple enough so that additional conditions could be added without undue complications or repetition.

We believe that in this sense the concept of a "system" is the right unifying choice for combinatorics and graph theory. A system consists of a finite set of objects called "vertices," another finite set of objects called "blocks," and an "incidence" function assigning to each block a subset of the set of vertices. Thus graphs are systems with blocksize two; designs are systems with constant blocksize satisfying certain conditions; matroids are also systems; and a system is the natural setting for matchings and inclusion-exclusion. Some important notions are studied in this most general setting, such as connectivity and orthogonality as well as the parameters and vector spaces of a system. Connectivity is important in both graph theory and matroid theory, and parallel theorems are now avoided. The vector spaces of a system have important applications in all of these topics, and again much duplication is avoided.

One other unifying technique employed is a single notation consistent throughout the book. In attempting to construct such a notation, one must face many different levels in the hierarchy of sets (elements, sets of elements, collections of sets, families of collections, etc.) as well as other objects (systems, functions, sets of functions, lists, etc.). We decided insofar as possible to use different types of letters for different types of objects. Since each topic covered usually involves only a few types of objects, there is a strong temptation to adopt a simpler notation for that section regardless of how it fits in with the rest of the book. We have resisted this temptation. Consequently, once the notation of system is mastered, the reader will be able to flip from chapter to chapter, understanding at glance the diverse roles played in the middle and later chapters by the concepts introduced in the earlier chapters.

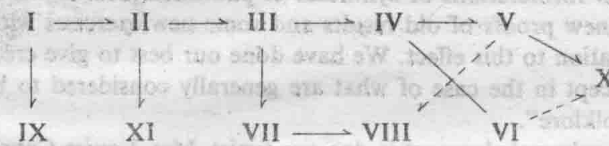
An undergraduate course in linear algebra is prerequisite to the comprehension of most of this book. Basic group theory is needed for sections *II*E and *XI*C. A deeper appreciation of sections *III*E, *III*G, *VII*C, and *VII*D will be gained by the reader who has had a year of topology. All of these sections may be omitted, however, without destroying the continuity of the rest of the text.

The level of exposition is set for the beginning graduate student in the mathematical sciences. It is also appropriate for the specialist in another mathematical field who wishes to learn combinatorics from scratch but from a sophisticated point of view.

It has been our experience while teaching from the notes that have evolved into this text, that it would take approximately three semesters of three hours classroom contact per week to cover all of the material that we have presented. A perusal of the Table of Contents and of the "Flow Chart of the

Sections" following this Preface will suggest the numerous ways in which a subset of the sections can be covered in a subset of three semesters. A List of Symbols and an Index of Terms are provided to assist the reader who may have skipped over the section in which a symbol or term was defined.

As indicated in the figure below, a one-semester course can be formed from Chapters I, II, IX, and XI. However, the instructor must provide some elementary graph theory in a few instances. The dashed lines in the figure below as well as in the Flow Chart of the Sections indicate a rather weak dependency.



If a two-semester sequence is desired, we urge that Chapters I, II, and III be treated in sequence in the first semester, since they comprise the theoretical core of the book. The reader should not be discouraged by the apparent dryness of Chapter II. There is a dividend which is compounded and paid back chapter by chapter. We recommend also that Chapters IV, V, and VI be studied in sequence; they are variations on a theme, a kind of minimax or maximin principle, which is an important combinatorial notion. Since Chapter X brings together notions from the first six chapters with allusions to Chapters VII and IX, it would be a suitable finale.

There has been no attempt on our part to be encyclopedic. We have even slighted topics dear to our respective hearts, such as integer programming and automorphism groups of graphs. We apologize to our colleagues whose favorite topics have been similarly slighted.

There has been a concerted effort to keep the technical vocabulary lean. Formal definitions are not allotted to terms which are used for only a little while and then never again. Such terms are often written between quotation marks. Quotation marks are also used in intuitive discussions for terms which have yet to be defined precisely.

The terms which do form part of our technical vocabulary appear in **bold-face** type when they are formally defined, and they are listed in the Index.

There are two kinds of exercises. When the term "**Exercise**" appears in bold-face type, then those assertions in italics following it will be invoked in subsequent arguments in the text. They almost always consist of straightforward proofs with which we prefer not to get bogged down and thereby lose too much momentum. The word "*Exercise*" (in italics) generally indicates a specific application of a principle, or it may represent a digression which the limitations of time and space have forced us not to pursue. In principle, all of the exercises are important for a deeper understanding of and insight into the theory.

Chapters are numbered with Roman numerals; the sections within each chapter are denoted by capital letters; and items (theorems, exercises, figures, etc.) are numbered consecutively regardless of type within each section. If an item has more than one part, then the parts are denoted by lower case Latin letters. For references within a chapter, the chapter number will be suppressed, while in references to items in other chapters, the chapter number will be italicized. For example, within Chapter III, Euler's Formula is referred to as F2b, but when it is invoked in Chapter VII, it is denoted by *III*F2b.

Relatively few of the results in this text are entirely new, although many represent new formulations or syntheses of published results. We have also given many new proofs of old results and some new exercises without any special indication to this effect. We have done our best to give credit where it is due, except in the case of what are generally considered to be results "from the folklore".

A special acknowledgement is due our typist, Mrs. Louise Capra, and to three of our former graduate students who have given generously of their time and personal care for the well-being of this book: John Kevin Doyle, Clare Heidema, and Charles J. Leska. Thanks are also due to the students we have had in class, who have learned from and taught us from our notes. Finally, we express our gratitude to our families, who may be glad to see us again.

Syracuse, N.Y.
April, 1977

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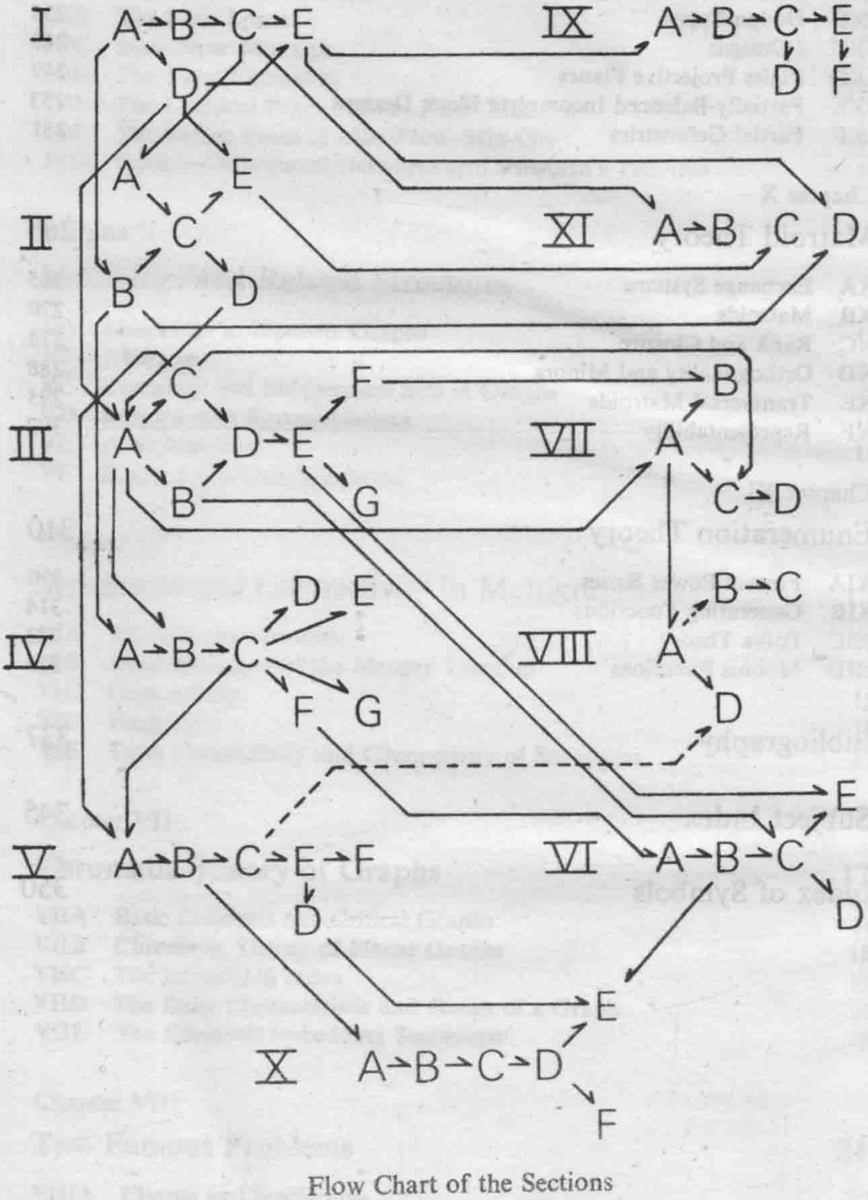


Flow Chart of the Sections

Contents

CHAPTER I

Finite Sets

IA Conventions and Basic Notation

The symbols ℕ, ℤ, ℚ, ℝ, 𝕂 will always denote, respectively, the natural numbers (including 0), the integers, the rational numbers, the real numbers, and the field of order 2. In each of these systems, 0 and 1 denote, respectively, the additive and multiplicative identities.

If U is a set, $\mathscr{P}(U)$ will denote the collection of all subsets of U. It is called the **power set** of U. In general, the more common, conventional terminology and notation of set theory will be used throughout except occasionally as noted. One such instance is the following usage: while "$U \subseteq W$" will continue to mean that U is a subset of W, we shall write "$U \subset W$" when $U \subseteq W$ and $U \neq W$. (Thus U can be empty if W is not empty.) The cardinality of the set U will be denoted by $|U|$, and $\mathscr{P}_m(U)$ will denote the collection of all subsets of U with cardinality m. A set of cardinality m is called an ***m*-set**.

The binary operation of **sum** (Boolean sum) of sets S and T in $\mathscr{P}(U)$ is denoted by $S + T$, where

$$S + T = \{x\colon x \in S \cup T;\ x \notin S \cap T\}.$$

In particular, $S + U$ is the complement of S in U, and no other notation for complementation will be required. Since the sum is the most frequently used set-operation in this text, we include a list of properties which can be easily verified.

For $R, S, T \in \mathscr{P}(U)$,

A1 $\quad S + T = T + S$

A2 $\quad (R + S) + T = R + (S + T)$

A3 $S + T = S \Leftrightarrow T = \varnothing$

A4 $S + T = \varnothing \Leftrightarrow S = T$

A5 $S + T = (S \cup T) + (S \cap T)$

A6 $R \cup (S + T) \supseteq (R \cup S) + (R \cup T)$

A7 $R \cap (S + T) = (R \cap S) + (R \cap T)$

A8 $R + (S \cap T) \supseteq (R + S) \cap (R + T)$

A9 $(R + S) \cap (R + T) \subseteq R + (S \cup T) \subseteq (R + S) \dot{\cup} (R + T)$

A10 *Exercise*. Show that the inclusions in A6, A8, and A9 cannot, in general, be reversed.

Because of A1 and A2, the sum $\sum_{S \in \mathscr{S}} S$ where $\mathscr{S} \subseteq \mathscr{P}(U)$ is well-defined if $\mathscr{S} \neq \varnothing$. If $\mathscr{S} = \varnothing$, we understand this sum to be $\varnothing$.

As usual, the **cartesian product** of sets $X_1, \ldots, X_m$ will be denoted by $X_1 \times \ldots \times X_m$. Thus

$$X_1 \times \ldots \times X_m = \{(x_1, \ldots, x_m): x_i \in X_i \text{ for } i = 1, \ldots, m\}.$$

A **function** f from X into Y is a subset of $X \times Y$ such that $|f \cap (\{x\} \times Y)| = 1$ for all $x \in X$. Following established convention, $f: X \to Y$ will mean that f is a function from X into Y. For each $x \in X$, $f(x)$ is the second component of the unique element of $f \cap (\{x\} \times Y)$. We shall adhere to the terms **injection** if $|f \cap (X \times \{y\})| \leq 1$ for all $y \in Y$; **surjection** if $|f \cap (X \times \{y\})| \geq 1$ for all $y \in Y$; and **bijection** if $|f \cap (X \times \{y\})| = 1$ for all $y \in Y$.

We say sets X and Y are **isomorphic** if there exists a bijection $b: X \to Y$, and we note that X and Y are isomorphic if and only if $|X| = |Y|$.

A **(binary) relation** on U is a subset of $U \times U$. Let R_i be a relation on U_i for $i = 1, 2$. We say that (U_1, R_1) is **isomorphic** to (U_2, R_2) if there exists a bijection $b: U_1 \to U_2$ such that $(x, y) \in R_1$ if and only if $(b(x), b(y)) \in R_2$. A binary relation R on U is **reflexive** if $(u, u) \in R$ for all $u \in U$; R is **symmetric** if $(u, v) \in R$ implies $(v, u) \in R$ for all $u, v \in U$; R is **transitive** if $(u, v) \in R$ and $(v, w) \in R$ together imply $(u, w) \in R$ for all $u, v, w \in U$. R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Problems involving categories being outside the scope of this book, we find it best to ignore them, and we shall freely use such terms as "equivalent" and "equivalence relation" in regard to objects from various categories and not only to elements of some given set. Such disregard for categorical problems will be particularly flagrant as we treat in turn various notions of "isomorphism." For example, the "relation" of "is isomorphic to" is clearly an "equivalence relation" on the category of sets.

We denote the set of all functions from X into Y by Y^X. Since $\varnothing \times Y = \varnothing$, $Y^\varnothing$ consists of a single function $\varnothing$ which is an injection; in case $Y = \varnothing$, it is a bijection, of course. If $S \subseteq X$, then the **restriction of** f **to** S, denoted by $f_{|S}$, belongs to Y^S and satisfies $f_{|S}(x) = f(x)$ for all $x \in S$.

A bijection $b: U \to U$ is called a **permutation** of U. The set of all permutations of U is denoted by $\Pi(U)$. The **identity** on U is the function $1_U \in \Pi(U)$ given by $1_U(x) = x$ for all $x \in U$.

The function $f: X \to Y$ induces two corresponding functions between $\mathscr{P}(X)$ and $\mathscr{P}(Y)$. One of these is also denoted by f, and $f: \mathscr{P}(X) \to \mathscr{P}(Y)$ is given by

$$f[S] = \{f(x): x \in S\}, \quad \text{for all } S \in \mathscr{P}(X).$$

(Note that the choice of parentheses or brackets to surround the argument determines which of the two functions denoted by the symbol f is intended.) The set $f[S]$ is the **image of** S **under** f. In particular, $f[X]$ is the **image of** f. The other function induced by f is the function $f^{-1}: \mathscr{P}(Y) \to \mathscr{P}(X)$ given by

$$f^{-1}[T] = \{x: f(x) \in T\}, \quad \text{for all } T \in \mathscr{P}(Y).$$

If f is a bijection, its inverse, also denoted by f^{-1}, is a function $f^{-1}: Y \to X$. By our convention, if $y \in Y$, $f^{-1}[y]$ $(= f^{-1}[\{y\}])$ denotes a subset of X, but if f is a bijection, $f^{-1}(y)$ denotes an element of X. f **maps** S **into** T if $f[S] \subseteq T$ and **onto** T if $f[S] = T$. We say f is a **constant function** if $|f[X]| \leq 1$.

Let $f: X \to Y$; $S, T \in \mathscr{P}(X)$; $U, W \in \mathscr{P}(Y)$. The following basic properties of functions and sets are readily verified:

A11 $$f[S \cup T] = f[S] \cup f[T]$$

A12 $$f[S \cap T] \subseteq f[S] \cap f[T]$$

A13 $$f^{-1}[U \cup W] = f^{-1}[U] \cup f^{-1}[W]$$

A14 $$f^{-1}[U \cap W] = f^{-1}[U] \cap f^{-1}[W]$$

A15 $$f[S + T] \supseteq f[S] + f[T]$$

A16 $$f^{-1}[U + W] = f^{-1}[U] + f^{-1}[W]$$

A17 *Exercise.* Show that the inclusions in A12 and A15 cannot, in general, be reversed.

Let X, Y, and Z be sets. Let $f \in Y^X$ and $g \in Z^Y$. The composite of f by g will be denoted by gf. Clearly $gf \in Z^X$. We conclude the present section with a rapid review of some elementary properties of functions and some terminology.

A18 If both f and g are injections (respectively, surjections, bijections), then so is gf.

A19 $$(gf)^{-1} = f^{-1}g^{-1} \in \mathscr{P}(X)^{\mathscr{P}(Z)}.$$

A20 g is an injection if and only if there exists $h \in Y^Z$ such that $hg = 1_Y$.

A21 Let g be an injection. If $gf_1 = gf_2$ for $f_1, f_2 \in Y^X$, then $f_1 = f_2$. The converse holds if $|X| \geq 2$.

A22 f is a surjection if and only if there exists $j \in X^Y$ such that $fj = 1_Y$.

A23 Let f be a surjection. If $g_1 f = g_2 f$ for $g_1, g_2 \in Z^Y$, then $g_1 = g_2$. The converse holds if $|Z| \geq 2$.

A24 f is a bijection if and only if there exists $b \in X^Y$ such that $bf = 1_X$ and $fb = 1_Y$. In this case $b = f^{-1}$, and so b is unique.

A25 If X is finite and $h \in X^X$, then h is a surjection if and only if h is an injection.

If $S \subseteq X$ and $h \in X^X$, we say h **fixes** S if $h[S] \subseteq S$. If $h_{|S} = 1_S$, we say h **fixes** S **pointwise.**

If $*$ is a binary operation on Y, then $*$ induces a binary operation on Y^X which is also denoted by $*$. Thus

$$(f_1 * f_2)(x) = f_1(x) * f_2(x), \quad \text{for all } f_1, f_2 \in Y^X, x \in X.$$

Note that if $*$ on Y enjoys any of the properties of associativity, commutativity, or existence of an identity, then that property is also enjoyed by $*$ on Y^X.

One final important convention: henceforth, **all arbitrarily chosen sets will be finite unless explicitly stated otherwise.**

A26 *Exercise.* Let $f: X \to Y$. Show that if f is an injection (respectively, surjection, bijection), then so is the induced function $f: \mathscr{P}(X) \to \mathscr{P}(Y)$, and conversely.

A27 *Exercise.* Let $f: X \to Y$. Show that if f is an injection (respectively, surjection, bijection), then $f^{-1}: \mathscr{P}(Y) \to \mathscr{P}(X)$ is a surjection (respectively, injection, bijection), and conversely.

IB Selections and Partitions

Let U be a set and let $S \in \mathscr{P}(U)$. The **characteristic function** of S is the function

$$c_S: U \to \mathbb{K}$$

given by

B1
$$c_S(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \in U + S. \end{cases}$$

B2 Proposition. *The function* $\sigma: \mathbb{K}^U \to \mathscr{P}(U)$ *given by*

$$\sigma(c) = \{x \in U: c(x) \neq 0\} \quad \textit{for all } c \in \mathbb{K}^U$$

is a bijection. Moreover, $\sigma^{-1}(S) = c_S$ *for all* $S \in \mathscr{P}(U)$.

PROOF. Clearly σ is an injection. If $S \in \mathscr{P}(U)$, then $\sigma(c_S) = S$. Hence σ is a surjection. □

B3 Exercise. Let $S, T \in \mathscr{P}(U)$. Prove that

$$c_S + c_T = c_{S+T} \quad \textit{and} \quad c_S c_T = c_{S \cap T},$$

and express $c_{S \cup T}$ in terms of c_S and c_T.

For a set U, a function $s \in \mathbb{N}^U$ is called a **selection** from U. If $x \in U$, the number $s(x)$ is the "number of times x is selected by s". The number

$$|s| = \sum_{x \in U} s(x)$$

is the **cardinality (weight) of the selection** s. If $|s| = m$, we say that s is an **m-selection.** The set of all m-selections from U is denoted by $\mathbb{S}_m(U)$, and we let

$$\mathbb{S}(U) = \bigcup_{m=0}^{\infty} \mathbb{S}_m(U) = \mathbb{N}^U.$$

If $S \in \mathscr{P}(U)$, we define the **characteristic selection** of S by

B4
$$s_S(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \in U + S. \end{cases}$$

The difference between B1 and B4 is subtle but important. In B4, the symbols 0 and 1 denote elements of $\mathbb{N}$ rather than $\mathbb{K}$. Of course, c_S and s_S are closely related, but since $1 + 1$ gives a different "answer" in $\mathbb{N}$ than in $\mathbb{K}$, the characteristic function and characteristic selection are not the same thing. In particular, the correspondence $S \to s_S$ gives a natural injection of $\mathscr{P}(U)$ into $\mathbb{S}(U)$ under which $S + T$ is not necessarily mapped onto $s_S + s_T$, even though $S \cap T$ is always mapped onto $s_S s_T$ for all $S, T \in \mathscr{P}(U)$. (Cf. B3.)

A subcollection $\mathscr{Q} \subseteq \mathscr{P}(U)$ of nonempty subsets of U is called a **partition** of U if

$$\sum_{Q \in \mathscr{Q}} Q = U$$

and

$$Q \cap R = \varnothing, \quad \text{for all } Q, R \in \mathscr{Q};\ Q \neq R.$$

The elements of $\mathscr{Q}$ are called the **cells** of $\mathscr{Q}$. If $|\mathscr{Q}| = m$, we call $\mathscr{Q}$ an **m-partition** of U. The collection of all m-partitions of U is denoted by $\mathbb{P}_m(U)$; $\mathbb{P}(U)$ denotes the collection of all partitions of U. A fundamental identity satisfied by any partition $\mathscr{Q} \in \mathbb{P}(U)$ is

B5
$$|U| = \sum_{Q \in \mathscr{Q}} |Q|.$$

There is a natural multiplication on $\mathbb{P}(U)$. Let $\mathcal{Q}, \mathcal{R} \in P(U)$ and let $\mathcal{Q}\mathcal{R}$ be the collection of nonempty subsets of the form $Q \cap R$ where $Q \in \mathcal{Q}$ and $R \in \mathcal{R}$.

B6 Exercise. Prove that *if $\mathcal{Q} \in \mathbb{P}_m(U)$ and $\mathcal{R} \in \mathbb{P}_n(U)$, then $\mathcal{Q}\mathcal{R} \in \mathbb{P}_p(U)$ for some $p \leq mn$.* Show, moreover, that this multiplication is commutative and associative and admits an identity in $\mathbb{P}(U)$.

The next result delineates the fundamental relationship between partitions and equivalence relations.

B7 Proposition. *A necessary and sufficient condition that a relation R on a set U be an equivalence relation is that there exist a partition $\mathcal{Q} \in \mathbb{P}(U)$ such that $(x, y) \in R$ if and only if x and y are elements of the same cell of $\mathcal{Q}$.*

PROOF. Let R be an equivalence relation on U. For each $x \in U$ let $S_x = \{w \in U: (x, w) \in R\}$. Since R is reflexive, $x \in S_x$ and so $S_x \neq \varnothing$ for each $x \in U$. Let $x, y \in U$ and suppose $w \in S_x \cap S_y$. Thus (x, w) and $(y, w) \in R$. Since R is symmetric, $(w, y) \in R$, and since R is transitive, $(x, y) \in R$. Now let $z \in S_y$; hence $(y, z) \in R$. Again by transitivity, $(x, z) \in R$ and $z \in S_x$. This proves that $S_y \subseteq S_x$. By a symmetrical argument we see that $S_x \subseteq S_y$. Thus exactly one of the following holds for any $x, y \in U$: $S_x = S_y$ or $S_x \cap S_y = \varnothing$. If $\mathcal{Q} = \{S: S = S_x \text{ for some } x \in U\}$, then $\mathcal{Q} \in \mathbb{P}(U)$.

Conversely, let $\mathcal{Q} \in \mathbb{P}(U)$. Define the relation R on U by: $(x, y) \in R$ if $x, y \in Q$ for some $Q \in \mathcal{Q}$. One readily verifies that R is an equivalence relation. □

B8 Proposition. *Let $f: B \to U$. Then $\{f^{-1}[x]: x \in f[B]\}$ is a $|f[B]|$-partition of B.*

PROOF. For each $b \in B$, $b \in f^{-1}[x]$ if and only if $x = f(b)$. Hence $\sum_{x \in f[B]} f^{-1}[x] = B$ and $f^{-1}[x] \cap f^{-1}[y] = \varnothing$ for $x \neq y$. Finally, $f^{-1}[x] \neq \varnothing$ if and only if $x \in f[B]$. □

B9 Proposition. *Let $f: B \to U$. Let $s: U \to \mathbb{N}$ be defined by $s(x) = |f^{-1}[x]|$. Then s is a $|B|$-selection from U.*

PROOF. Clearly $s \in \mathbb{S}(U)$. We have that

$$|s| = \sum_{x \in U} |f^{-1}[x]| = \sum_{x \in f[B]} |f^{-1}[x]| = |B|.$$

The first equality here is the definition of $|s|$; the second follows from the fact that $|\varnothing| = 0$ and $f^{-1}[x] = \varnothing$ for $x \notin f[B]$; the third equality follows from B5 and B8. □

If $f: B \to U$, then the **partition of** f is $\{f^{-1}[x]: x \in f[B]\}$, and the **selection of** f is the function $s: U \to \mathbb{N}$ given by $s(x) = |f^{-1}[x]|$.

B10 *Exercise.* Prove that the functions $f: B \to U$ and $g: C \to U$ have the same selection if and only if there is a bijection $b: B \to C$ such that $f = gb$.

B11 Exercise. Prove that *the functions $f: B \to U$ and $h: B \to W$ have the same partition if and only if there is a bijection $b: f[B] \to h[B]$ such that $bf = h$.*

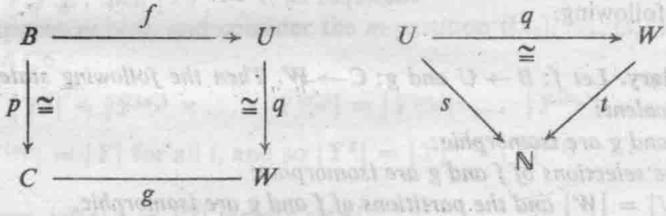
B12 *Exercise.* Let $f: X \to Y$. Define $f_1: \mathbb{S}(Y) \to \mathbb{S}(X)$ by $f_1(s) = sf$ for all $s \in \mathbb{S}(Y)$. Show that f is an injection (respectively, surjection, bijection) if and only if f_1 is a surjection (respectively, injection, bijection).

B13 *Exercise.* Let $f: X \to Y$. Define $f_2: \mathbb{P}(Y) \to \mathbb{P}(X)$ as follows: if $\mathcal{Q} \in \mathbb{P}(Y)$, then $f_2(\mathcal{Q})$ consists of the nonempty members of the collection $\{f^{-1}[Q]: Q \in \mathcal{Q}\}$. First verify that $f_2(\mathcal{Q}) \in \mathbb{P}(X)$; then show that f is an injection (respectively, surjection, bijection) if and only if f_2 is a surjection (respectively, injection, bijection).

The remainder of this section is concerned with the notion of "isomorphism" between objects of the kinds we have been considering.

Functions $f: B \to U$ and $g: C \to W$ are **isomorphic** if there exist bijections $p: B \to C$ and $q: U \to W$ such that $f = q^{-1}gp$. The pair (p, q) is called a **function-isomorphism.** The selections $s \in \mathbb{S}(U)$ and $t \in \mathbb{S}(W)$ are **isomorphic** if there exists a bijection $q: U \to W$ such that $s = tq$. Such a bijection is called a **selection-isomorphism.** (These two definitions are illustrated by the commutative diagrams B14. In this and other such diagrams bijections are indicated by the symbol $\cong$.) Partitions $\mathcal{Q} \in \mathbb{P}(B)$ and $\mathcal{R} \in \mathbb{P}(C)$ are **isomorphic** if there exists a bijection $p: B \to C$ such that $p[Q] \in \mathcal{R}$ for all $Q \in \mathcal{Q}$. The bijection p is a **partition-isomorphism.**

B14



B15 Exercise. Prove that *in each of the above definitions, "isomorphism" is an equivalence relation.*

B16 Proposition. *Let $f: B \to U$ and $g: C \to W$. Let $p: B \to C$ and $q: U \to W$ be bijections.*

(**a**) *If (p, q) is a function-isomorphism from f to g, then p is a partition-isomorphism from the partition of f to the partition of g and q is a selection-isomorphism from the selection of f to the selection of g.*