Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1390

G. Da Prato L. Tubaro (Eds.)

Stochastic Partial Differential Equations and Applications II

Proceedings, Trento 1988

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1390

G. Da Prato L. Tubaro (Eds.)

Stochastic Partial Differential Equations and Applications II

Proceedings of a Conference held in Trento, Italy
February 1–6, 1988

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo Hong Kong

Editors

Giuseppe Da Prato
Scuola Normale Superiore
56100 Pisa, Italy

Luciano Tubaro
Dipartimento di Matematica, Università di Trento
38050 Povo, Italy

Mathematics Subject Classification (1980): 60H 15, 60G 35, 93E 11

ISBN 3-540-51510-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-51510-0 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210 – Printed on acid-free paper

Lecture Notes in Mathematics

For information about Vols. 1–1173 please contact your bookseller or Springer-Verlag.

Vol. 1174: Categories in Continuum Physics, Buffalo 1982. Seminar. Edited by F.W. Lawvere and S.H. Schanuel. V, 126 pages. 1986.

Vol. 1175: K. Mathiak, Valuations of Skew Fields and Projective Hjelmslev Spaces. VII, 116 pages. 1986.

Vol. 1176: R.R. Bruner, J.P. May, J.E. McClure, M. Steinberger, H_∞ Ring Spectra and their Applications. VII, 388 pages. 1986.

Vol. 1177: Representation Theory I. Finite Dimensional Algebras. Proceedings, 1984. Edited by V. Dlab, P. Gabriel and G. Michler. XV, 340 pages. 1986.

Vol. 1178: Representation Theory II. Groups and Orders. Proceedings, 1984. Edited by V. Dlab, P. Gabriel and G. Michler. XV, 370 pages. 1986.

Vol. 1179: Shi J.-Y. The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups. X, 307 pages. 1986.

Vol. 1180: R. Carmona, H. Kesten, J.B. Walsh, École d'Été de Probabilités de Saint-Flour XIV – 1984. Édité par P.L. Hennequin. X, 438 pages. 1986.

Vol. 1181: Buildings and the Geometry of Diagrams, Como 1984. Seminar. Edited by L. Rosati. VII, 277 pages. 1986.

Vol. 1182: S. Shelah, Around Classification Theory of Models. VII, 279 pages. 1986.

Vol. 1183: Algebra, Algebraic Topology and their Interactions. Proceedings, 1983. Edited by J.-E. Roos. XI, 396 pages. 1986.

Vol. 1184: W. Arendt, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, U. Schlotterbeck, One-parameter Semigroups of Positive Operators. Edited by R. Nagel. X, 460 pages. 1986.

Vol. 1185: Group Theory, Beijing 1984. Proceedings. Edited by Tuan H.F. V, 403 pages. 1986.

Vol. 1186: Lyapunov Exponents. Proceedings, 1984. Edited by L. Arnold and V. Wihstutz. VI, 374 pages. 1986.

Vol. 1187: Y. Diers, Categories of Boolean Sheaves of Simple Algebras. VI, 168 pages. 1986.

Vol. 1188: Fonctions de Plusieurs Variables Complexes V. Séminaire, 1979–85. Edité par François Norguet. VI, 306 pages. 1986.

Vol. 1189: J. Lukeš, J. Malý, L. Zajíček, Fine Topology Methods in Real Analysis and Potential Theory. X, 472 pages. 1986.

Vol. 1190: Optimization and Related Fields. Proceedings, 1984. Edited by R. Conti, E. De Giorgi and F. Giannessi. VIII, 419 pages. 1986.

Vol. 1191: A.R. Its, V.Yu. Novokshenov, The Isomonodromic Deformation Method in the Theory of Painlevé Equations. IV, 313 pages. 1986.

Vol. 1192: Equadiff 6. Proceedings, 1985. Edited by J. Vosmansky and M. Zlámal. XXIII, 404 pages. 1986.

Vol. 1193: Geometrical and Statistical Aspects of Probability in Banach Spaces. Proceedings, 1985. Edited by X. Femique, B. Heinkel, M.B. Marcus and P.A. Meyer. IV, 128 pages. 1986.

Vol. 1194: Complex Analysis and Algebraic Geometry. Proceedings, 1985. Edited by H. Grauert. VI, 235 pages. 1986.

Vol. 1195: J.M. Barbosa, A.G. Colares, Minimal Surfaces in $\mathbb{R}^3$. X, 124 pages. 1986.

Vol. 1196: E. Casas-Alvero, S. Xambó-Descamps, The Enumerative Theory of Conics after Halphen. IX, 130 pages. 1986.

Vol. 1197: Ring Theory. Proceedings 1985. Edited by F.M.J. van Oystaeyen. V, 231 pages. 1986.

Vol. 1198: Séminaire d'Analyse, P. Lelong – P. Dolbeault – H. Skoda. Seminar 1983/84. X, 260 pages. 1986.

Vol. 1199: Analytic Theory of Continued Fractions II. Proceedings, 1985. Edited by W.J. Thron. VI, 299 pages. 1986.

Vol. 1200: V.D. Milman, G. Schechtman, Asymptotic Theory of Finite Dimensional Normed Spaces. With an Appendix by M. Gromov. VIII, 156 pages. 1986.

Vol. 1201: Curvature and Topology of Riemannian Manifolds. Proceedings, 1985. Edited by K. Shiohama, T. Sakai and T. Sunada. VII, 336 pages. 1986.

Vol. 1202: A. Dür, Möbius Functions, Incidence Algebras and Power Series Representations. XI, 134 pages. 1986.

Vol. 1203: Stochastic Processes and Their Applications. Proceedings, 1985. Edited by K. Itô and T. Hida. VI, 222 pages. 1986.

Vol. 1204: Séminaire de Probabilités XX, 1984/85. Proceedings. Edité par J. Azéma et M. Yor. V, 639 pages. 1986.

Vol. 1205: B.Z. Moroz, Analytic Arithmetic in Algebraic Number Fields. VII, 177 pages. 1986.

Vol. 1206: Probability and Analysis, Varenna (Como) 1985. Seminar. Edited by G. Letta and M. Pratelli. VIII, 280 pages. 1986.

Vol. 1207: P.H. Bérard, Spectral Geometry: Direct and Inverse Problems. With an Appendix by G. Besson. XIII, 272 pages. 1986.

Vol. 1208: S. Kaijser, J.W. Pelletier, Interpolation Functors and Duality. IV, 167 pages. 1986.

Vol. 1209: Differential Geometry, Peñiscola 1985. Proceedings. Edited by A.M. Naveira, A. Ferrández and F. Mascaró. VIII, 306 pages. 1986.

Vol. 1210: Probability Measures on Groups VIII. Proceedings, 1985. Edited by H. Heyer. X, 386 pages. 1986.

Vol. 1211: M.B. Sevryuk, Reversible Systems. V, 319 pages. 1986.

Vol. 1212: Stochastic Spatial Processes. Proceedings, 1984. Edited by P. Tautu. VIII, 311 pages. 1986.

Vol. 1213: L.G. Lewis, Jr., J.P. May, M. Steinberger, Equivariant Stable Homotopy Theory. IX, 538 pages. 1986.

Vol. 1214: Global Analysis – Studies and Applications II. Edited by Yu.G. Borisovich and Yu.E. Gliklikh. V, 275 pages. 1986.

Vol. 1215: Lectures in Probability and Statistics. Edited by G. del Pino and R. Rebolledo. V, 491 pages. 1986.

Vol. 1216: J. Kogan, Bifurcation of Extremals in Optimal Control. VIII, 106 pages. 1986.

Vol. 1217: Transformation Groups. Proceedings, 1985. Edited by S. Jackowski and K. Pawalowski. X, 396 pages. 1986.

Vol. 1218: Schrödinger Operators, Aarhus 1985. Seminar. Edited by E. Balslev. V, 222 pages. 1986.

Vol. 1219: R. Weissauer, Stabile Modulformen und Eisensteinreihen. III, 147 Seiten. 1986.

Vol. 1220: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin. Proceedings, 1985. Edité par M.-P. Malliavin. IV, 200 pages. 1986.

Vol. 1221: Probability and Banach Spaces. Proceedings, 1985. Edited by J. Bastero and M. San Miguel. XI, 222 pages. 1986.

Vol. 1222: A. Katok, J.-M. Strelcyn, with the collaboration of F. Ledrappier and F. Przytycki, Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities. VIII, 283 pages. 1986.

Vol. 1223: Differential Equations in Banach Spaces. Proceedings, 1985. Edited by A. Favini and E. Obrecht. VIII, 299 pages. 1986.

Vol. 1224: Nonlinear Diffusion Problems, Montecatini Terme 1985. Seminar. Edited by A. Fasano and M. Primicerio. VIII, 188 pages. 1986.

Vol. 1225: Inverse Problems, Montecatini Terme 1986. Seminar. Edited by G. Talenti. VIII, 204 pages. 1986.

Vol. 1226: A. Buium, Differential Function Fields and Moduli of Algebraic Varieties. IX, 146 pages. 1986.

Vol. 1227: H. Helson, The Spectral Theorem. VI, 104 pages. 1986.

Vol. 1228: Multigrid Methods II. Proceedings, 1985. Edited by W. Hackbusch and U. Trottenberg. VI, 336 pages. 1986.

Vol. 1229: O. Bratteli, Derivations, Dissipations and Group Actions on C*-algebras. IV, 277 pages. 1986.

Vol. 1230: Numerical Analysis. Proceedings, 1984. Edited by J.-P. Hennart. X, 234 pages. 1986.

Vol. 1231: E.-U. Gekeler, Drinfeld Modular Curves. XIV, 107 pages. 1986.

continued on page 265

PREFACE

These are the Proceedings of the second meeting on *Stochastic Partial Differential Equations and Applications,* the first having been in October 1985 (with Proceedings published in the Lecture Notes in Mathematics n. 1236). It seems that our wishes three years ago — that this occasion for a more direct communication among the researchers in this area of Mathematics occur every two or three years — have been satisfied. Evidently two successes do not imply that there will be a third one, but they give us good hopes ...!

The range of applications of SPDE becomes ever wider: filtering theory, biological models, control theory, field theory in Physics. They offer new problems and give at the same time hints for the solution. This time the *Applications* are better represented, although some lectures do not appear in these Proceedings because they have been published elsewhere.

We wish to thank all the participants because it is due to them that the meeting has been successful. Finally we wish to thank the CIRM (Centro Internazionale per la Ricerca Matematica) for its financial support. Special thanks again go the secretary, Mr. A. Micheletti, for his help (more than help) before, during and after the meeting.

Giuseppe Da Prato (Scuola Normale Superiore, Pisa)

Luciano Tubaro (University of Trento)

CONTENTS

S. Albeverio, R. Hoegh-Krohn, H. Holden, T. Kolsrud – *A covariant Feynman-Kac formula for unitary bundles over euclidean space* 1

A. Bensoussan – *On the integrated formulation of Zakai and Kushner equations* 13

V.S. Borkar, S.K. Mitter – *Lattice approximation in the stochastic quantization of* $(\phi^4)_2$ *fields* 24

M. Chaleyat-Maurel, D. Michel – *The support of the density of a filter in the uncorrelated case* 33

P.L. Chow, J.L. Menaldi – *Variational inequalities for the control of stochastic partial differential equations* 42

D.A. Dawson, L.G. Gorostiza – *Generalized solutions of stochastic evolution equations* 53

A. Dembo, O. Zeitouni – *On the relation of anticipative Stratonovich and symetric integrals: a decomposition formula* 66

A. Frigerio – *Some applications of quantum probability to stochastic differential equations in Hilbert space* 77

I. Gyöngy – *The stability of stochastic partial differential equations and applications. Theorems on supports* 91

G. Kallianpur, V. Pérez-Abreu – *Weak convergence of solutions of stochastic evolution equations on nuclear spaces* 119

P. Kotelenez – *A stochastic reaction-diffusion model* 132

H.-H. Kuo – *Stochastic partial differential equations of generalized Brownian functionals* 138

P.L. Lions – *Viscosity solutions of fully nonlinear second order equations and optimal stochastic control in infinite dimensions. Part II: optimal control of Zakai's equation* 147

S. Meleard, S. Roelly-Coppoletta – *A generalized equation for a continuous measure branching process* 171

D. Nualart, A.S. Üstünel – *Mesures cylindriques et distributions sur l'espace de Wiener* 186

D. Nualart, M. Zakai – *A summary of some identities of the Malliavin calculus* 192

D. Ocone, E. Pardoux – *A Lie algebraic criterion for non-existence of finite dimensionally computable filters* 197

M. Piccioni – *A generalization of Wahba's theorem on the equivalence between spline smoothing and Bayesian estimation* 205

P. Protter – *A connection between the expansion of filtrations and Girsanov's theorem* 221

K.-U. Schaumlöffel – *White noise in space and time as the time-derivative of a cylindrical Wiener process* 225

W. Smolenski, R. Sztencel – *Large deviations for non-linear radonifications of white noise* 230

J. Zabczyk – *Symmetric solutions of semilinear stochastic equations* 237

List of Participants 257

A COVARIANT FEYNMAN-KAC FORMULA FOR UNITARY BUNDLES OVER EUCLIDEAN SPACE

Introduction. The purpose of the present article is to derive a kind of covariant Feynman-Kac formula, and with some more specificity, this refers to a Euclidean Brownian motion lifted to a unitary bundle. To motivate it, let us first go back to the Schrödinger equation

(1) $$i\partial\psi/\partial t=-1/2\Delta\psi+V\psi=-1/2\sum_{j=1}^{d}\partial_j^2\psi+V\psi,\quad \psi(\cdot,0)=\psi_0.$$

(Here and below, physical constants are all equal to one; also $\partial_\mu\equiv\partial/\partial x^\mu$.) Regarding the potential V, which expresses interaction, as the electrostatic potential A_0, this equation becomes a special case of

(2) $$i(\partial/\partial t+iA_0)\psi=-1/2\sum_{j=1}^{d}(\partial_j+iA_j)^2\psi,\quad \psi(\cdot,0)=\psi_0,$$

where $A=(A_1,..,A_d):\mathbb{R}^d\to\mathbb{R}^d$ is the magnetic potential. Let $x_0=t$ and

(3) $$D_\mu=\partial_\mu+iA_\mu,\ 0\le\mu\le d.$$

Then the D_μ are covariant derivatives on a (trivial) line bundle $\mathbb{R}^{d+1}\times\mathbb{C}$, associated with the bundle $\mathbb{R}^{d+1}\times S^1$, and (2) can be written

(4) $$iD_0\psi=-1/2\sum_{j=1}^{d}D_j^2\psi,\quad \psi(\cdot,0)=\psi_0.$$

Replacing t by -it we obtain instead the diffusion equation

(5) $$\partial\varphi/\partial t=1/2\sum_{j=1}^{d}D_j^2\varphi+V\varphi,\quad \varphi(\cdot,0)=\varphi_0,$$

which can be solved using stochastic analysis (and then the solution to (4) is obtained by analytic continuation in t). Indeed, using the Cameron-Martin-Girsanov and Feynman-Kac formulae for the drift and potential terms respectively, we get

(6) $$\varphi(x,t)=\mathbb{E}_x\{\exp[i(\int_0^t A(b_s)\cdot db_s + \int_0^t 1/2\,\mathrm{div}A(b_s)ds) - \int_0^t V(b_s)ds]\varphi(b_t)\},$$

where b_t is a Brownian motion in $\mathbb{R}^d$, $\mathbb{E}_x$ denotes its expectation given that $b_0=x$, and the db-integral is taken in the sense of Ito. In Stratonovich notation this is simply written

(7) $$\varphi(x,t)=\mathbb{E}_x\{\exp[i\int_0^t A(b_s)\circ db_s - \int_0^t V(b_s)ds]\varphi(b_t)\},$$

and the first integral actually represents a lifting of b to the bundle $\mathbb{R}^d\times S^1$. (The variable t should be seen as a separate entity here.)
This formula is of course well-known, see e.g. Simon [**7**], Ch. V. 16, or Elworthy [**3**], Example VII.13.F. What we have in mind here is an extension of this formula to the case when more general unitary groups than $S^1=U(1)$ is considered. In particular will the groups in general be non-abelian.
The derivation we are about to give is, we think, elementary and straight-forward. Another merit is that it gives you an idea of how to extend non-standard stochastic analysis to geometric settings (see [**1**]). It is, however, probably possible also to derive it from the rather sophisticated machinery presented in Elworthy's book [**3**]. See also Potthoff [**5**].
Let us mention finally, that the mentioned formula plays an important part in the construction of so-called Higgs-fields, see [**2**] (and also [**6**] and further references therein). In fact this construction has been our motivation.

1.

We start a bit more generally than is actually needed. We have a Riemannian manifold N, and consider a trivial principal bundle $E=N\times G$, where G is a compact Lie group. There is a unitary, irreducible and locally faithful representation ρ of G in GL(W) where W is a complex vector space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The Lie algebra of G will be denoted by $\mathfrak{g}=T_eG$, where e is the identity element in G. There is a $\mathfrak{g}$-valued one-form ω, the connection form, such that in local co-ordinates

(8) $$\omega=\sum A_\mu dx^\mu,$$

where the A_μ are $\mathfrak{g}$-valued functions. The corresponding covariant derivatives are then

(9) $$D_\mu\varphi\equiv\partial_\mu\varphi+\rho(A_\mu)\varphi,\quad 1\le\mu\le d,$$

where $\partial_\mu=\partial/\partial x^\mu$ and the same letter ρ is used to denote the induced representation of $\mathcal{G}$.

The covariant energy (Dirichlet) form is, for a function $\varphi: N \to W$,

(10) $$\int_N |D\varphi|^2 \, dvol,$$

whereas the classical energy form is

(11) $$\int_N |d\varphi|^2 \, dvol,$$

with d denoting outer differentiation. If we have a potential, i.e. a real-valued, and, to avoid complications, non-negative and smooth function V on N, we obtain the total energy by adding the term

(12) $$\int_N V|\varphi|^2 \, dvol,$$

to (10, 11).

2.

Let N be an oriented manifold and denote by $\mathcal{C}N$ all curves in N, i.e. all equivalence classes up to reparametrisation of piecewise smooth and continuous maps c: $[0,1] \to N$. We write $c^- = c(0)$, $c^+ = c(1)$, and denote by c^{-1} the inverse of c, i.e. c with reversed orientation. Whenever $c_1^+ = c_2^-$ we can compose c_1 and c_2 (c_1 followed by c_2) and we write c_1c_2 for this curve. When defined the composition is associative, so $\mathcal{C}N$ is a *partial group*. We will use the notation $\mathcal{C}_xN$ for the curves c in $\mathcal{C}N$ with $c^- = x$.

Let now G and $\mathcal{G}$ be as in the introduction, and denote by E the trivial bundle over N with fibre G, i.e. $E = N \times G$. With a *connection* τ we understand a *lifting* of curves in N to curves in E, preserving the partial group structures. We therefore demand

(13) $\tau: \mathcal{C}N \times G \to \mathcal{C}E$,

(14) $\tau(\cdot,g): \mathcal{C}_xN \to \mathcal{C}_{(x,g)}E$,

i.e.

(15) $\tau(c,g)^- = (c^-,g)$.

Now for some $h \in G$ we have $\tau(c,g)^+ = (c^+,h)$. We will write $h = gm(c)$:

(16) $\tau(c,g)^+ = (c^+, gm(c))$.

We want τ to be a homomorphism under the partial group structures:

(17) $\tau(c_1c_2,g) = \tau(c_1,g)\tau(c_2,gm(c_1))$,

and

(18) $\tau(c,g)^{-1}=\tau(c^{-1},gm(c))$.

From (16) and (17) follows

$$(c_2^+,gm(c_1c_2))=((c_1c_2)^+,gm(c_1c_2))=\tau(c_1c_2,g)^+$$

$$=(\tau(c_1,g)\tau(c_2,gm(c_1)))^+=\tau(c_2,gm(c_1))^+=(c_2^+,gm(c_1)m(c_2)).$$

Similarly (13), (14) and (16) yield

$$(c^-,gm(c)m(c^{-1}))=((c^{-1})^+,gm(c)m(c^{-1}))$$

$$=\tau(c^{-1},gm(c))^+=(\tau(c,g)^{-1})^+=\tau(c,g)^-=(c^-,g).$$

Consequently m must be a homomorphism of $\mathcal{C}N$ into G, or - better - a representation of $\mathcal{C}N$ in G:

(19) $m\colon \mathcal{C}N\to G$.

(20) $m(c_1c_2)=m(c_1)m(c_2)$,

(21) $m(c^{-1})=m(c)^{-1}$.

We call m a *multiplicative curve integral* (MCI). The lifting of c to a curve in E, given the point $g\in G$, is now defined as

(22) $\tau(c,g)(t)=\tau(c^t,g)^+=(c(t),gm(c^t))$,

where c^t $(0\le t\le 1)$ denotes the curve

(23) $c^t(s)=c(st)$, $0\le s\le 1$.

It is not difficult to check that there is a one-to-one correspondence between connections τ and MCIs m. We have already seen how τ gives rise to m. Conversely, given m, we simply define τ by eqn (16). Then (13-18) hold.

Suppose now that we have an MCI which is smooth in the sense that $t\to m(c^t)$ is smooth whenever c is. Then we can define

(24) $\alpha(c)=\int_0^1 m(c^t)^{-1}\,dm(c^t)\in \mathfrak{g}$.

Then α is in the obvious sense a (smooth) additive *curve integral* (ACI). Conversely, given an ACI we obtain an MCI by solving the differential èquations

(25) $m(c^t)^{-1}\,dm(c^t) = d\alpha(c^t),\ m(c^0) = e.$

Hence we have a one-to-one correspondence between smooth ACIs and MCIs. Now, given an ACI α we can write

(26) $$\alpha(c) = \int_c \omega,$$

where ω is a $\mathcal{G}$-valued one form: $\omega \in \Omega^1(N,\mathcal{G})$. (If $X \in T_xN$ is determined by the curve c, define $\omega(X) = d\alpha(c^t)/dt|_{t=0}$.) Thus ω is a connection form as in section 1. It is evident that any $\omega \in \Omega^1(N,\mathcal{G})$ defines an ACI α by eqn. (26). Summing up, we see that there is a one-to-one correspondence between (smooth) connections τ defined by eqns. (13-18) and connection forms $\omega \in \Omega^1(N,\mathcal{G})$.

3.

From now on N will be $\mathbb{R}^d$. Then the outer differential can be identified with the ordinary gradient.

Let us fix some more notation. To start with, we will work on the lattice

(27) $\delta\mathbb{Z}^d \equiv \{\delta x : x \in \mathbb{Z}^d\},\ \delta > 0,$

and we write 'dx' for Lebesgue measure on $\delta\mathbb{Z}^d$, i.e. δ^d times the counting measure. The basic Hilbert space is

(28) $\mathbf{L}^2(\delta\mathbb{Z}^d) \equiv L^2(\delta\mathbb{Z}^d, dx; W),$

consisting of all functions $f: \delta\mathbb{Z}^d \to W$ such that $(f,f)_\delta < \infty$, where

(29) $$(f,g)_\delta \equiv \int_{\delta\mathbb{Z}^d} \langle f(x), g(x)\rangle\, dx \equiv \delta^d \sum_{x \in \delta\mathbb{Z}^d} \langle f(x), g(x)\rangle.$$

(Recall that W is the space associated with the representation of G.)

It is convenient to use the gradient symbol '∇' also for discrete quantities. We define

(30) $$(\nabla f, \nabla g)_\delta \equiv \int_{\delta\mathbb{Z}^d} \langle \nabla f(x), \nabla g(x)\rangle\, dx \equiv \delta^{d-2} \sum\sum_{\langle xy\rangle} \langle f(x)-f(y), g(x)-g(y)\rangle,$$

so that

(31) $$\int_{\delta\mathbb{Z}^d} |\nabla f|^2\, dx = \delta^{d-2} \sum\sum_{\langle xy\rangle} |f(x)-f(y)|^2.$$

Here $\langle xy\rangle$ denotes that we only sum over nearest neighbours, i.e. x and y with $|x-y| = \delta$.

4.

The basic, free, energy form is the one in Eq. (30). Define the discrete Laplace operator by

(32) $$\Delta f(x) = \Delta_\delta f(x) = -2d\delta^{-2}[f(x) - 1/2d \sum_{|x-y|=\delta} f(y)].$$

Then

(33) $$-(f,\Delta f)_\delta = \delta^{d-2} \sum_x \langle f(x), 2df(x) - \sum_{|x-y|=\delta} f(y)\rangle$$

$$= \delta^{d-2} \sum\sum_{\langle xy\rangle} \langle f(x), f(x)-f(y)\rangle = \delta^{d-2} \sum\sum_{\langle xy\rangle} |f(x)-f(y)|^2 = \int_{\delta\mathbb{Z}^d} |\nabla f|^2 \ dx.$$

[Here we have used that in a double sum $\Sigma\Sigma a_i(b_i-b_j)$ we have an (i,j)-term: $a_i(b_i-b_j)$, and a (j,i)-term: $a_j(b_j-b_i)=-a_j(b_i-b_j)$. Now their sum is $(a_i-a_j)(b_i-b_j)$.]

It follows from general theory, see e.g. [**4**], that Δ_δ can be associated with a stochastic process. We shall denote it by $b=(b_t)_{t\geq 0}$ or b^δ if we wish to emphasise the dependence of δ. The relation is

(34) $$\lim_{t\to 0^+} t^{-1}\{f(x)-\mathbb{E}_x[f(b^\delta(t))]\} = -\Delta_\delta f(x).$$

Here $\mathbb{E}_x$ is the expectation operator associated with the probability measure $\mathbb{P}_x$ governing b when starting at a point x in the lattice. From the translation invariance of the Laplacian it is clear that b is likewise. In fact

(35) $$\mathbb{P}_x[b_t \in M] = \lambda_{\delta,t}(M-x),$$

where

(36) $$\lambda_{\delta,t} \equiv e^{-2td/\delta^2} \sum_{n=0}^{\infty} \frac{(2td/\delta^2)^n}{n!} (\varepsilon_\delta)^{*n} = \exp[-(2td/\delta^2)(1-\varepsilon_\delta)],\ t>0,$$

is the corresponding convolution semi-group of probability measures. Here $(\varepsilon_\delta)^{*n}$ is the n-fold convolution power of the measure

(37) $$\varepsilon_\delta \equiv 2^{-d} \sum_{|x|=\delta} \varepsilon(x,\cdot).$$

with $\varepsilon(x,\cdot)$ denoting the Dirac measure (unit mass) at the point x. Summing up, our basic process b is a random walk with continuous time. More precisely is b a compound Poisson - and therefore a pure jump - process with state space $\delta\mathbb{Z}^d$. It is the continuous time/discrete space analogue of the Brownian motion.

The Fourier transform of (36) is

(38) $$\int e^{-ix\cdot\xi}\lambda_{\delta,t}(dx)=\exp\Big[-(2td/\delta^2)[1-1/d\sum_{n=1}^{d}\cos\delta\xi_n]\Big],$$

which as $\delta\to0^+$ tends to

(39) $$\exp[-t|\xi|^2]=\int_{\mathbb{R}^d} e^{-ix\cdot\xi}(4\pi t)^{-1}\exp[-|x|^2/4t]\,dx.$$

Hence $b^{\delta}(t)\to b^{0}(t)$ in law for each t as $\delta\to0^+$, where $b^0=(b^0(t))$ is a Brownian motion in $\mathbb{R}^d$ (by Lévy's classical theorem). It follows from this that as processes $b^{\delta}\to b^0$ in law (in the sense of convergence of finite-dimensional distributions).

It will be convenient below to have a notation for the (unnormalised) transition expectations

(40) $$\mathbb{E}_{t;x,y}\equiv\mathbb{E}[\cdot\,|b_0=x,\ b_t=y],$$

so that

(41) $$\mathbb{E}_x[f(b_t)]=\int \mathbb{E}_{t;x,y}[f(b_t)]\,dy=\int \mathbb{E}_{t;x,y}[f(y)]\,dy=\int \mathbb{E}_{t;x,y}[\mathbb{1}]\,f(y)\,dy,$$

where $\mathbb{1}$ is the identity (matrix) on W, and the integrals are over $\delta\mathbb{Z}^d$.

4.

We will now consider the form on $\mathbf{L}^2(\delta\mathbb{Z}^d)$ obtained when replacing ∇ by covariant derivation in (29). Accordingly, let m be an MCI (§1). Then, with $D\equiv D^m$ denoting the covariant derivative given by m, the energy form

(42) $$(\varphi,A\varphi)_\delta=\int_{\delta\mathbb{Z}^d}|D\varphi|^2dx,$$

where $A=-\mathrm{tr}D^*D$, is defined by

(43) $$(\varphi,A\varphi)_\delta=\delta^{d-2}\sum_{\langle xy\rangle}|\varphi(x)-\rho(m_{xy})\varphi(y)|^2,$$

and m_{xy} denotes m acting on the directed line segment from x to y.
The following result shows how to obtain the covariant energy form (42) from its 'classical' counterpart (29).

THEOREM 4.1. The kernel of the semi-group associated with A via m is given by

(44) $$e^{-tA}(x,y)=\mathbb{E}_{t;x,y}[\rho(m_b)],\ t\geq0,$$

i.e.

(45) $$e^{-tA}f(x)=\mathbb{E}_x[\rho(m_{b(t)})f(b(t))],\ t\geq 0.$$

Proof. We show first that $\mathbb{E}_{t;x,y}[\rho(m_b)]$ defines a semi-group, and write $T_t f(x)$ for the right-hand side of (45). Then

(46) $$T_{t+s}f(x)=\int \mathbb{E}_{t+s;x,z}[\rho(m_b)]f(z)\,dz=\int \mathbb{E}[\rho(m_b)|b_0=x,b_{t+s}=z]f(z)\,dz$$
$$=\int\int \mathbb{E}[\rho(m_b)|b_0=x,b_s=y,b_{t+s}=z]f(z)\,dydz,$$

where all integrals are over $\delta\mathbb{Z}^d$.
Let now β and γ denote b restricted to $[0,s]$ and $[s,t+s]$, respectively. By the multiplicativity of ρ and m,

(47) $$\rho(m_b)=\rho(m_{\beta\gamma})=\rho(m_\beta m_\gamma)=\rho(m_\beta)\rho(m_\gamma),$$

and by the Markov property of b, $\rho(m_\beta)$ and $\rho(m_\gamma)$ are independent given b(s). Moreover, since b is time homogeneous, i.e. has stationary increments, γ has the same law as b restricted to $[0,t]$. Consequently

(48) $$T_{t+s}f(x)=\int\int \mathbb{E}[\rho(m_b)|b_0=x,b_s=y]\mathbb{E}[\rho(m_b)|b_0=y,b_t=z]f(z)\,dydz$$
$$=\int \mathbb{E}_{s;x,y}[\rho(m_b)|b_0=x,b_s=y]T_tf(y)\,dy=T_sT_tf(x),$$

which is the semi-group property.

We will now show that (T_t) is symmetric on $\mathbf{L}^2(\delta\mathbb{Z}^d)$. We have, suppressing all δs for simplicity,

(49) $$(f,T_tg)=\int\int\langle f(x),\ \mathbb{E}_{t;x,y}[\rho(m_b)]g(y)\rangle\,dxdy$$
$$=\int\int\langle \mathbb{E}_{t;x,y}[\rho(m_b)^*]f(x),g(y)\rangle\,dxdy$$
$$=\int\int\langle \mathbb{E}_{t;x,y}[\rho(m(b^{-1}))]f(x),g(y)\rangle\,dxdy,$$

where we have used that m is an MCI and ρ is unitary. Now b is a symmetric Markov process ([**1**,**4**]), so b and b^{-1} are equal in law. Hence

(50) $$\mathbb{E}_{t;x,y}[\rho(m(b^{-1}))]$$
$$=\mathbb{E}[\rho(m(b^{-1}))|b(0)=x,b(t)=y]$$
$$=\mathbb{E}[\rho(m(b^{-1})|b^{-1}(0)=y,b^{-1}(t)=x]$$
$$=\mathbb{E}[\rho(m_b)|b(0)=y,b(t)=x]=\mathbb{E}_{t;y,x}[\rho(m_b)],$$

whence

(51) $$(f,T_tg)=\int\int\langle \mathbb{E}_{t;y,x}[\rho(m_b)]f(x),g(y)\rangle\,dxdy=$$

$$= \int\int \langle T_t f(y), g(y) \rangle \, dy = (T_t f, g).$$

It now follows that each T_t is positive:

(52) $(f, T_{2t} f) = (f, T_t T_t f) = (T_t f, T_t f) \geq 0.$

From the obvious 'diamagnetic' bound

(53) $|T_t(x,y)| \leq e^{-t\Delta}(x,y),$

follows that (T_t) is a is strongly continuous semigroup, so by general operator theory T_t is of the form $\exp[-tB]$ for a uniquely defined positive and self-adjoint operator B on $\mathbf{L}^2(\delta\mathbb{Z}^d)$. Our next item is to show that B = A.
If N is the number of steps needed to reach y from x we can write

(54) $\mathbb{E}_{t;x,y}[\rho(m_b)] = \mathbb{E}_{t;x,y}[\rho(m_b);\ N \leq 1] + \mathbb{E}_{t;x,y}[\rho(m_b);\ N > 1].$

The second term is, as one sees from Eq. (38), o(t) as $t \to 0$, and in the first term $\rho(m_b)$ can be replaced by $\rho(m_{xy})$. Consequently, by eqns. (30) and (32), and using $\rho(m_{xx}) = \rho(e) = \mathbb{1}$,

(55) $-d/dt|_{t=0} \mathbb{E}_x[\rho(m_{b(t)}) f(b(t))]$

$$= -d/dt|_{t=0} \int \mathbb{E}_{t;x,y}[\rho(m_b)] f(y) \, dy$$

$$= \int \rho(m_{xy}) (-d/dt|_{t=0})\ \mathbb{E}_{t;x,y}[\mathbb{1}] f(y) \, dy$$

$$= \sum_{|x-y| \leq \delta} \rho(m_{xy}) (-d/dt|_{t=0})\ \mathbb{E}_{t;x,y}[\mathbb{1}] f(y)$$

$$= 2d\, \delta^{-2} [\rho(m_{xx}) f(x) - 1/2d \sum_{|x-y| = \delta} \rho(m_{xy}) f(y)]$$

$$= 2d\, \delta^{-2} [f(x) - 1/2d \sum_{|x-y| = \delta} \rho(m_{xy}) f(y)],$$

so

(56) $-d/dt|_{t=0} \int_{\delta\mathbb{Z}^d} \langle f(x), \mathbb{E}_x[\rho(m_{b(t)}) f(b(t))] \rangle \, dx$

$$= \sum\sum_{\langle xy \rangle} \langle f(x), f(x) - \rho(m_{xy}) f(y) \rangle.$$

To see that the latter term equals $(f, Af)_\delta$ it suffices to show (cf. the reasoning that led to (33)) that

(57) $\langle f(y), f(y)-\rho(m_{yx})f(x)\rangle = -\langle \rho(m_{xy})f(y), f(x)-\rho(m_{xy})f(y)\rangle$,

because then the sum of the (x,y) and the (y,x) term in the sum above is

(58) $\langle f(x), f(x)-\rho(m_{xy})f(y)\rangle + \langle f(y), f(y)-\rho(m_{yx})f(x)\rangle = |f(x)-\rho(m_{xy})f(y)|^2$.

The properties of ρ and m give

(59) $\rho(m_{yx})^* = \rho(m_{yx})^{-1} = \rho(m_{xy})$,

so

(60) $\langle f(y), f(y)-\rho(m_{yx})f(x)\rangle$

$= -\langle f(y), \rho(m_{yx})f(x)-f(y)\rangle$

$= -\langle \rho(m_{yx})^* f(y), f(x)-\rho(m_{yx})^{-1}f(y)\rangle =$

$= -\langle \rho(m_{xy})f(y), f(x)-\rho(m_{xy})f(y)\rangle$,

as was to be proved. ■

From Theorem 4.1 we obtain an expression for the resolvent of A:

COROLLARY 4.2. The resolvent kernel associated with A is

(61) $$G_\alpha(x,y) \equiv (\alpha+A)^{-1}(x,y) = \int_0^\infty e^{-\alpha t}\, \mathbb{E}_{t;x,y}[\rho(m_b)]\,dt, \alpha>0.$$

Remarks. (a) Note that b^δ takes its values in the lattice $\delta\mathbb{Z}^d$. When we use m to lift its trajectories, we have tacitly considered the process as one in $\mathbb{R}^d$, and the curve $b^\delta(\cdot)$, is then obtained by connecting the points of successive jumps. In particular this means that m(b) is really a Stratonovich integral (as it has to be to conform with eqn (7)).
(b) It is not at all necessary to assume that G is a Lie group. This is merely for the interpretation with covariant derivation. What is important is that G has a unitary, etc., and continuous representation.
(c) It should be clear from the proof above that other Markov jump processes, not necessarily translation invariant, can be lifted using the same ideas.

5.

To complete our investigations, we should now put δ=0 in the formulae obtained above. (This could of course also be handled by using results from non-standard analysis, and let δ be infinitesimal.)
First of all, the proof of Theorem 4.1 shows that the right-hand sides in eqns (44-45) define a strongly continuous contraction semi-group also for δ=0, i.e. for