Lecture Notes in Mathematics 1745

V. D. Milman G. Schechtman (Eds.)

Geometric Aspects of Functional Analysis

Israel Seminar

GAFA
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Editors

Vitali D. Milman
Department of Mathematics
Tel Aviv University
69978 Tel Aviv, Israel

E-mail: vitali@math.tau.ac.il

Gideon Schechtman
Department of Mathematics
Weizmann Institute of Science
76100 Rehovot, Israel

E-mail: gideon@wisdom.weizmann.ac.il

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Geometric aspects of functional analysis / Israel Seminar (GAFA) 1996 - 2000. V. D. Milman ; G. Schechtman (ed.). - Berlin ; Heidelberg ; New York ; Barcelona ; Hong Kong ; London ; Milan ; Paris ; Singapore ; Tokyo : Springer, 2000
(Lecture notes in mathematics ; Vol. 1745)
ISBN 3-540-41070-8

Mathematics Subject Classification (2000): 40-06, 46B07, 52-06, 60-06

ISSN 0075-8434
ISBN 3-540-41070-8 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH

Printed in Germany

Typesetting: Camera-ready TEX output by the author
SPIN: 10724232 41/3142-543210 - Printed on acid-free paper

Preface

During the last two decades the following volumes containing papers presented at the Israel Seminar in Geometric Aspects of Functional Analysis appeared

1983-84 Published privately by Tel Aviv University
1985-86 Springer Lecture Notes, Vol. 1267
1986-87 Springer Lecture Notes, Vol. 1317
1987-88 Springer Lecture Notes, Vol. 1376
1989-90 Springer Lecture Notes, Vol. 1469
1992-94 Operator Theory: Advances and Applications, Vol. 77, Birkhauser
1994-96 MSRI Publications, Vol. 34, Cambridge University Press.

The first six were edited by Lindenstrauss and Milman while the last, which also contains material from the program in Convex Geometry and Geometric Analysis held at MSRI in 1996, was edited by Ball and Milman.

The current volume reflects some of the new directions in Banach Space Theory in the last few years. These include the tighter connection with classical convexity and as a result the added emphasis on convex bodies which are not necessarily centrally symmetric. Initially, emerging from the functional analysis point of view, symmetric convex bodies were the natural object of investigation but, as it becomes more and more clear, a large portion of the theory carries over to the non-symmetric case and this sometimes sheds new light even on the symmetric case. A similar situation, which is also reflected in some of the articles of this volume, is the treatment of bodies which have only very weak convex–like structure - they are only p-convex for some $0 < p < 1$. Another topic which is represented here is the use of some new probabilistic tools; in particular transportation of measures methods and new inequalities emerging from Poincare–like inequalities. Finally, several of the papers here deal with improving and finding the best, or best order, constants in several results. This is another topic which has received considerable attention recently.

All the papers here are original research papers and were subject to the usual standards of refereeing.

As in previous volumes of the GAFA Seminar, we also list here all the talks given in the seminar as well as talks in related workshops and conferences. We believe this gives a sense of the main directions of research in our area.

We are grateful to Ms. Diana Yellin for taking excellent care of the typesetting aspects of this volume.

Vitali Milman
Gideon Schechtman

Contents

The Transportation Cost for the Cube

M. Anttila

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

Abstract. The transportation method for proving concentration of measure results works directly for the cube. Here we find the best constant that can be found using this method which turns out to be better than those obtained by previous methods and which cannot be far from that which is best possible.

1 Introduction

In this paper we prove a deviation inequality for the cube using a method developed by K. Marton to show similar results for Markov chains. Talagrand named this method the transportation method when simplifying Marton's arguments for certain product spaces.

Let us consider the cube, $[0,1]^n \subset \mathbf{R}^n$, and denote by P the n-dimensional Lebesgue measure on it. If B is any measurable subset of the cube, let B_t be its expansion,

$$B_t = \{x \in [0,1]^n : d(x,B) \leq t\},$$

where $d(x,B)$ denotes the Euclidean distance from x to B. We shall prove a deviation inequality of the form

$$1 - P(B_t) \leq e^{-ct^2}, \tag{1}$$

provided B does not have too small probability, where c is a constant dependent on $P(B)$.

Concentration results of this form have been known for the cube for some time. Indeed, it was pointed out in [TIS] that inequality (1), with bound $\frac{1}{P(B)}e^{-\frac{\pi}{2}t^2}$, can be obtained directly from concentration in Gauss space via a measure preserving Lipschitz map. Here our objective is to point out that the transportation method works directly for the cube and, more importantly, to ascertain the best constant that can be found using this method. This constant is better than those previously obtained and cannot be far from best possible. Finding this constant gives rise to a "text-book example" of a variational problem which has a surprisingly neat solution.

Marton's original method uses an inequality bounding the so-called $\bar{d}$-distance by informational divergence to prove a concentration of measure result for certain Markov chains (see [M] for definitions and a detailed account).

Supported by EPSRC-97409672.

The important thing in her method is that her one-dimensional inequality can be inducted on dimension and quickly implies a concentration of measure result. Marton's method certainly works for product spaces. However, Talagrand simplifies it and strengthens the result for certain product spaces in [T], by considering l_2, rather than l_1, distance in the inequality. More precisely, Talagrand's inequality bounds something called transportation cost, with the square of the l_2-distance as the cost function. The definition of transportation cost now follows.

Suppose we have two probability measures μ_1 and μ_2 on measurable spaces Ω_1 and Ω_2 respectively. The basic idea is to look at all bijections $b : \Omega_1 \to \Omega_2$ which transport μ_1 to μ_2, i.e. for which

$$\mu_1(A) = \mu_2\big(b(A)\big) \quad \text{whenever } A \subset \Omega_1.$$

For a given function $C : \Omega_1 \times \Omega_2 \to \mathbf{R}^+ \cup \{\infty\}$, $(C(x, b(x))$ measures the cost of moving a unit mass from x to $b(x))$, we seek to minimise

$$\int_{\Omega_1} C\big(x, b(x)\big)\, d\mu_1(x).$$

If μ_1 or μ_2 has atoms then there may be no such function b. So, formally, the transportation cost is defined in terms of an integral over the product space $\Omega_1 \times \Omega_2$ with respect to a probability measure with marginals μ_1 and μ_2. However, in our case $\Omega_1 = \Omega_2 = [0,1]^n$ and our measures will be the Lebesgue measure on the cube itself and a weighted Lebesgue measure on one of its subsets, so no such formality is needed here.

As already mentioned, we shall use the square of the Euclidean distance as our "cost function", C, just as Talagrand did for Gaussian measure. So now we can define the transportation cost, $\tau(\mu_1, \mu_2)$, to be the minimum, over all functions b as above, of

$$\int_{\Omega_1} |x - b(x)|_2^2\, d\mu_1(x).$$

The main result of this article is the following:

Theorem (Bound on Transportation Cost) *If A is a subset of $[0,1]^n$ and μ is the normalised restriction of the Lebesgue measure, P, to A (i.e. has density $1_A/P(A)$ with respect to P), then*

$$\tau(\mu, P) \;\leq\; \frac{2}{\pi^2} \log \frac{1}{P(A)}.$$

From this it is easy to get a concentration estimate using the following short argument. Let $B \subset [0,1]^n$. The cost of transporting $[0,1]^n$ to the complement of the expanded B, B_t^c, is clearly greater than that of transporting B

(a subset of $[0,1]^n$) to B_t^c. The Theorem gives an upper bound on the former and the latter is greater than $P(B)t^2$. So

$$P(B)t^2 \leq \frac{2}{\pi^2} \log \frac{1}{P(B_t^c)}.$$

Rearranging this we have

$$P(B_t^c) \leq e^{\frac{-\pi^2}{2} P(B)t^2}.$$

However, this bound can be improved by applying the Theorem to B as well as to B_t^c as in [M] and [T]. This gives the slightly better estimate

$$P(B_t^c) \leq \exp\left\{ \frac{-\pi^2}{2} \left(t - \sqrt{\frac{2}{\pi^2} \log \frac{1}{P(B)}} \right)^2 \right\}.$$

As already mentioned, we will see, in the proof of the Theorem, that $\frac{\pi^2}{2}$ is the best constant that we can find using this method. Before we begin the proof, however, we observe that our constant is not far from best possible.

The following tells us that c in (1) cannot be greater than 6. Let K be the cube of volume 1, now centered at zero. We regard K as a probability space and define on it the random variable $X_\theta : x \mapsto \langle x, \theta \rangle$, where $\theta = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)$, so that the density of X_θ is obtained by scanning across K with hyperplanes perpendicular to θ. Since

$$X_\theta(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i(x),$$

where $X_i : x \mapsto x_i \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ are random variables with zero mean and variance $\rho^2 = \frac{1}{12}$, the Central Limit Theorem tells us that as $n \to \infty$

$$\begin{aligned} \operatorname{Prob}(X_\theta > t) &\to \frac{1}{\sqrt{2\pi}\rho} \int_t^\infty e^{-\frac{y^2}{2\rho^2}}\, dy \\ &\geq \frac{\rho}{\sqrt{2\pi} t} e^{-\frac{t^2}{2\rho^2}} \\ &= \frac{\text{const}}{t} e^{-6t^2}. \end{aligned}$$

Now we need only notice that the left hand side is precisely $P(H_t^c)$ for $H \subset [0,1]^n$ given by the intersection of the cube with the halfspace through zero perpendicular to θ:

$$H = \{ x \in [0,1]^n : \langle x, \theta \rangle \leq 0 \}. \qquad \square$$

2 Proof of the Theorem

We wish to show by induction that

$$\tau(\mu, P) \leq \frac{2}{\pi^2} \log \frac{1}{P(A)} \quad \text{for } A \subset [0,1]^n,$$

where the left hand side is the minimum cost of transporting A to $[0,1]^n$.

We show in Section 2.1 that a result of the form

$$\tau(\mu, P) \leq c \log \frac{1}{P(A)} \quad \text{for } A \subset [0,1]^n, \tag{2}$$

can be obtained from the following one-dimensional inequality for absolutely continuous $f : [0,1] \to [0,1]$,

$$\int_0^1 \left(f(t) - t\right)^2 f'(t)\, dt \;\leq\; c \int_0^1 f'(t) \log f'(t)\, dt, \tag{3}$$

where c is the same constant in both inequalities.

We continue in the ensuing sections to use the Calculus of Variations to show that (3) holds for all appropriate f if (and only if) $c \geq \frac{2}{\pi^2}$. More precisely we show that there is an optimising function satisfying an appropriate Euler-Lagrange equation and then we analyse the solutions of this equation.

2.1 The Inductive Step

Let us choose one of the n coordinate directions, e_1 say. We denote by $g(t)$ the $(n-1)$-dimensional volume of A intersected with the "slice of the cube at $t \in [0,1]$":

$$\{x \in [0,1]^n : \langle x, e_1 \rangle = t\}.$$

The idea is that we transport in the e_1 direction, the $(n-1)$-dimensional slices via an increasing function, f, such that the proportion of A between the slices at t and $t+\delta$ is equal to the proportion of $[0,1]^n$ between $f(t)$ and $f(t+\delta)$:

$$\frac{\delta g(t)}{P(A)} = f(t+\delta) - f(t). \tag{4}$$

The weighted cost of transporting in this way in one dimension is clearly

$$\int_0^1 \left(f(t) - t\right)^2 \frac{g(t)}{P(A)}\, dt. \tag{5}$$

We then use the inductive hypothesis, (2), to transport in each $(n-1)$-dimensional slice. The total of the transportation costs in all of the slices is at most

$$\int_0^1 c \log \frac{1}{g(f^{-1}(s))}\, ds.$$

After substituting $s = f(t)$, this is

$$-\int_0^1 cf'(t)\log g(t)\,dt. \tag{6}$$

We see from (4) that $f'(t) = \frac{g(t)}{P(A)}$. So to complete the inductive step we combine (5) and (6) and ask whether

$$\int_0^1 \left(f(t)-t\right)^2 f'(t)\,dt - \int_0^1 cf'(t)\log\left[f'(t)P(A)\right]\,dt \ \le\ c\log\frac{1}{P(A)}.$$

When rearranged, this is

$$\int_0^1 \left(f(t)-t\right)^2 f'(t)\,dt - \int_0^1 cf'(t)\log f'(t)\,dt \ \le\ c\log\frac{1}{P(A)}\int_0^1 \left(1-f'(t)\right)\,dt,$$

which simplifies to

$$\int_0^1 \left(f(t)-t\right)^2 f'(t)\,dt \ \le\ c\int_0^1 f'(t)\log f'(t)\,dt, \tag{7}$$

since $f(0)=0$ and $f(1)=1$.

The same inequality handles the one-dimensional case because we can transport in exactly the same way in dimension one, where $g(t) = 1_A(t)$ and where clearly we will not be required to transport further within $(n-1)$-dimensional sheets. So the transportation cost is at most (5). Further, since $f'(t) = \frac{1_A(t)}{P(A)}$, we have

$$c\log\frac{1}{P(A)} = c\int_0^1 f'(t)\log f'(t)\,dt. \quad \square$$

It is not difficult to find some c for which (7) holds (and hence such that (2),

$$\tau(\mu, P) \le c\log\frac{1}{P(A)},$$

is true). For example, if we rewrite the left hand side of (7) as below, we see that (7) holds with $c = 2$ by using standard methods from information theory and the Csiszár-Kullback-Pinsker inequality. We mention here also that the logarithmic Sobolev inequality for the cube implies (2) with $c = \frac{1}{\pi}$, see [OV]. However we wish to find the smallest c.

We begin by rewriting (7). Notice that

$$\int_0^1 \left(f(t)-t\right)^2\left(f'(t)-1\right)\,dt = 0, \quad \text{since } f(0)=0 \text{ and } f(1)=1.$$

So we can rewrite the left hand side of (7) as

$$\int_0^1 \left(f(t)-t\right)^2\,dt.$$

If we consider instead the deviation of f from t, $h(t) = f(t) - t$, then (7) becomes

$$\int_0^1 h^2(t)\,dt \le c \int_0^1 \big(1 + h'(t)\big) \log\big(1 + h'(t)\big)\,dt. \tag{8}$$

Our problem is to find the smallest constant, c, such that the functional in (8),

$$\mathcal{F}_c(h) = \int_0^1 \Big[c\big(1 + h'(t)\big) \log\big(1 + h'(t)\big) - h^2(t)\Big]\,dt, \tag{9}$$

is non-negative for all h in the admissible class of functions given by

$$\mathcal{C} = \big\{h \text{ absolutely continuous} : \; h(0) = h(1) = 0,\; h' > -1\big\}.$$

This variational problem is the subject of the following sections.

2.2 The Variational Problem

Recall that our aim is to find the smallest c such that the functional $\mathcal{F}_c$ is non-negative for all functions in $\mathcal{C}$. First we will show that for all $c > 0$, a minimiser of $\mathcal{F}_c$ exists and satisfies the Euler-Lagrange equation:

$$\big(1 + h'(t)\big)\,h(t) + \frac{c}{2}\,h''(t) = 0. \tag{10}$$

Then we will find that if $c > \frac{2}{\pi^2}$, the only solution of (10) which satisfies the boundary conditions of $\mathcal{C}$, is the trivial one, $h = 0$. Hence $\mathcal{F}_c \ge 0$ for such c. To show that $c = \frac{2}{\pi^2}$ is the smallest constant for which $\mathcal{F}_c$ is non-negative, we will consider specific functions in our admissible class.

A classical theorem of Tonelli on the existence of minimisers of a one-dimensional variational integral,

$$\mathcal{F}(v) = \int_I F(x, v, v')\,dx,$$

can be found, for example, in [BGH]. The standard conditions are that the Lagrangian, $F(x, v, p)$, is continuous, convex in p and has superlinear growth in p at ∞ (i.e. is such that there exists a function $\theta(p)$ such that

$$F(x, v, p) \ge \theta(p) \quad \text{for all } (x, v, p) \in I \times \mathbf{R} \times \mathbf{R}$$
$$\text{and } \frac{\theta(p)}{|p|} \to \infty \quad \text{as } |p| \to \infty).$$

The superlinearity condition clearly does not hold for our Lagrangian,

$$F_c(x, v, p) = c(1 + p)\log(1 + p) - v^2,$$

because the "v" term could make F_c very small. However, it is not hard to see that the standard arguments can be adapted to demonstrate the existence of minimisers in our case. In fact, our Lagrangian has certain invariance properties which, if anything, make our problem easier than the general one. We include here a very rough explanation.

We wish to show that there exists a function $u \in \mathcal{C}$ such that

$$\mathcal{F}_c(u) = \inf\{\mathcal{F}_c(v) : v \in \mathcal{C}\}.$$

Call this infimum λ, say. From the boundary conditions on $\mathcal{C}$, we have that $|v| \le 1$ for $v \in \mathcal{C}$, so $\mathcal{F}_c$ is bounded below. Hence we can find a minimising sequence, $\{u_k\} \subset \mathcal{C}$, such that $\mathcal{F}_c(u_k) \to \lambda$.

The following properties of our Lagrangian allow us to take the functions in this minimising sequence to be positive and concave. Since F_c comprises only the square of the function, v, and its derivatives, rotating a negative section of the function by 180^o leaves the functional unaltered. Further, if we approximate any positive function in the minimising sequence by a piecewise linear function and make this concave in steps, it is clear that in doing so the functional, $\mathcal{F}_c$, decreases. This follows since v increases and since

$$(1+p)\log(1+p) \quad \text{is convex for } p > -1.$$

We can use the Ascoli-Arzelà Theorem to show that a subsequence of $\{u_k\}$ converges uniformly to a continuous function u, say. Equiboundedness is clear. To prove equicontinuity, we need to show that u_k' cannot get too large on $[0,1]$. But since we restricted u_k to being positive and concave, we need only show that u_k' is not too large near zero.

Notice that since $u_k(0) = u_k(1) = 0$, we can write $\mathcal{F}_c(x, u_k, u_k')$ as

$$\int_0^1 c\big[(1+u_k')\log[1+u_k'] - u_k'\big] - u_k^2 \, dx.$$

So if, for $\varepsilon > 0$, $u_k(\varepsilon) = L\varepsilon$, it is not hard to see, using the restriction $|u_k| < 1$ and that $(1+p)\log[1+p] - p > 0$ for all positive p, that

$$\mathcal{F}_c(x, u_k, u_k') \ge \varepsilon c\big[(1+L)\log[1+L] - L\big] - 1.$$

This in turn gives us an upper bound on $L\varepsilon$ which tends to zero as $\varepsilon \to 0$.

To show that $u' > -1$, and hence $u \in \mathcal{C}$, requires noticing that $(1+p)\log(1+p)$ has infinite derivative at $p = -1$ and so a minimiser will not have derivative equal to -1, except possibly at 1. Finally, the concavity of the functions in the minimising sequence ensures that $u_k' \to u'$ *a.e.* Then $\mathcal{F}_c(u_k) \to \mathcal{F}_c(u)$ dominatedly. □

That any minimiser satisfies the Euler-Lagrange equation (10) is standard, see e.g. [BGH]. The only possible issue in our case is that the functional must be defined for all functions in a neighbourhood of the minimiser, u. But since we just saw that our Lagrangian forces $u' > -1$, this does not pose a problem.

2.3 Periodicity Analysis

It remains to show that for $c > \frac{2}{\pi^2}$, the only solution of the Euler-Lagrange equation is the trivial one (hence $\mathcal{F}_c \geq 0$ for such c) and that conversely there are functions in our admissible class for which $\mathcal{F}_c < 0$ if $c < \frac{2}{\pi^2}$.

Recall that the Euler-Lagrange equation is given by

$$\big(1 + h'(t)\big)\ h(t) + \frac{c}{2}\ h''(t) = 0. \tag{11}$$

If we rearrange and multiply both sides by $h'(t)$, (11) becomes

$$h(t)h'(t) = -\frac{c\ h''(t)}{2(1 + h'(t))}\ h'(t)$$

and this integrates to

$$-h^2(t) + M^2 = c\Big[h'(t) - \log\big(1 + h'(t)\big)\Big], \quad \text{where } M = \sup\ |h|. \tag{12}$$

If we define the function $\Omega : (-1, \infty) \to [0, \infty)$ to be

$$\Omega(s) = s - \log(1 + s),$$

then (12) can be written in terms of Ω as

$$\frac{1}{c}(-h^2 + M^2) = \Omega(h'). \tag{13}$$

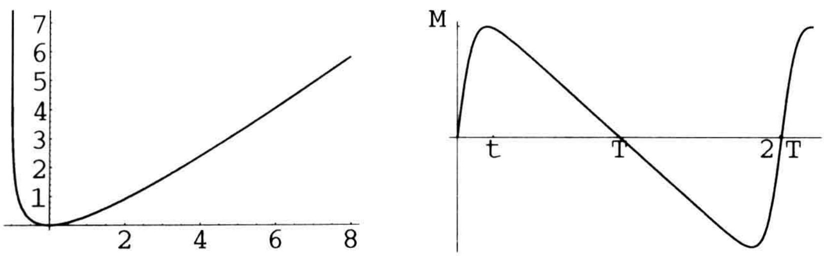

Fig. 1. The function Ω, (left), and a solution, h, of (12), (right).

It is not difficult to see that a solution of (12) either increases to M or is periodic. Since we have the restriction that any function in our admissible class is zero at 1, we need only consider periodic solutions. So if for $c > \frac{2}{\pi^2}$ every non-trivial solution has period greater than 2, then we know that there is no non-trivial solution in our admissible class for such c. Hence we will have $\mathcal{F}_c \geq 0$ for $c > \frac{2}{\pi^2}$.

Let $2T$ denote the period of a solution, h, of (12). Suppose that h attains its maximum at the point $t \in (0, T)$. Then we can express t as an integral over h between 0 and M:

$$t = \int_0^t ds = \int_0^M \frac{1}{h'}\, dh. \tag{14}$$

Similarly, for the second section of the semiperiod, on which $h' \leq 0$ we have

$$T - t = \int_M^0 \frac{1}{h'}\, dh. \tag{15}$$

So if we denote the two branches of Ω^{-1}, Ω_+^{-1} and Ω_-^{-1}, using (13), we can express h' in terms of the inverses Ω_+^{-1} and Ω_-^{-1}, depending on the sign of h'.

Hence from (14) and (15), we know the semiperiod of a periodic solution of (12) to be

$$T = \int_0^M \frac{1}{\Omega_+^{-1}(\frac{1}{c}(M^2 - h^2))}\, dh - \int_0^M \frac{1}{\Omega_-^{-1}(\frac{1}{c}(M^2 - h^2))}\, dh. \tag{16}$$

We shall see below that

$$\frac{1}{\Omega_+^{-1}(x)} - \frac{1}{\Omega_-^{-1}(x)} \geq \frac{\sqrt{2}}{\sqrt{x}} \quad \text{for } x \geq 0. \tag{17}$$

Applying this to (16) we have

$$T \geq \int_0^M \frac{\sqrt{2}}{\sqrt{\frac{1}{c}(M^2 - h^2)}}\, dh = \frac{\pi}{\sqrt{2}}\sqrt{c}.$$

Hence for $c > \frac{2}{\pi^2}$ the return time, T, is strictly greater than 1 and we are done.

To prove (17) we fix $x \in [0, \infty)$ and define $s, t \geq 0$ by

$$\Omega_+^{-1}(x) = t \quad \text{and} \quad \Omega_-^{-1}(x) = -s.$$

Then

$$x = t - \log(1 + t) = -s - \log(1 - s) \tag{18}$$

and we need to show that

$$\frac{1}{s} + \frac{1}{t} \geq \frac{\sqrt{2}}{\sqrt{x}},$$

i.e.

$$\frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right) \geq \frac{1}{\sqrt{2x}}.$$

By the AM/GM inequality, the left hand side is at least $\frac{1}{\sqrt{st}}$, so it suffices to show that under (18),

$$st \leq 2x.$$

By (18) this will follow if we show that for any $s, t \geq 0$,

$$st \leq t - \log(1+t) - s - \log(1-s),$$

i.e.

$$\log(1+t) + \log(1-s) \leq t - s - st.$$

But the left hand side is

$$\log\left((1+t)(1-s)\right) = \log(1+t-s-st) \leq t - s - st. \qquad \square$$

Finally, to show that $\frac{2}{\pi^2}$ is the best constant, we find that there are specific admissible functions for which the inequality, (8),

$$\int_0^1 h^2(t)\,dt \leq c\int_0^1 \left(1+h'(t)\right)\log\left(1+h'(t)\right)\,dt,$$

does not hold, if $c < \frac{2}{\pi^2}$.

Let $j(t) = \delta \sin \pi t$ where, among other things, δ is sufficiently small to ensure that $j \in C$. Substituting this function into (8) we have

$$\int_0^1 \delta^2(\sin \pi t)^2\,dt \leq c\int_0^1 (1+\delta\pi\cos\pi t)\log(1+\delta\pi\cos\pi t)\,dt. \tag{19}$$

For small δ, the right hand side is

$$c\int_0^1 \delta\pi\cos\pi t + \frac{(\delta\pi\cos\pi t)^2}{2}\,dt + O(\delta^3).$$

So we can rewrite (19) as

$$\int_0^1 \delta^2(\sin\pi t)^2\,dt \leq c\int_0^1 \frac{(\delta\pi\cos\pi t)^2}{2}\,dt + O(\delta^3).$$

Dividing both sides by δ^2 and letting $\delta \to 0$, we get

$$\int_0^1 (\sin\pi t)^2\,dt \leq c\frac{\pi^2}{2}\int_0^1 (\cos\pi t)^2\,dt$$

which does not hold if $c < \frac{2}{\pi^2}$. $\square$

After this article had already been circulated, M. Ledoux communicated an alternative method of finding the same constant, c, in (2). His argument depends upon a reflection trick to modify the logarithmic Sobolev inequality on the interval to the periodic case for which it is known that spectral methods yield the log-Sobolev constant. This can then be transferred to a transportation constant using the methods of Otto and Villani [OV].

This work will form part of a Ph.D. thesis which is being supervised by Keith Ball.

References

[BGH] Buttazzo G., Giaquinta M., Hildebrandt S. (1998) One-Dimensional Variational Problems. Oxford Lecture Series in Mathematics and its Applications, 15, Clarendon Press, Oxford

[M] Marton K. (1996) Bounding $\bar{d}$-distance by informational divergence: a method to prove measure concentration. The Annals of Probability 24(2):857–866

[OV] Otto F., Villani C. (1999) Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality. Preprint

[T] Talagrand M. (1996) Transportation cost for Gaussian and other product measures. Geometric and Functional Analysis 6(3):587–599

[TIS] Tsirel'son B.S., Ibragimov I.A., Sudakov V.N. (1976) Norms of Gaussian sample functions. Proceedings of the Third Japan-USSR Symposium on Probability Theory, Lecture Notes in Mathematics, 550