Lecture Notes in Physics

Edited by J. Ehlers, München, K. Hepp, Zürich and
H. A. Weidenmüller, Heidelberg
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Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics

Vol. I
General Lectures. Fundamental Numerical Techniques

July 3-7, 1972
Universities of Paris VI and XI
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Editors' Preface

This issue of Lecture Notes in Physics is the first of two volumes constituting the Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics, which was held at the University of Paris VI, from July 3 to 7, 1972. Three general lectures and forty eight short individual communications were presented at this conference; the complete proceedings are published here. The general lectures were given by Professor A. DORODNICYN, Director of the Computing Center of the Academy of Sciences of the Soviet Union, who presented the Soviet works dealing with the solution of Navier-Stokes equations; by P. MOREL, professor at the University of Paris VI and Director at the Laboratory of Dynamical Meteorology of the National Center of scientific research (C.N.R.S.), who presented the Problems of numerical simulation of geophysical flows; by Professor R.D. RICHTMYER of the University of Colorado, U.S.A., who spoke on Methods for (generally unsteady) Flows with Shocks.

The individual communications have been separated into two groups: Fundamental Numerical Techniques and Problems of Fluid Mechanics; in each group they are published in the alphabetic order of the author, or of the first of the authors.

Volume I contains the three general lectures and the thirteen communications on Fundamental Numerical Techniques. Volume II contains the thirty five communications on Problems of Fluid Mechanics.

This Conference follows the conferences with the same topic hold at Novossibirsk, U.S.S.R. in 1969, and at Berkeley, U.S.A. in 1970 (the proceedings of which appeared in Lecture Notes in Physics, Vol. 8). The French Organizing Committee was sponsored by Commissariat à l'Energie Atomique, Electricité de France, Union des Chambres Syndicales des Industries de Pétrole, in France, and also by the Office of Naval Research and Air Force Office of Scientific Research, in the U.S.A. The Universities of Paris VI and Paris XI, and the Centre National de la Recherche Scientifique also helped the Committee in a much appreciated manner.

We wish to thank all the persons who contributed to the success of the Conference, the participants for their scientific contributions, our colleagues and younger researchers for their help in the organization and Mrs. M.T. CARTIER and Miss S. DELABEYE for their excellent secretarial work.

Finally we wish to express our appreciation to Dr. W. BEIGLBÖCK and the Springer-Verlag Company for the rapid publication of these proceedings in the series of Lecture Notes in Physics.

January 25, 1973

HENRI CABANNES
ROGER TEMAM

Contents

Volume I

olume II

REVIEW OF METHODS FOR SOLVING THE NAVIER-STOKES EQUATIONS

A. A. Dorodnicyn

During the last 10 - 12 years the numerical solution of the complete equations of motion of a viscous fluid - the Navier-Stokes equations - has been investigated intensively.

Of course, interest in the Navier-Stokes equations arose much earlier. Although the theory of the boundary layer permits us to estimate the influence of viscosity in many practically important cases (perhaps in most of them), nevertheless some phenomena, important in practice, are not described by boundary layer theory. Among them must firstly be mentioned flow with separation. This phenomenon occurs sometimes in flows around ships, planes, or rockets in spite of all possible means being used to avoid it.

It is to be noted that in many complicated cases we cannot imagine the picture of viscous flow even qualitatively and numerical calculations sometimes reveal quite unexpected features of motion.

Finally there is the problem of turbulence. I don't think that complete information on the structure of the turbulence - one which will permit us to construct an adequate theory of turbulance - can ever be derived from experimental observations.

The main difficulty here consists in the non-local character of the relations between the stress-tensor and the deformation tensor. The non-linearity of this relation seems to introduce less complication.

I think the numerical solution of the Navier-Stokes equations is the only way to obtain sufficient information and the application of the methods of mathematical simulation to this information will possibly bring us to some mathematical models of turbulence with good precision.

In fact, modern computers are not yet sufficiently effective for solving the Navier-Stokes equations with values of Reynolds numbers corresponding to fully developed turbulence.

But even "pessimistic" forecasts of progress in computers gives us the hope that in some 10 - 15 years such a possibility will become quite real.

The beginning of the work on the numerical calculations of the viscous fluid motion was caused by the state of computers which just 10 or 12 years ago achieved the level permitting a start to the solution of the problem.

Nevertheless the problem of solution of the Navier-Stokes equations continues to be one of the most difficult problems in the mechanics of a continuous medium and it always requires the fullest possible use of computers.

I. The Methods of Solution

Two properties of the Navier-Stokes equations are the main source of difficulty in their numerical solution: 1. high order of the system (order 4 in the simplest case of an incompressible fluid), and 2. the unbounded domain of the solution (for the most practically interesting cases), together with the elliptical character of the equations.

When the Reynolds number is large a third difficulty appears - strong irregularity in the distribution of physical values (velocity, temperature, etc.).

As the system of Navier-Stokes equations is nonlinear its solution can be found only by means of some iterational procedure. It is well known in numerical analysis that it is advantageous to work with differential equations of the second order - since to solve them the numerical procedure can be reduced to the solution of a three-diagonal algebraic system of equations, and this means the number of necessary arithmetical operations falls from an order $O(n^3)$ to an order $O(n)$, where n is the number of unknown values.

It is quite natural, therefore, to try to construct the iterational procedure in such a way that a sequence of separated second-ordered differential equations has to be solved at each step of the iteration. For this purpose we always used the modification of boundary conditions on solid surfaces which gives the values of the stream-function and vorticity on boundaries at each step of the iteration.

To make the idea clear we consider the plane motion of an incompressible fluid.

For the solution of steady-flow problems three methods were used; to each of them a different computational procedure can be applied.

1. Method of successive approximations

In this method the steady system of Navier-Stokes equations is solved directly.

The system can be written in the form

$$\Delta\omega - 2\frac{\partial\omega}{\partial\xi} = 2\left[\frac{\partial\phi}{\partial\eta}\cdot\frac{\partial\omega}{\partial\xi} - \frac{\partial\phi}{\partial\xi}\cdot\frac{\partial\omega}{\partial\eta}\right] \qquad (I)$$

$$\Delta\phi = F(\xi,\eta)\omega$$

where ξ,η - are some "canonical" coordinates (usually, the potential and stream-function in ideal flow), ω - dimensionless vorticity, ϕ - dimensionless additional stream-function (the full stream-function $\psi = \phi+\eta$).

On solid surfaces, the equations of which in canonical coordinates have the form: η = const, the real boundary conditions are

$$\phi = \text{const}, \quad \frac{\partial\phi}{\partial\eta} = -1 \qquad (II)$$

For reducing the system (I) to a sequence of second-order equations we modify the second boundary condition from (II)

$$\omega = \alpha(\xi)\left[1 + \frac{\partial\phi}{\partial\eta}\right] + \omega$$

and construct successive approximations in the following way

$$\Delta\omega_{n+1} - 2\frac{\partial\omega_{n+1}}{\partial\xi} = 2\left[\frac{\partial\phi_n}{\partial\eta}\cdot\frac{\partial\omega_n}{\partial\xi} - \frac{\partial\phi_n}{\partial\xi}\cdot\frac{\partial\omega_n}{\partial\eta}\right]$$

$$\Delta\phi_{n+1} = F(\xi,\eta)\omega_{n+1} \qquad (III)$$

$$\phi_{n+1} = \text{const}, \qquad \omega_{n+1} = \alpha(\xi)\left(1 + \frac{\partial\phi_n}{\partial\eta}\right) + \omega_n$$

So we see that at each step of the iteration two separated equations of the second order are to be solved.

The coefficient $\alpha(\xi)$ in the boundary conditions (we call it the "relaxation parameter") is introduced to secure the convergence of successive approximations.

If these approximations are convergent the real boundary condition (second in (II)) will automatically be satisfied.

2. "Real" stabilization method

The solution of the steady motion problem is obtained in this method as the limit of the unsteady Navier-Stokes equations when time $t \to \infty$.

The system of equations in canonical coordinates can be written

$$F(\xi,\eta)\frac{\partial\omega}{\partial t} = \Delta\omega - 2\left(\frac{\partial\psi}{\partial\eta}\cdot\frac{\partial\omega}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\cdot\frac{\partial\omega}{\partial\eta}\right)$$
$$\Delta\psi = F(\xi,\eta)\omega \qquad \text{(IV)}$$

(with corresponding choice of time unit).

This system is used naturally for the solution of real unsteady problems, but when we are interested only in finding the final steady solution, the boundary conditions on the solid surface can be modified in such a way that for each time-step the values of vorticity (ω) and stream function (ψ) will be known and equations for ω and ψ become separated.

This modified boundary condition is

$$\frac{\partial\omega}{\partial t} = \alpha(\xi,t)\frac{\partial\psi}{\partial\eta} \qquad (\eta = \text{const}) \qquad \text{(V)}$$

Obviously, the real condition $\partial\psi/\partial\eta = 0$ will be satisfied when the process tends to a steady state.

The real nonstationary system of Navier-Stokes equations is not convenient for numerical calculation, as it is a combinatior of parabolical and elliptical equations. When the steady state only is of interest we are not obliged to use this real system. Any nonstationary system can be used with the same steady part although perhaps it has no physical sense. So to avoid the bad properties of the real system we use the third method called:

3. "Artificial" stabilization method

In this method the solution of the steady problem is obtained as the limit ($t \to \infty$) of the solution of the system of parabolic equations:

$$\frac{\partial\omega}{\partial t} = \Delta\omega - 2\left(\frac{\partial\psi}{\partial\eta}\cdot\frac{\partial\omega}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\cdot\frac{\partial\omega}{\partial\eta}\right)$$
$$\frac{\partial\psi}{\partial t} = \Delta\psi - F(\xi,\eta)\omega \qquad \text{(VI)}$$

The boundary condition on a solid wall has the same form (V).

II. The Computational Procedure

In the methods of successive approximation, as we have seen, the problem is reduced to successive solutions of the Helmholz and Poisson equations. There are many effective methods for the numerical solution of these classical equations, especially taking into account that the forms of the domains in canonical coordinates are very simple (strips or planes with sections).

In our calculation we usually used the matrix factorization method. This is well known too, so it is not necessary to describe it here.

As the most interesting problems connected with the Navier-Stokes equations deal with flow in an infinite domain, the question arises how to transmit the condition from infinity to some finite distance.

In the case of the flow in a channel with parallel walls at infinity the boundary conditions for sufficiently large ξ can be written in the form:

$$\frac{\partial \omega}{\partial \xi} = 0 \quad , \quad \frac{\partial \psi}{\partial \xi} = 0 \qquad \text{when} \quad |\xi| = X_o >> 1$$

As the flow in this case tends to Poiseuille flow the comparison with it can serve as some kind of control of accuracy.

When the asymptotic expression for the solution can be established (as, for instance, in the case of a semi-infinite flat plate), the boundary conditions at a large but finite distance can be deduced from these expressions. For example, in the flow around the flat plate the asymptotic expressions are:
for $\xi >> 1$

$$\omega \sim \frac{1}{\sqrt{x}} \quad , \qquad \phi \sim \sqrt{x}$$

Hence

$$\frac{\partial \omega}{\partial x} \cong -\frac{\omega}{2x} \quad , \qquad \frac{\partial \phi}{\partial x} \cong \frac{\phi}{2x}$$

and the boundary condition taken for calculation was:
when $\xi = X_o >> 1$

$$\frac{\partial \omega}{\partial x} = -\frac{\omega}{2\,X_o} \quad , \qquad \frac{\partial \phi}{\partial x} = \frac{\phi}{2\,X_o} \quad .$$

Again the well known Blasius solution for the boundary layer flow past a flat plate could be used as a control of the accuracy of the calculations.

In the problems of external flow around some finite body, far from this body the solution of the Navier-Stokes equations asymptotically tends to the solution of the Oseen equations. This asymptotic equation gives the relation between the values of normal derivatives ω or ϕ and these functions themselves. In finite-difference approximation this relation will have the form:

$$\frac{\partial \vec{\omega}}{\partial n} = A\vec{\omega} \quad , \quad \frac{\partial \vec{\phi}}{\partial n} = B\vec{\phi} + C\vec{\omega}$$

Here by the "vector" $\partial\vec{\omega}/\partial n$ we understand the set of values of $\partial\omega/\partial n$, $\vec{\omega}$ - the set of values of ω on the boundary, chosen far enough from the body. The matrices A, B, and C are calculated from the general solution of the Oseen equations for the outer domain.

These approaches to the approximation of conditions at infinity are, of course, applicable to any of the methods described.

The "artificial stabilization method" provides the greatest "freedom" in the choice of computational procedures. Just as for systems of parabolic equations the latest methods of decomposition of computational operator can be applied in the most natural way. More precisely, these are the methods which reduce the solution of a algebraic linear system of high order to the solution of some set of systems of lower order.

For the system (VI) we used several methods; two of them deserve to be mentioned here.

Alternating directions method in which the space derivativatives in ξ and η directions are written in turn in implicit form.

The first time "half step":

$$\frac{\psi_{n+1} - \psi_n}{\Delta t} = \frac{\partial^2 \psi_{n+1}}{\partial \xi^2} + \frac{\partial^2 \psi_n}{\partial \eta^2} - F(\xi,\eta)\omega_n$$

$$\frac{\omega_{n+1} - \omega_n}{\Delta t} = \frac{\partial^2 \omega_{n+1}}{\partial \xi^2} + \frac{\partial^2 \omega_n}{\partial \eta^2} - 2\left[\frac{\partial \psi_{n+1}}{\partial \eta} \cdot \frac{\partial \omega_{n+1}}{\partial \xi} - \frac{\partial \psi_{n+1}}{\partial \xi} \cdot \frac{\partial \omega_n}{\partial \eta}\right]$$

$$\frac{\omega_{n+1} - \omega_n}{\Delta t} = \alpha(\xi, t_{n+1}) \frac{\partial \psi_{n+1}}{\partial \eta} \qquad (\eta = \text{const})$$

The second time "half step":

$$\frac{\psi_{n+2} - \psi_{n+1}}{\Delta t} = \frac{\partial^2 \psi_{n+1}}{\partial \xi^2} + \frac{\partial^2 \psi_{n+2}}{\partial \eta^2} - F(\xi,\eta)\omega_{n+1}$$

$$\frac{\omega_{n+2} - \omega_{n+1}}{\Delta t} = \frac{\partial^2 \omega_{n+1}}{\partial \xi^2} + \frac{\partial^2 \omega_{n+2}}{\partial \eta^2} - 2\left[\frac{\partial \psi_{n+2}}{\partial \eta} \cdot \frac{\partial \omega_{n+1}}{\partial \xi} - \frac{\partial \psi_{n+2}}{\partial \xi} \cdot \frac{\partial \omega_{n+2}}{\partial \eta}\right]$$

$$\frac{\omega_{n+2} - \omega_{n+1}}{\Delta t} = \alpha(\xi, t_{n+2}) \frac{\partial \psi_{n+2}}{\partial \eta} \qquad (\eta = \text{const})$$

For simplicity we write here the space derivatives in differential form. In calculations they are, of course, replaced by corresponding finite-difference expressions. It is important to note that the three-point approximation is used, for instance,

$$\frac{\partial^2 \omega_n}{\partial \xi^2} \cong \frac{\omega_n^{k+1} - 2\omega_n^{k} + \omega_n^{k-1}}{\Delta \xi^2}$$

Just this three-point scheme reduces the problem to the solution of a three-diagonal system of algebraic linear equations.

Locally onedimensional method. Here the following sequence of systems is solved:

The first half-step

$$\frac{\psi_{n+1} - \psi_n}{\Delta t} = 2 \frac{\partial^2 \psi_{n+1}}{\partial \xi^2}$$

$$\frac{\omega_{n+1} - \omega_n}{\Delta t} = 2 \frac{\partial^2 \omega_{n+1}}{\partial \xi^2} - 4 \frac{\partial \psi_{n+1}}{\partial \eta} \cdot \frac{\partial \omega_{n+1}}{\partial \xi}$$

The second half-step

$$\frac{\psi_{n+2} - \psi_{n+1}}{\Delta t} = 2 \frac{\partial^2 \psi_{n+2}}{\partial \eta^2} - 2F(\xi,\eta)\omega_{n+1}$$

$$\frac{\omega_{n+2} - \omega_{n+1}}{\Delta t} = 2 \frac{\partial^2 \omega_{n+2}}{\partial \eta^2} + 4 \frac{\partial \psi_{n+2}}{\partial \xi} \cdot \frac{\partial \omega_{n+2}}{\partial \eta}$$

The boundary conditions are approximated as in previous cases.

We see in this method there is no approximation of differential equations at each half-step. But two half-steps together give

$$\frac{\psi_{n+2} - \psi_n}{2\,\Delta t} = \frac{\partial^2 \psi_{n+1}}{\partial \xi^2} + \frac{\partial^2 \psi_{n+2}}{\partial \eta^2} - F(\xi,\eta)\omega_{n+1}$$

$$\frac{\omega_{n+2} - \omega_n}{2\,\Delta t} = \frac{\partial^2 \omega_{n+1}}{\partial \xi^2} + \frac{\partial^2 \omega_{n+2}}{\partial \eta^2} - 2\left[\frac{\partial \psi_{n+1}}{\partial \eta} \cdot \frac{\partial \omega_{n+1}}{\partial \xi} - \frac{\partial \psi_{n+2}}{\partial \xi} \cdot \frac{\partial \omega_{n+2}}{\partial \eta}\right]$$

which is the approximation of the differential system.

To give "equal weight" to both variables ξ and η the alternating of them is used at successive full steps, let us say: first full step - at first the derivatives with respect to ξ remain in the equations and at the second half step the derivatives with respect to η remain; second full step - first half-step, derivatives in η remain, second half-step, derivatives in ξ remain and so on: ξ, η; ξ, η; ξ, η....

In the real stabilization method the procedures described can be applied only to the equation for the vorticity. For the stream-function some other method must be used, which gives for each time step the solution of the elliptical Poisson equation, by the matrix-factorization method, for instance.

This method is not good enough for large Reynolds numbers, because it requires a large computer memory. Therefore some other methods must be used (for instance, based on Fourier expansions) which permit some economy of memory, although requiring many more arithmetical operations.

Finally I shall mention one important computational procedure which permits us to simplify calculations in the case of a flow domain of complicated shape.

The simple shapes of domain (for example - rectangles) permit us to reduce the amount of calculation and the logic of the computer program also. If some complicated domain can be subdivided in a set of simple sub-domains, it is reasonable to carry out the solution for each sub-domain connecting them on the boundaries. For illustration of the idea let us consider the flow in Borda's mouthpiece (Fig. 1). We construct the solution of the Navier-Stokes system in sub-domains 1 and 2 (SD1 and SD2 on Fig. 1). On the common border (dotted line) the following conditions must be fulfilled:

$$\frac{\partial \omega_1}{\partial \eta} = \frac{\partial \omega_2}{\partial \eta} \quad ; \quad \omega_1 = \omega_2 \quad ; \quad \frac{\partial \psi_1}{\partial \eta} = \frac{\partial \psi_2}{\partial \eta} \quad ; \quad \psi_1 = \psi_2$$

Instead of this system of four conditions we write two conditions for each sub-domain:

SD1

$$\frac{\partial \psi_1}{\partial \eta} = \beta(\psi_1 - \psi_2) + \frac{\partial \psi_1}{\partial \eta}$$

$$\frac{\partial \omega_1}{\partial \eta} = \gamma(\omega_1 - \omega_2) + \frac{\partial \omega_1}{\partial \eta}$$

SD2

$$\frac{\partial \psi_2}{\partial \eta} = \beta(\psi_1 - \psi_2) + \frac{\partial \psi_2}{\partial \eta}$$

$$\frac{\partial \omega_2}{\partial \eta} = \gamma(\omega_1 - \omega_2) \quad \frac{\partial \omega_2}{\partial \eta}$$

Using successive approximations we have

$$\frac{\partial \psi_{1,n+1}}{\partial \eta} = \beta(\psi_{1,n} - \psi_{2,n}) + \frac{\partial \psi_{1,n}}{\partial \eta}$$

$$\frac{\partial \omega_{1,n+1}}{\partial \eta} = \gamma(\omega_{1,n} - \omega_{2,n}) + \frac{\partial \omega_{1,n}}{\partial \eta}$$

and similarly for the second sub-domain. If the initial approximation ensures that

$$\frac{\partial \psi_{1,0}}{\partial \eta} \equiv \frac{\partial \psi_{2,0}}{\partial \eta} \quad , \quad \frac{\partial \omega_{1,0}}{\partial \eta} \equiv \frac{\partial \omega_{2,0}}{\partial \eta}$$

then the equality between $\partial\omega_1/\partial\eta$ and $\partial\omega_2/\partial\eta$ will always be fulfilled (the same for $\partial\psi_1/\partial\eta$ and $\partial\psi_2/\partial\eta$). If the successive approximations converge then automatically the condition $\omega_1 = \omega_2$, $\psi_1 = \psi_2$ will be satisfied. We see that at each step of the iteration the equations for each sub-domain are solved separately.

The convergence of the method (by proper choice of the relaxation parameters β or γ) is easily proved for Laplace's equation. For the Navier-Stokes system the convergence was verified only by calculations.

Quite naturally the question can arise: which method is better?

I answer quite definitely - the stationary method of successive approximation (if convergent, of course). Its most satisfying property is that the rate of convergence does not depend on the accuracy of the approximation (upon the number of nodal points in finite-difference grid). In any stabilization method the time-step decreases when the finite-difference grid becomes denser.

Nevertheless the application of different methods gives some possibility for better understanding of the processes in viscous fluids. I will make some further remarks on this.

III. Some Results of Calculation

Now the number of different cases calculated is already so huge all over the world that there is no reason to show many "picturesque images". I limit myself therefore to a few results which I selected, considering them to be "suggestive" in some respect.

As the first example consider the motion in an expanding channel (Fig. 2). The flow picture here has no surprises.

At small Reynolds numbers separation does not occur. When the Reynolds number increases the separation appears, the point of detachment very soon stabilizes, while the reattachment point moves off roughly proportionally to Reynolds number (Fig. 3).

I give this example because it was a test case in our calculations for which different methods were tried out. In Fig. 4, for example, the comparison of calculations by the method of successive approximation and by artificial stabilization method (using locally one-dimensional procedure) is represented. Here Re - 100 (32 π).

This use of different methods disclosed one interesting phenomenon: all the methods fail when Re is close to 200.

The properties of the methods are very different, and the easiest way to explain their simultaneous failure is to suppose that the solution of the steady

flow equations does not exist for such large Reynolds numbers.

Of course, numerical methods can never give the proof of some mathematical fact, but they are suggestive at least.

In connection with this we intend to undertake wide numerical experiments with the use of different methods for the same cases of flow with separation. All the calculations for this case were made by Miss N. Meller - the senior scientific collaborator of the Computing Center.

Figures 4 and 5 show the flow around fixed and rotating cylinders (the example is taken from the work of Dr. V. Lulka).

The separation vanishes when the rotation speed is large enough.

The third example illustrates the motion in a "pit" - in the channel with local expansion. Here the method of subdividing of domain was used (dotted lines on Fig. 6). The results of the calculation were kindly put at my disposal by the Bulgarian postgraduate Miss E. Mateyeva.

The picture of flow itself is interesting here. It shows how careful one must be when trying to use the integral conservation laws (mass, impulse, energy) on the basis of some flow picture suggested a priori.

I am not showing here any examples of compressible flow calculations, for although they include some interesting results, we are still in the initial stage of investigating methods in this case.

IV. The Problem of Large Reynolds Numbers

In assessing the possibility of the numerical solution of the Navier-Stokes equations in the present state of computer capacity, we can say that this is realizable for an incompressible fluid in plane or axisymmetrical flow, and perhaps even easy for practical calculations when Reynolds numbers are not too large (of the order one hundred, approximately). Unfortunately, the most interesting problems in practice relate to much larger Reynolds numbers.

The difficulties here, I would say, are purely computational. In a finite - difference approximation to the nonlinear term of equations the truncation error is in general of the order $Re \cdot h$ ($h = \Delta x$ or Δy), when the order of main viscous term is supposed to be of the order 1. Obviously, an approach to the real solution can be reached when $Re \cdot h < \varepsilon \ll 1$, otherwise the mathematical viscosity exceeds the real one and we don't know what the numerical solution means.

It seems that by using central-difference expressions for derivatives we can reduce the error to the order $Re \cdot h^2$ and the condition of approximation will be $Re \cdot h^2 < \varepsilon \ll 1$, which is much more favorable than the previous one. However, the truncation error in this case contains the third derivatives of vorticity, and we can't be sure that ε is independent of Re. To imagine what it means, let us evaluate, for example, the work for solution of an unsteady problem in the case of Poiseuille motion between two plates. According to C. C. Lin's calculation the breakdown of stability of Poiseuille motion occurs when $Re > 6000$. Bearing in mind that we wish to obtain the picture of development of turbulence by numerical calculations, we need to satisfy the condition, let us say, $Re \cdot h^2 = 0.1$, that is, $h \cong 1/250$. The length of channel must be at least 20 times more than the width. So our finite - difference grid must have at least 5000 nodal points.

Even with all the achievements of modern computer technology this is a horrible problem. And it must be noted that the main obstacle now is the volume of fast memory - not the speed of computers. Even with the speeds which are already attainable in a good computer, the problem would have been attacked, if the fast memory contained something like 10^7 words.

These estimates, unfavorable as they are for the time being, nevertheless show a good prospect for the future (and even the near future). Computers will soon have such a memory, together with an increase of speed.

In conclusion I would like to make a remark. For practical application of numerical solutions of the Navier-Stokes equations it is not always necessary to apply them over the whole domain of flow. We have to learn to combine the solutions of this system in different approximations: boundary layer, ideal fluid approximations. To use "dogmatically" the complete Navier-Stokes equations for the whole field of flow is not only difficult, but even unreasonable: the accuracy of results can deteriorate instead of increasing.

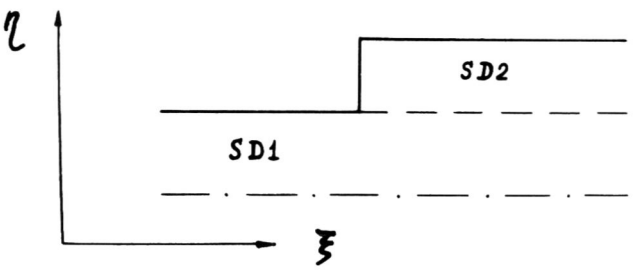

Fig. 1

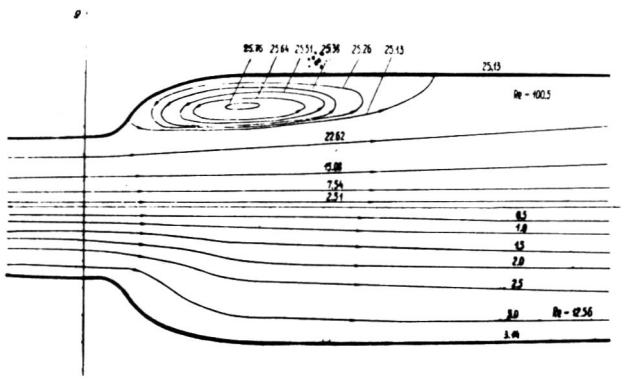

Fig. 2